New Gilbreath Conjectures, Sum and Erase, Dissecting Polygons, and Other New Sequences

Experimental Math Seminar, Rutgers, September 14 2022

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Outline of talk

- Scott Shannon's circle counting problems
- Dissecting regular polygons into rectangles (with Gavin Theobald)
- New Gilbreath Conjectures
- Éric Angelini's Sum and Erase sequence
- Report on Status of OEIS
- [Sequences with No 3-Term Arithmetic Progressions]

Circle Counting Problems (Scott Shannon)

Scott Shannon's Circle Counting Problems

BACKGROUND

Lars Blomberg, Scott Shannon, N. J. A. Sloane, A long-running project, counting regions in "Stained Glass Windows"

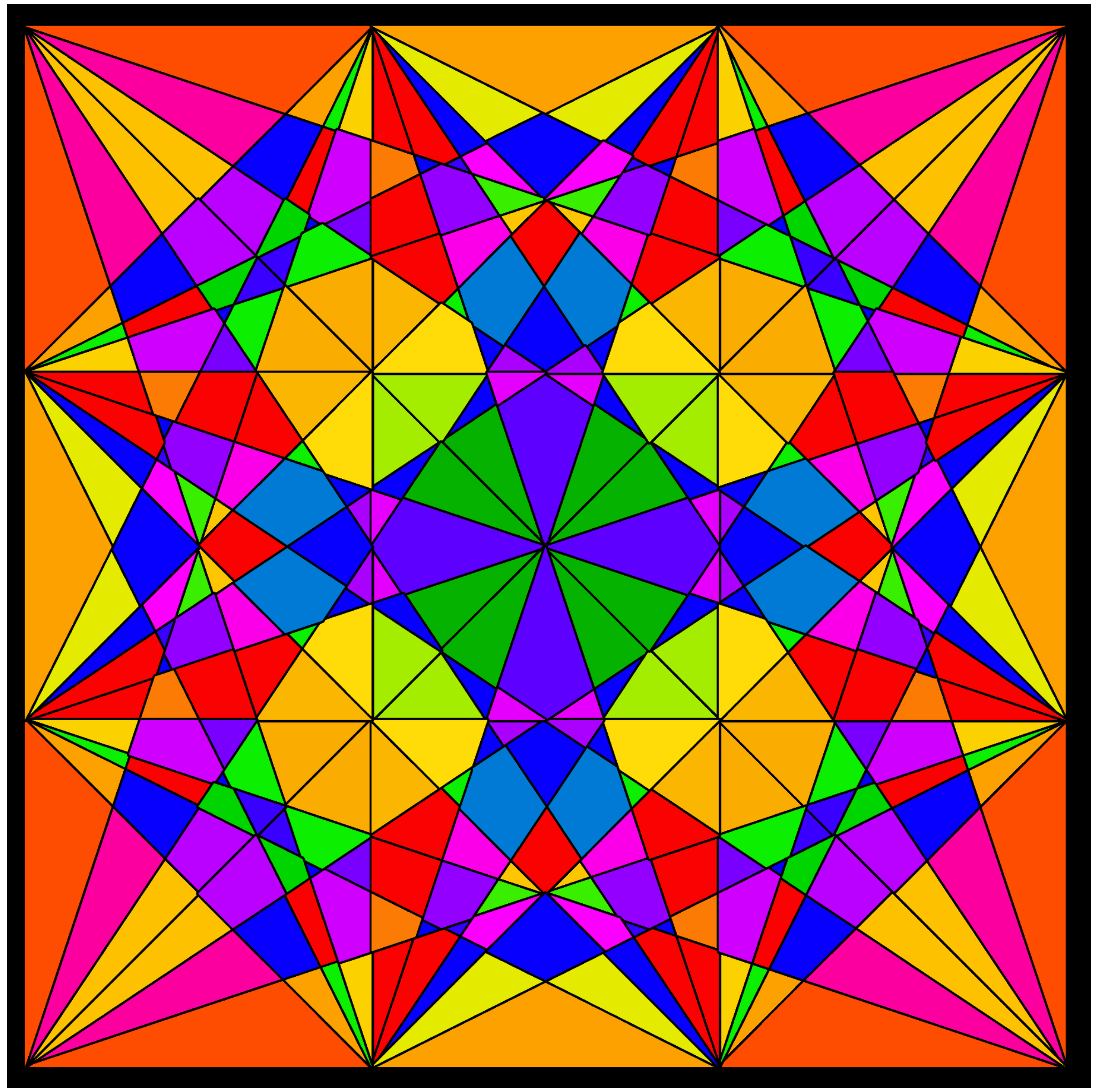
Typical problem:

n x n grid of points

Join each pair of boundary points by a chord

In resulting graph, count vertices, edges, regions.

A331452 + many more



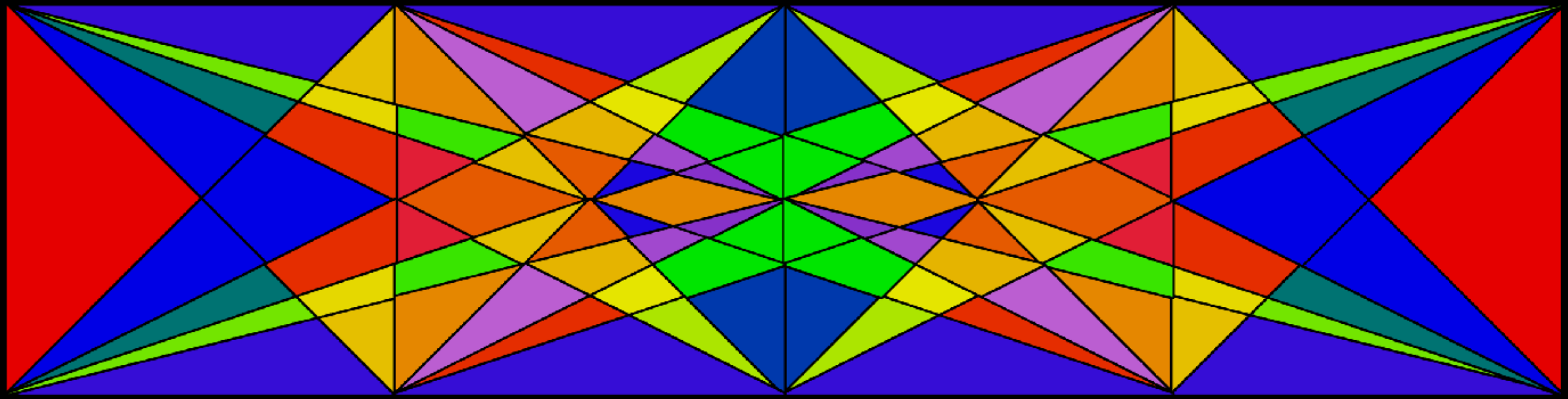
4 x 4 grid

See: Blomberg, Shannon, NJAS, Graphical enumeration and stained glass windows I, Integers, 2022.

Problem LastYear_1

A306302

BC(1, 4)



104 cells (70 triangles, 34 quadrilaterals)
but no pentagons or hexagons - why?!

