Open Problems in the OEIS

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Guest Lecture, Zeilberger Experimental Math Class, May 2 2016

- Puzzles
- Strange recurrences
- Number theory
- Counting problems
PUZZLES

61, 21, 82, 43, 3, ?

(A087409)
Low-Hanging Fruit from the OEIS

Some new problems for the ghosts of Fermat, Gauss, Euler, ...
Strange Recurrences

- Modified Fibonacci
- Reed Kelley
- A recurrence that looks ahead
- Van Eck’s sequence
**Modified Fibonacci**

\[ a(n) = a(n-1) + a( (a(n-1)-1) \ mod \ n ) \] with \( a(0)=a(1)=1 \)

\[ \text{A268176, Christian Perfect, Jan 2016} \]

Similar to \[ \text{A125204, also not analyzed} \]

Explain!
Reed Kelley’s Sequence  A214551

14th century Narayana cows sequence A930:

\[ a(n) = a(n - 1) + a(n - 3) \]

1, 1, 1, 2, 3, 4, 6, 9, 13, 19, 28, ...

Reed Kelley, 2012:

\[ a(n) = \frac{a(n - 1) + a(n - 3)}{\gcd\{a(n - 1), a(n - 3)\}} \]

1, 1, 1, 2, 3, 4, 3, 2, 3, 2, 2, 5, 7, 9, 14, 3, ...

(Have guesses, but nothing is proved.)
A recurrence that looks ahead

\[ a(2k) = k + a(k), \quad a(2k+1) = k + a(6k+4) \] with \( a(1) = 0 \).

**A271473**, suggested by 3x+1 sequence **A6370**
and new **A266569**

Explain!
Jan Ritsema van Eck’s Sequence

0, 0, 1, 0, 2, 0, 2, 2, 1, 6, 0, 5, 0, 2, 6, 5, 4, 0, 5, 3, 0, 3, 2, 9, 0, 4, 9, 3, 6, 14, 0, 6, 3, 5, 15, 0, 5, 3, 5, 2, 17, 0, 6, 11, 0, 3, 8, 0, ...

a(n): how far back did we last see a(n-1)?
or 0 if a(n-1) never appeared before.

Van Eck: A181391
A181391 as a graph:

Pin plot of A181391

Van Eck: A181391
Thm. (Van Eck) There are infinitely many zeros.

Proof: (i) If not, no new terms, so bounded.
    Let $M = \text{max term}$.
    Any block of length $M$ determines the sequence.
    Only $M^M$ blocks of length $M$.
    So a block repeats.
    So sequence becomes periodic.
    Period contains no 0’s.

Van Eck: A181391
Proof (ii). Suppose period has length $p$ and starts at term $r$.

Therefore period really began at term $r - 1$.

Therefore period began at start of sequence.
But first term was 0, contradiction.

Van Eck: A181391
It seems that:

\[ \limsup \frac{a(n)}{n} = 1 \]

Gaps between 0’s roughly \( \log_{10} n \)

Every number eventually appears

Proofs are lacking!

Van Eck: A181391
Conjecture:
There is no nontrivial cycle

Trivial cycle

Nontrivial cycle

( David Applegate: Only trivial cycles of length up through 14 )
Number Theory

- Sum of primes in sum of previous terms
- $3^n + 1 = \text{square} + \text{square}$
- Yosemite graph
- Leroy Quet's prime-producing sequence
- 999999000000
- A memorable prime
a(n) = sum of prime factors of sum of all previous terms (with repetition, starting 1, 1)

1, 1, 2, 4, 6, 9, 23, 25, 71, 73, 48, 263, 265, 120, 911, 913, 552, 192, 85, 27, 35, 53, 296, 66, 455, 289, 48, 188, 5021, 5023, 159, 190, 379, 946, 900, 600, 97, 204, 118, 512, 87, 148, 3886, 23291, 23293, 71, 896, 11812, 60, 41359,

\[1 + 1 + 2 + 4 + 6 = 14 = 2 \times 7 \text{ gives } 2 + 7 = 9\]

A268868, David Sycamore, Feb 2016

Explain!
Generalize!
Odd numbers \( n \) such that \( 3^n + 1 \) is sum of 2 squares