Blomberg, Shannon, NJAS,
Graphical enumeration and stained
glass windows I, Integers, 2022.

Farey Trees (Scott Shannon & N.J.A.S., Dec 2022)

See A358948, A358949 for counts of regions and vertices

Farey Series

0 1

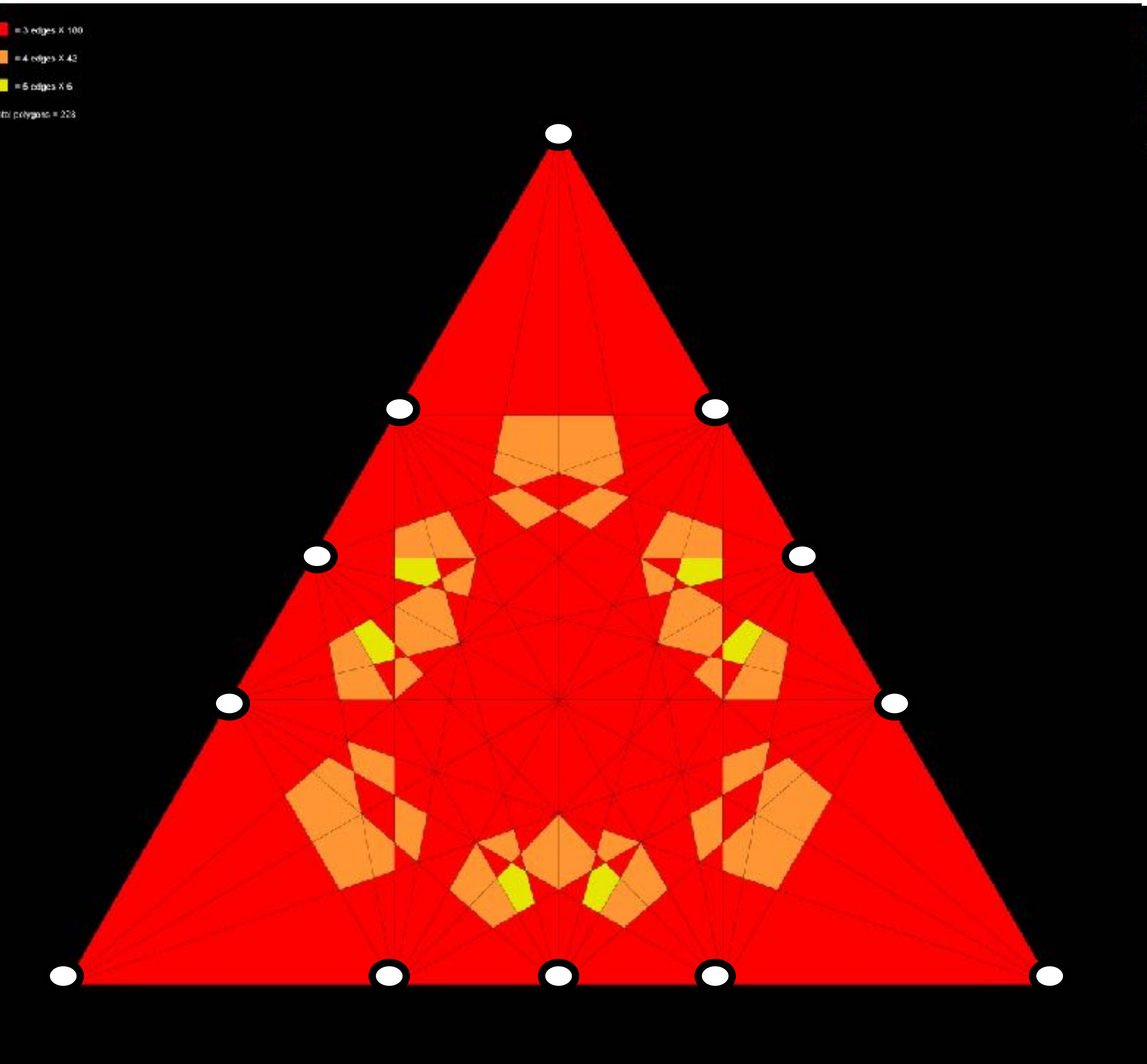
0 1/2 1

0 1/3 1/2 2/3 1

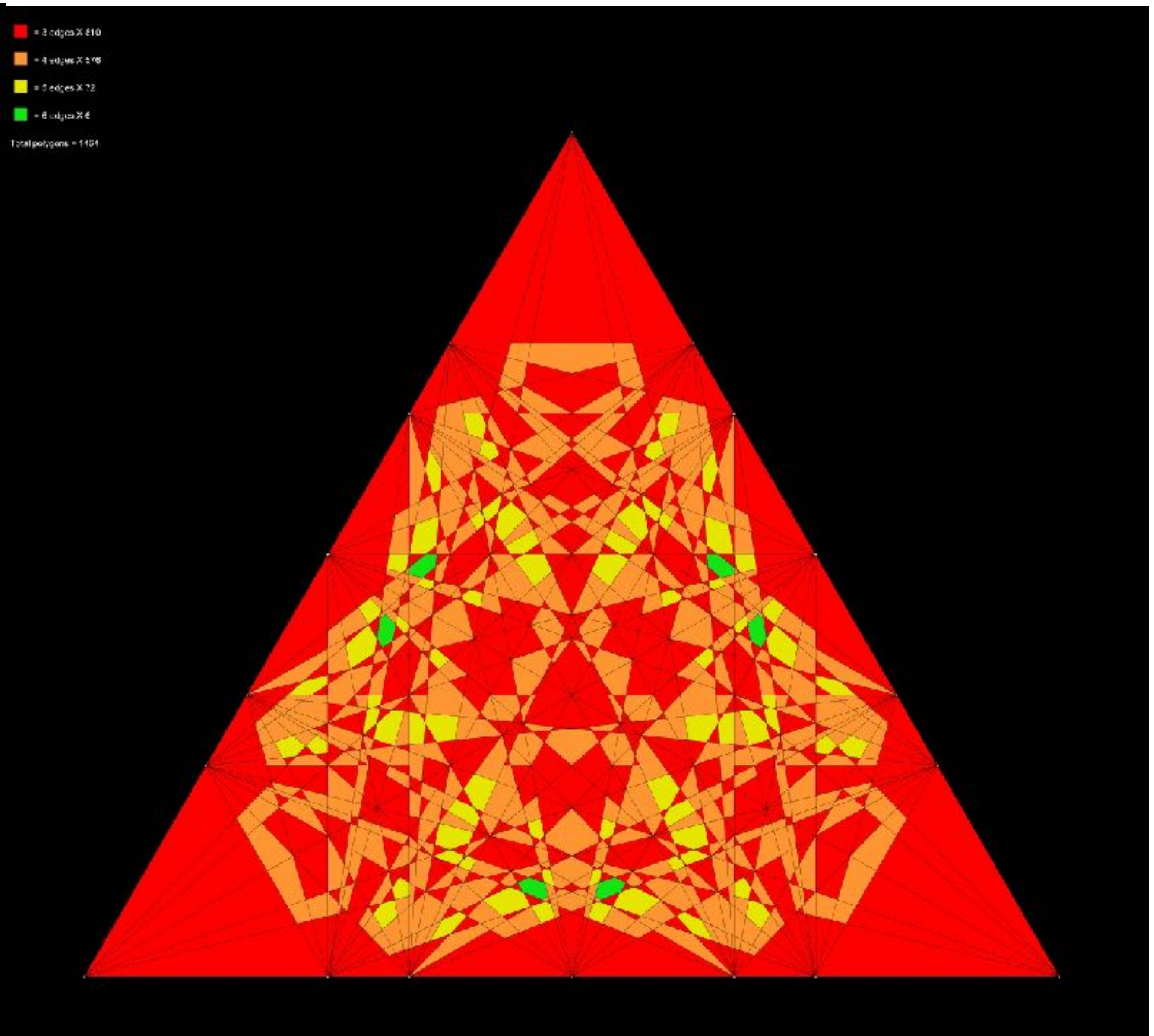
0 1/4 1/3 1/2 2/3 3/4 1

A006842

Order 3



Order 4



Farey Tree of Order 6 (Scott Shannon & N.J.A.S., A358949(6) = 23770 vertices Dec 2022)

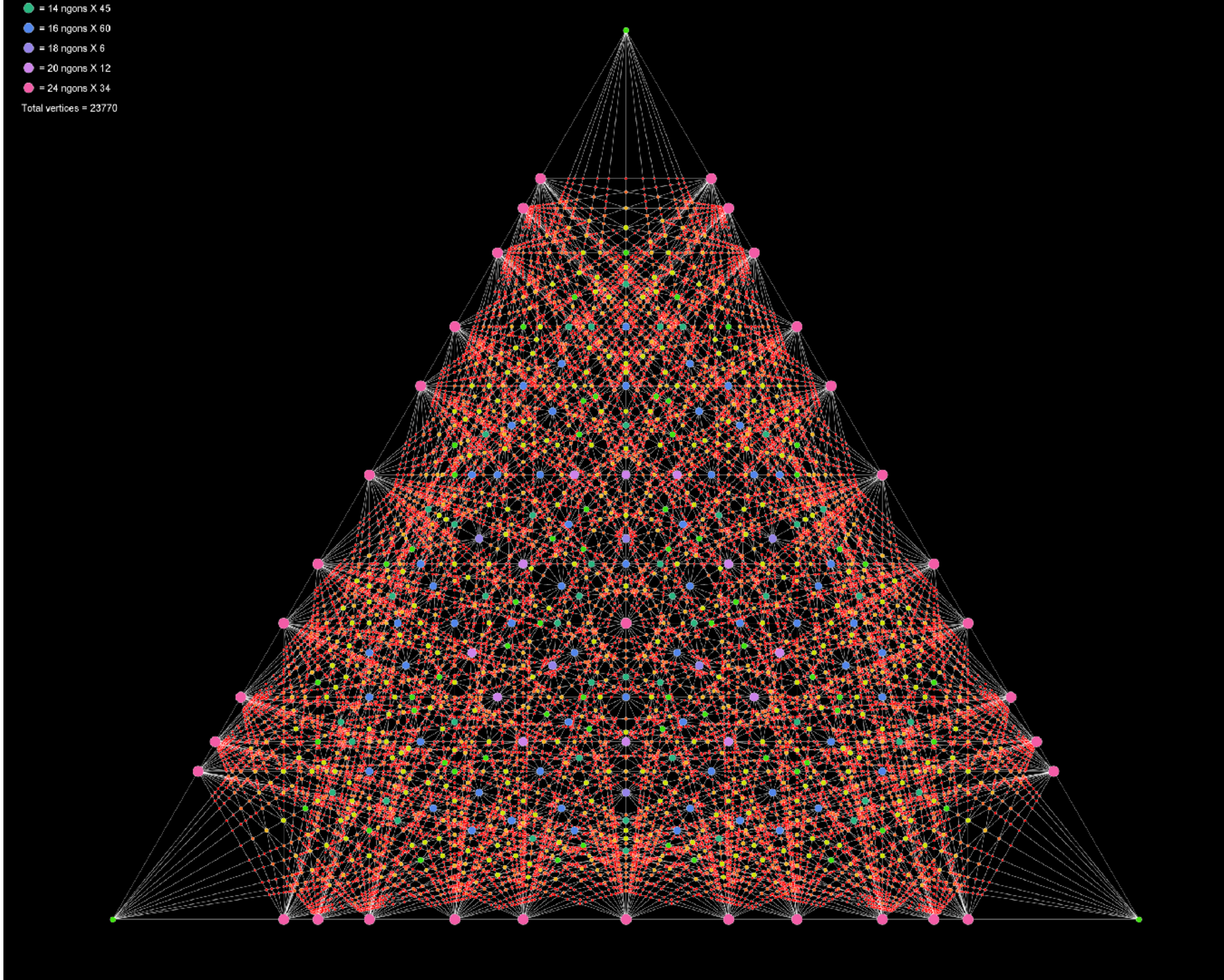
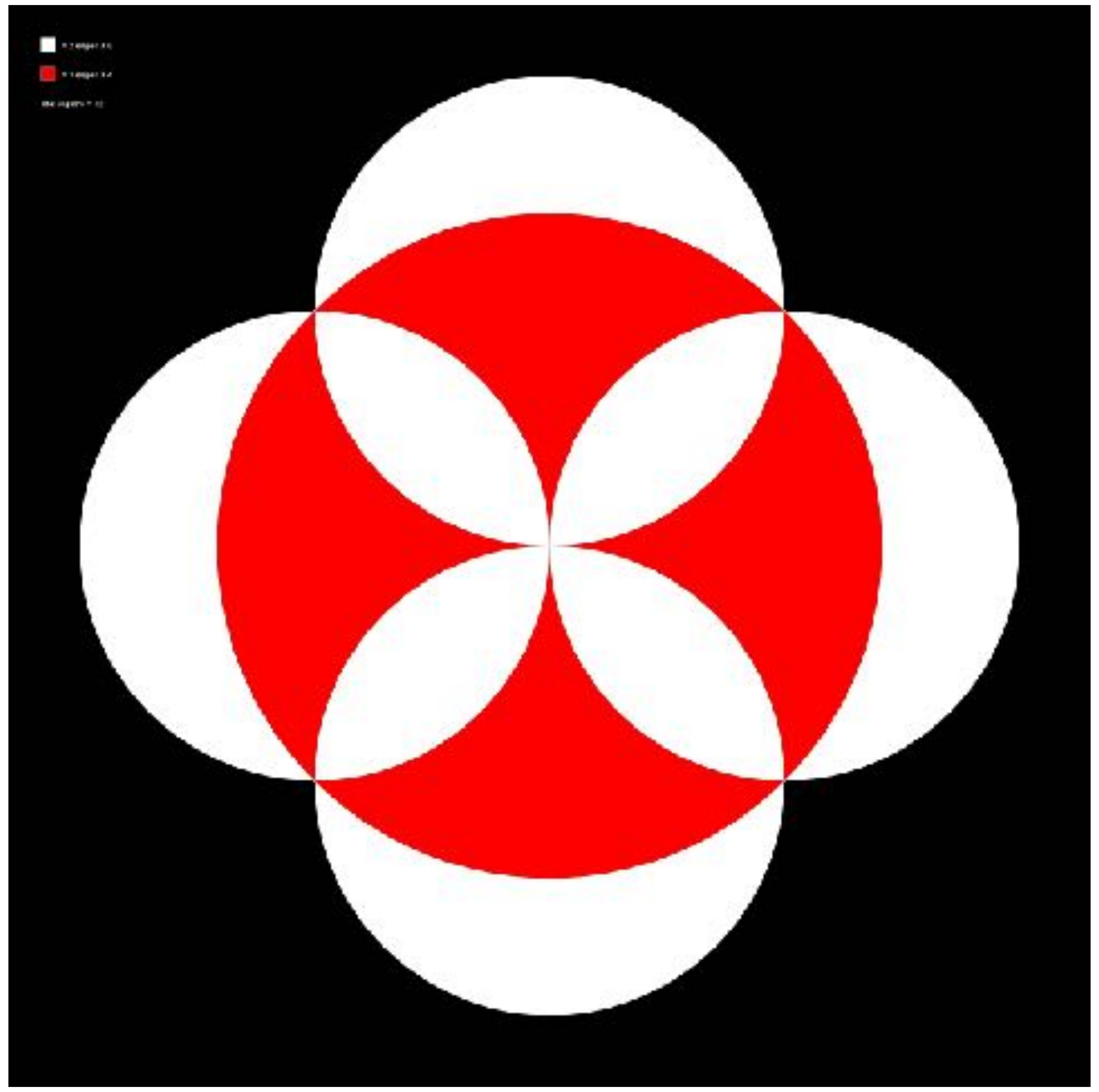