5, 13, 65, 149, 281, 409, 421, 449, 461, 577

\[ 3^5 + 1 = 244 = 10^2 + 12^2 \]

Found by Keenan Curtis, u/grad, Wake Forest U.
Only 10 terms known

\textcolor{red}{\textbf{A272069 = A404 intersect A34472}}

April 19 2016
Yosemite Graph?? \hspace{1cm} (A272412)

Numbers n such that sum of divisors \((A203(n))\) is a Fibonacci number \((\text{in A45})\)

Random combination of 2 sequences, except look at the graph:

Altug Alkan, Apr 29 2016

Have 10000 terms but need a lot more
Hostadter’s Q-sequence
A5185

Leroy Quet’s Prime-generating sequence
A134204

Franklin Adams-Watters
A166133

“About your cat, Mr. Schrödinger—I have good news and bad news.”

(The New Yorker, March 2015)
Leroy Quet’s Prime-Producing Sequence

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\[ q = \text{smallest missing prime such that } n \text{ divides } p + q \]

10 divides 31 + 29

\[ p + q = kn \]
\[ q = -p + kn \]

Dirichlet: OK unless \( p \) divides \( n \)

Does the sequence exist?

800,000,000 terms exist
If \( \left\lfloor n^{1/k} \right\rfloor \mid n \) for all \( k \) then \( n \leq 9999999000000 \) (conj.)

1, 2, 4, 6, 12, 36, 132, 144, 156, 900, 3600, 4032, 7140, 18360, 44100, 46440, 4062240, 9147600, 999999000000

No more terms below \( 10^{16} \)
9999999000000 (cont.)

Th. 1

\[ \left\lfloor \sqrt{n} \right\rfloor \mid n \iff n = \begin{bmatrix} \frac{M}{2} \\ \frac{M}{2} \end{bmatrix} \text{ for some } M \]

(the quarter-squares, A002620)

Pf.

\[ \left\lfloor \sqrt{n} \right\rfloor = m + 1 \iff m^2 + 1 \leq n \leq (m + 1)^2 \]

Say \( n = m^2 + 1 + i \)

So \( i = m - 1 \) or \( 2m \), \( n = m(m + 1) \) or \( (m + 1)^2 \)

\( M = 2m + 1 \) or \( 2m + 2 \)

Example:

\[ 9999999000000 = \begin{bmatrix} \frac{1999999}{2} \\ \frac{1999999}{2} \end{bmatrix} \]
999999000000 (cont.)

Th. 2

\[ \left\lfloor \frac{n^{1/3}}{} \right\rfloor \mid n \iff n = m^3 + 1 + \lambda(m + 1), \quad 0 \leq \lambda \leq 3m \]

for some \( m \) \quad (A261011)

Example: With \( m = 9999, \lambda = 29897 \),

\[ m^3 + 1 + \lambda(m + 1) = 999999000000 \]

If both Th 1 and Th 2 apply, get A261417:

1, 2, 4, 6, 9, 12, 36, 56, 64, 90, 100, 110, 132, 144, 156, 210, 400, 576, 702, 729, 870, ...

And so on?
A Memorable Prime

12345678910987654321

When is 123...n-1 n n-1...321 prime?

It is a square: 11...1^2 for n ≤ 9.

Prime for n=10, 2446 (Shyam Gupta, PRP only), ...

Or, in base b, when is 123...b-1 b b-1...321 prime?

Prime for b =

2, 3, 4, 6, 9, 10, 16, 40, 104, 8840 (PRP)

(David Broadhurst, Aug 2015, A260343)
Counting Problems

• Sequences with no final repeats
• Lines in the plane; or in general position
• Points in \( \{0,1\}^n \) with no right angles
• Alex Meiburg’s A260273
Sequences with no final repeats

Number of binary sequences, length \(n\), not of form \(XY^k, k > 1\)

Good: 00001, 11001
Bad: 00000, 00011, 00101

2, 2, 4, 6, 12, 20, 40, 74, 148, 286, ...  \(\text{(A122536)}\)

\[
a(2n + 1) = 2a(2n), \quad a(2n) = 2a(2n - 1) - b(n)
\]

where \(b(n) = \text{number of robust sequences } S\)

[ SS without initial symbol has no final repeats ]