Image: Scott Shannon

Scott Shannon's First Circle Counting Problem (1)

Take n x n grid of points. For every pair of points, draw a circle with that diameter.

In resulting graph, count circles (C), vertices (V), edges (E), regions (R)



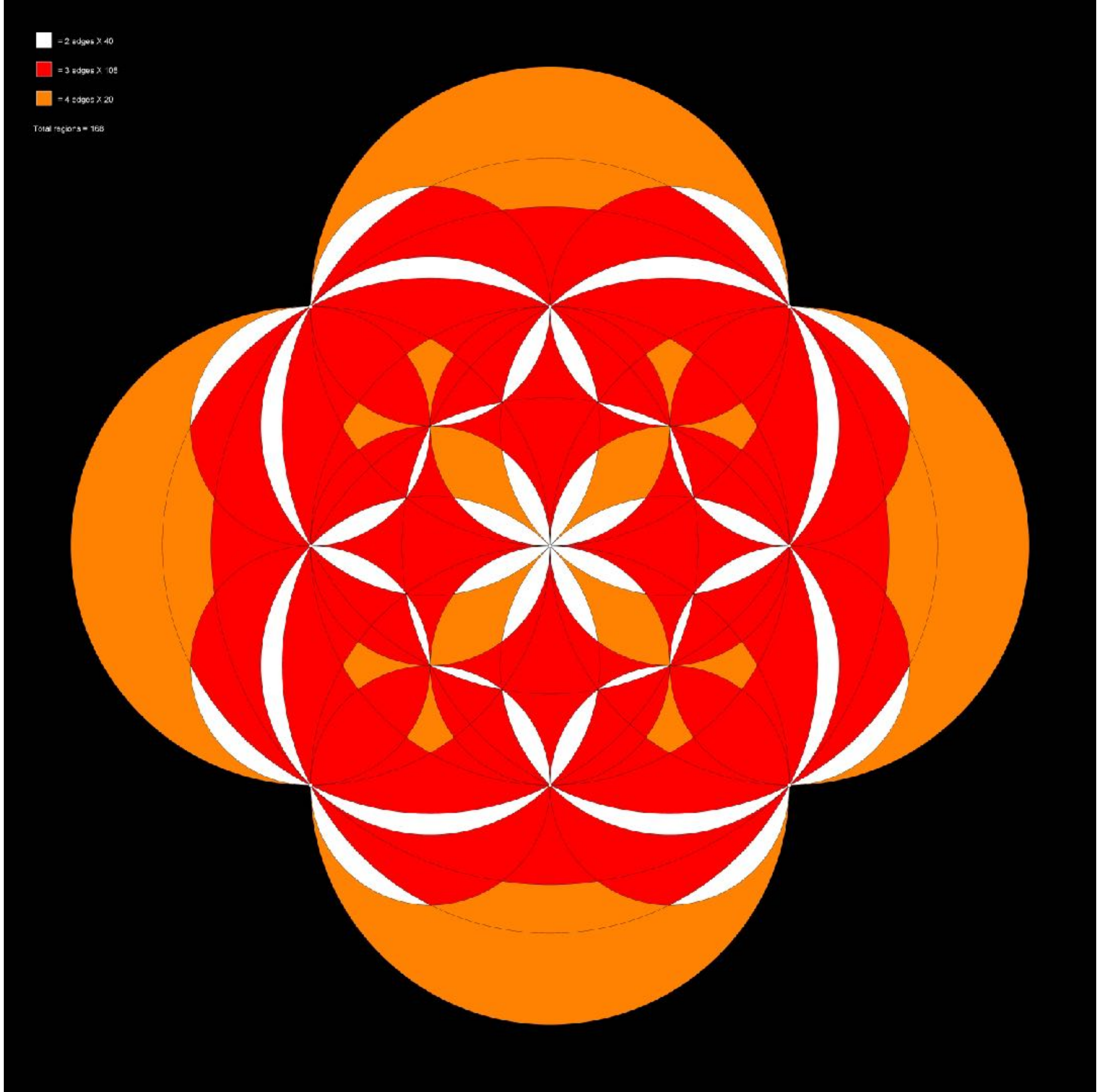
2 x 2 grid

n	C	V	R
2	5	5	12
3	26	77	168
4	79	1045	1536
5	185	6885	8904
	A360350	A360351	A360352

Have 9 terms for each sequence. No formulas are known.

Scott Shannon's Circle Counting Problem (2)

3 x 3 grid
26 circles, 77 vertices, 168 regions

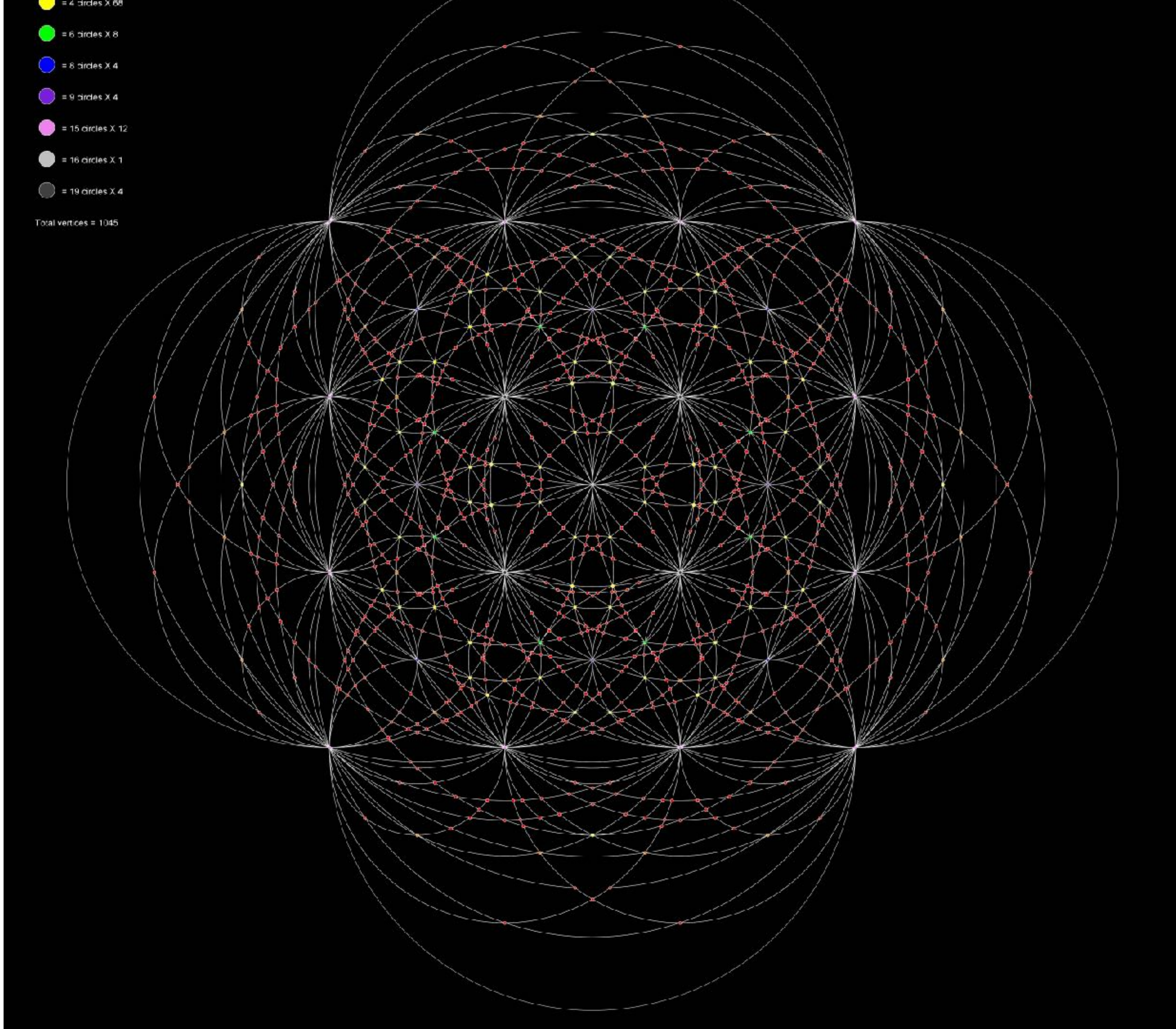


A360352
(regions)

Scott Shannon's Circle Counting Problem (3)

4 x 4 grid
79 circles
1045 vertices
1536 regions

A36035 (vertices)



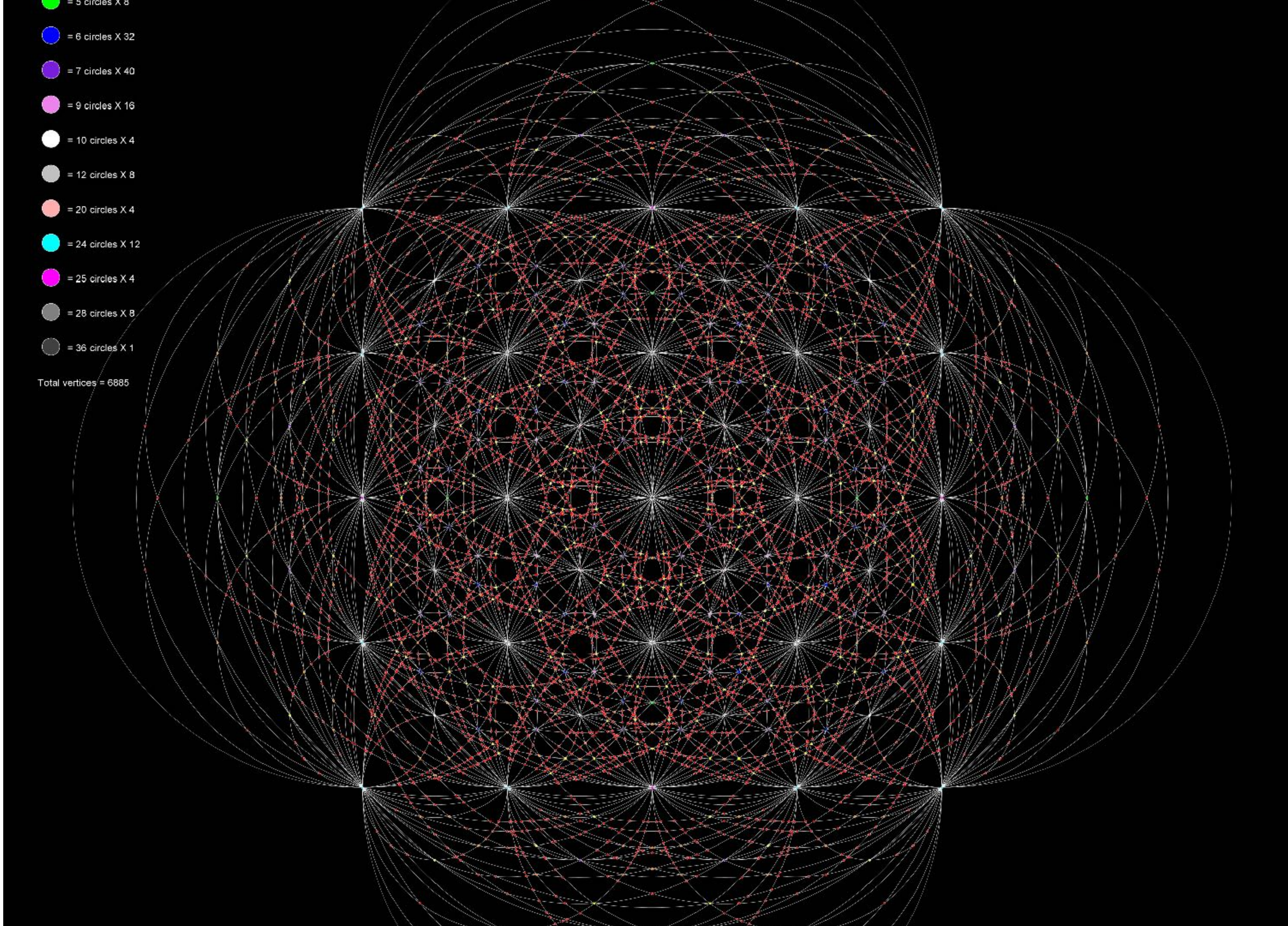
Scott Shannon's Circle

Counting Problem (4)