\(S = 32232\) is not robust: SS = 32232 32232

Have 200 terms.  Conj.  \(a(n) / 2^\uparrow n \to 0.27004339...\)
No. of ways to arrange \( n \) lines in the plane

\[ \begin{align*}
\alpha(1) &= 1 \\
\alpha(2) &= 2 \\
\alpha(3) &= 4 \\
\alpha(4) &= 9 \\
\alpha(5) &= 47 \\
\alpha(6) &= 791 \\
\alpha(7) &= 37830
\end{align*} \]
\( a(5) = 47 \). Summary:

\[
\begin{align*}
& P_5 : 1, \quad P_4 L : 1, \quad P_3 P_2 : 1, \quad P_3 S_2 : 4, \quad P_2 L : 6, \\
& P_2 S_3 : 14, \quad P_2 S_3 : 3, \quad S_5 : 1, \quad S_4 L : 1, \quad G_5 : 6
\end{align*}
\]

\( \begin{array}{c}
(5.1) \quad P_5 \\
(5.2) \quad P_4 L \\
(5.3) \quad P_3 P_2 \\
(5.4)-(5.7) \\
(5.8)-(5.13)
\end{array} \)

\( P_3 S_2 : \)

\( P_2^2 L : \)

A241600 (cont.)
A90338

A subset: $n$ lines in general position

$1, 1, 1, 1, 6, 43, 922, 38609$

Wild and Reeves, 2004

5 lines in general position: 6 ways

$a(5) = 6$
Points in \( \{0,1\}^n \) with no right angles

\[ a(n) = \text{max no of points in } \{0,1\}^n \text{ such that all angles } PQR \text{ are less than } 90 \text{ degs.} \]

**A089676**, Classic problem, only 10 terms known!

1, 2, 2, 4, 5, 6, 8, 9, 10, 16

\[ a(3) = 4: \{000, 011, 101, 110\}. \]
\[ a(4) = 5: \{0011, 0101, 0110, 1000, 1111\}. \]
\[ a(5) = 6: \{00000 00011 00101 01001 10001 11110\} \]

**NEED MORE TERMS!**

Prompted by Prof. Jeff Kahn’s lecture on The Probabilistic Method, March 28 2016

Monday, May 2, 16
Alex Meiburg’s  A260273
Define $M(n)$: E.g. $n = 57 = 111001$

Can see 0, 1, 10, 11, 100 but not 101 so $M(57)=5$

$M(n)$ = smallest missing number in binary exp. of $n$  \hspace{1cm} (A261922)

$M'(n)$ = smallest missing positive number in binary exp. of $n$

$$a(1)=1; \quad a(n+1) = a(n) + M'(a(n))$$

1, 3, 5, 8, 11, 15, 17, 20, 23, 27, 31, 33, 36, ...

$$a(n) \sim \frac{n}{2} \log_2(n)$$

Conjecture (Meiburg):
\[ T(n,k) = \text{no. of binary numbers of length } n \text{ with } M(x) = k \]
Sum \( k \cdot T(n,k) = \text{A261016} \):

1, 6, 18, 46, 107, 241, 535, 1178, 2569, 5546, 11859, 25156, 53058, ...

Divide by \( 2^n \): average step size in Meiburg’s sequence

What is this sequence? Have 58 terms from Hiroaki Yamanouchi.

\[
a(n) \approx 2^n \left( \frac{n}{4} + 4.3 \right)
\]

Need analysis of \text{A261019}, \text{A261016} and related sequences!
Smallest Prime Beginning
With the “igits” of
Previous Prime
A262283

2, 3, 5, 7, 11, 13, 31, 17, 71, 19, 97, 73, 37, 79, 907, 701, 101, ...

s = digits of a(n) without leading digit,
a(n+1) = smallest missing prime beginning with s.

Show 23 etc never appear!

A. Murthy, F. Adams-Watters, A. Heinz, R. Zumkeller, NJAS

(Binary analog A262374 etc)
Circulant determinant equals number

Generalize:

\[
\begin{vmatrix}
2 & 4 & 7 \\
7 & 2 & 4 \\
4 & 7 & 2
\end{vmatrix} = 247.
\]

N. I. Belukhov, 2011.

247, 370, 378, 407, 481, 518, 592, 629, 1360, 3075, 26027, ...

47 terms are known (A219324).
We need editors!

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