5 x 5 grid
185 circles
6885 vertices
8904 regions

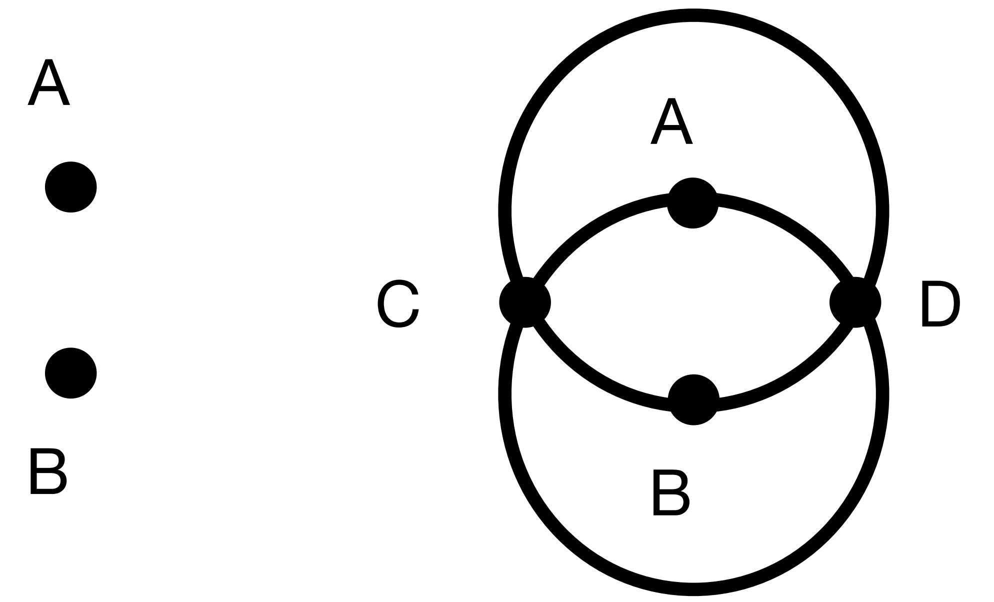
An astronomer's nightmare

A360351
(vertices)



Scott Shannon's Second Circle Counting Problem (1)

Given two points A and B, draw 2 circles of radius |AB| centered at A and B, creating 2 new intersection points C and D:



Iterate!
How many vertices (V), circles (C), regions (R)?

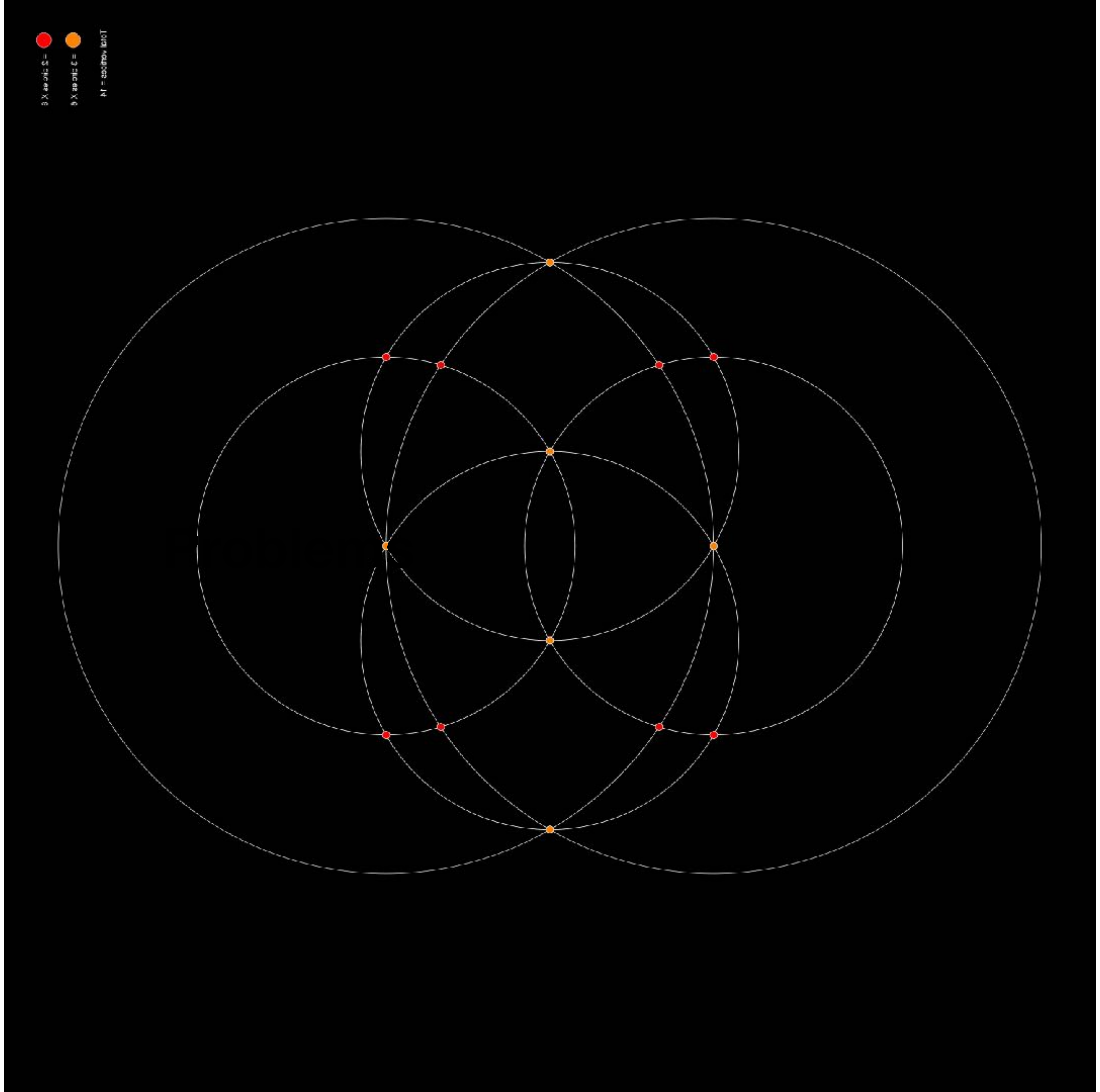
Need more terms!
Need formulas.
Basic combinatorial question.

stage	V	C	R
0	2	0	0
1	4	2	3
2	14	6	21
3	6562	?	7169
	A359569		A359570

Scott Shannon's Second Circle Counting Problem (2)

Stage 2
14 vertices
6 circles
21 regions

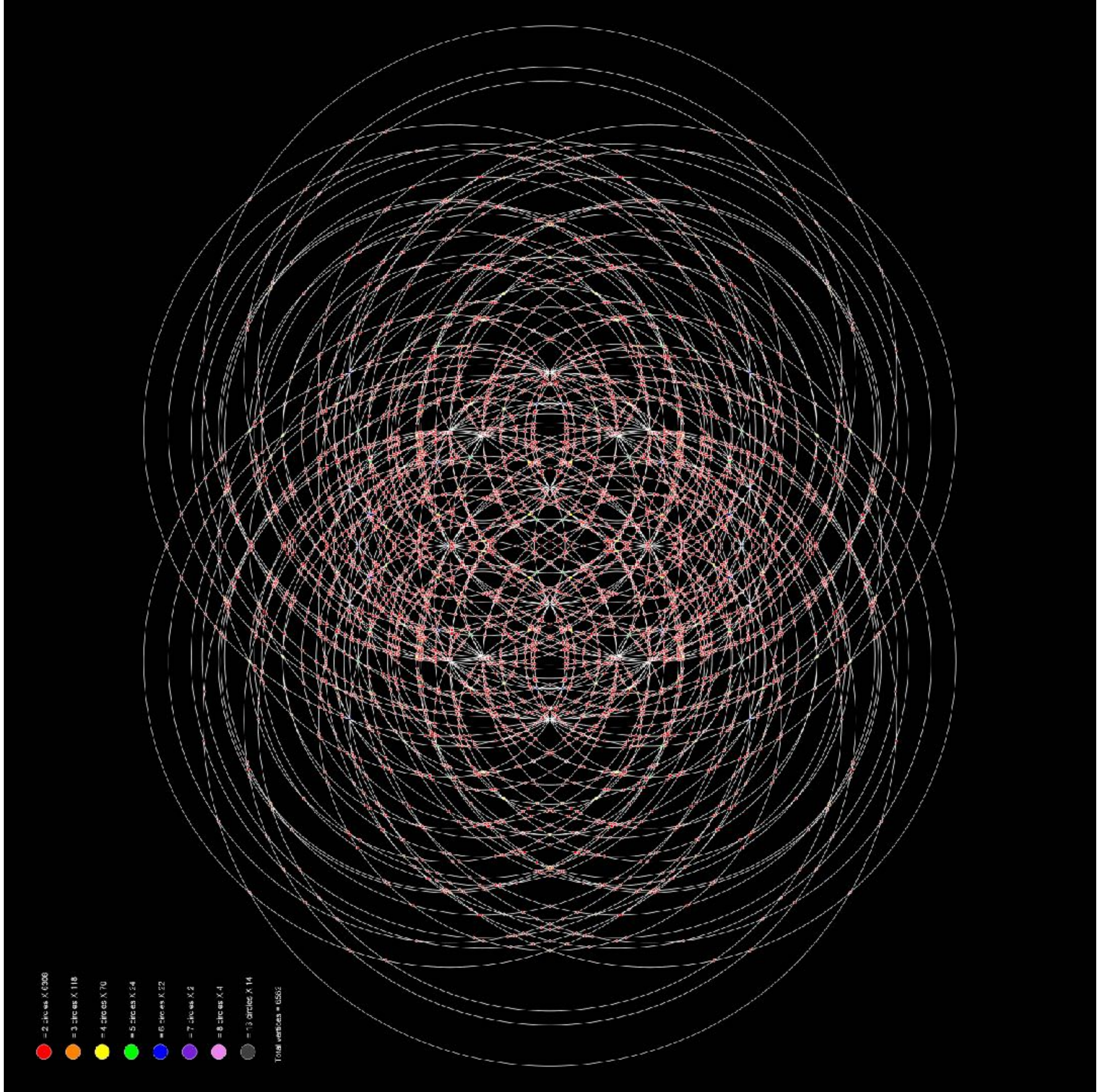
A359569
(vertices)



Scott Shannon's Second Circle Counting Problem (3)

Stage 3
6562 vertices
? circles
7169 regions

A359569
(vertices)

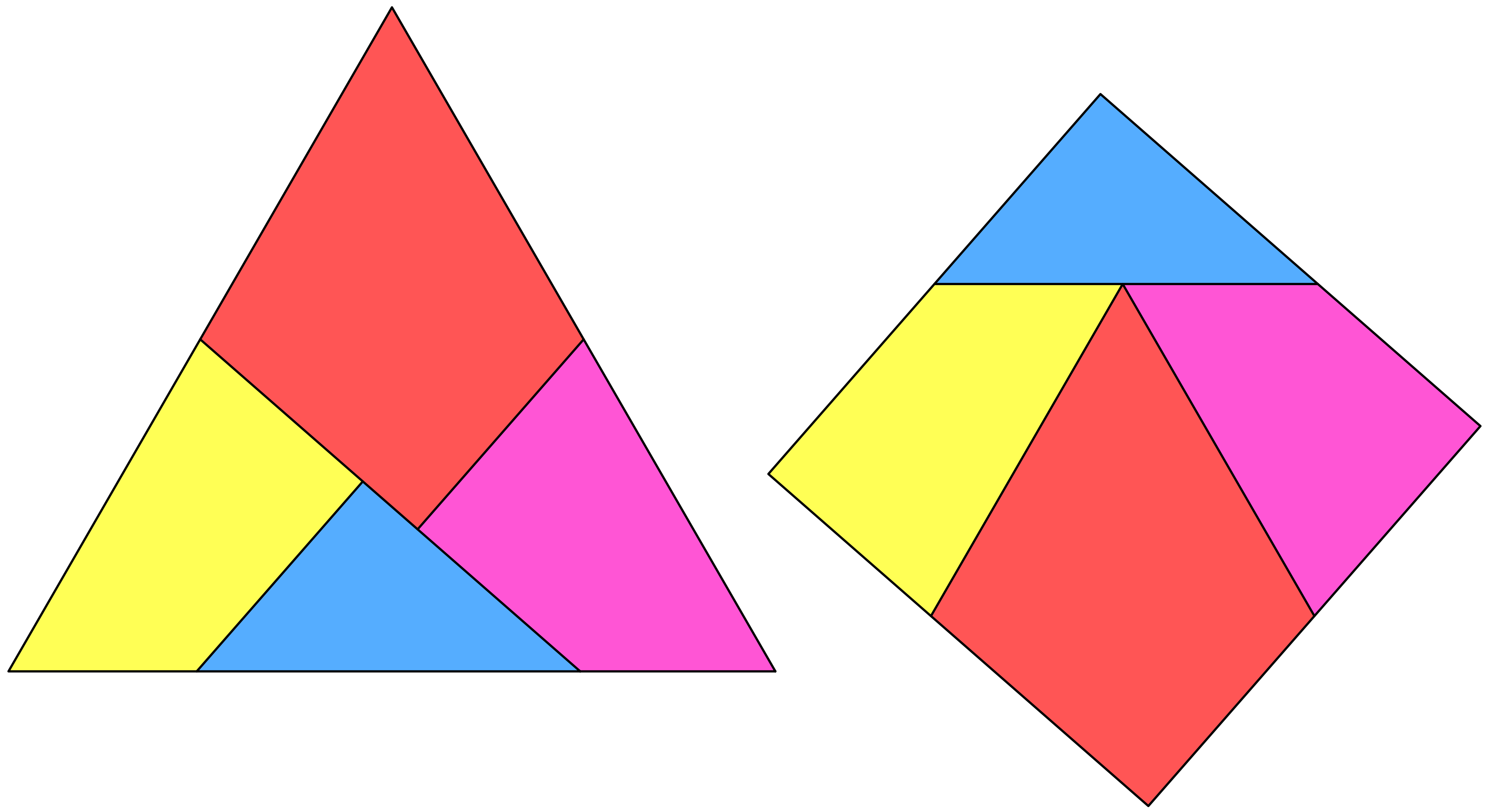


Dissecting Regular Polygons Into Rectangles (With Gavin Theobald)

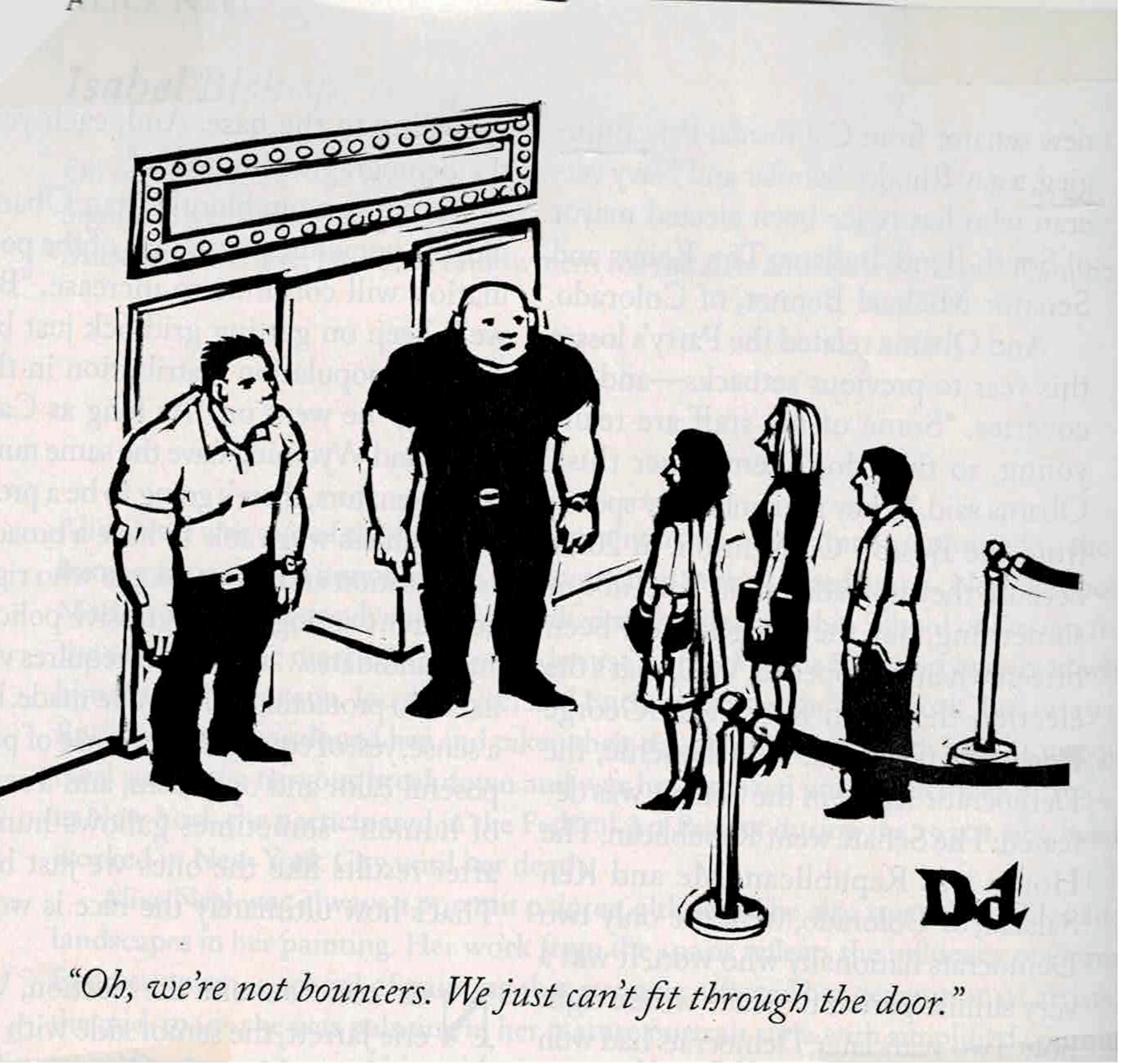
Dissecting polygons into squares and rectangles

Joint work with Gavin Theobald, plus contributions from others

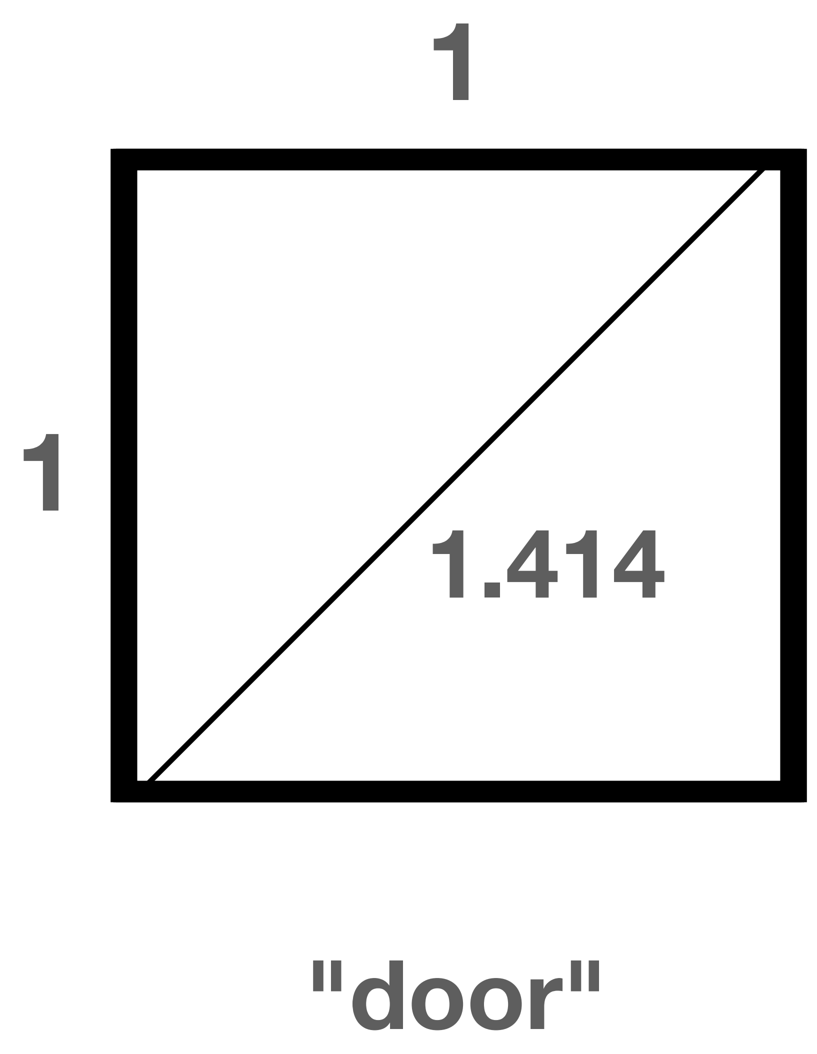
Classical problem: s(n) = min number of pieces needed to dissect regular n-gon to a square.
Open for 100+ years: show s(3) = 4

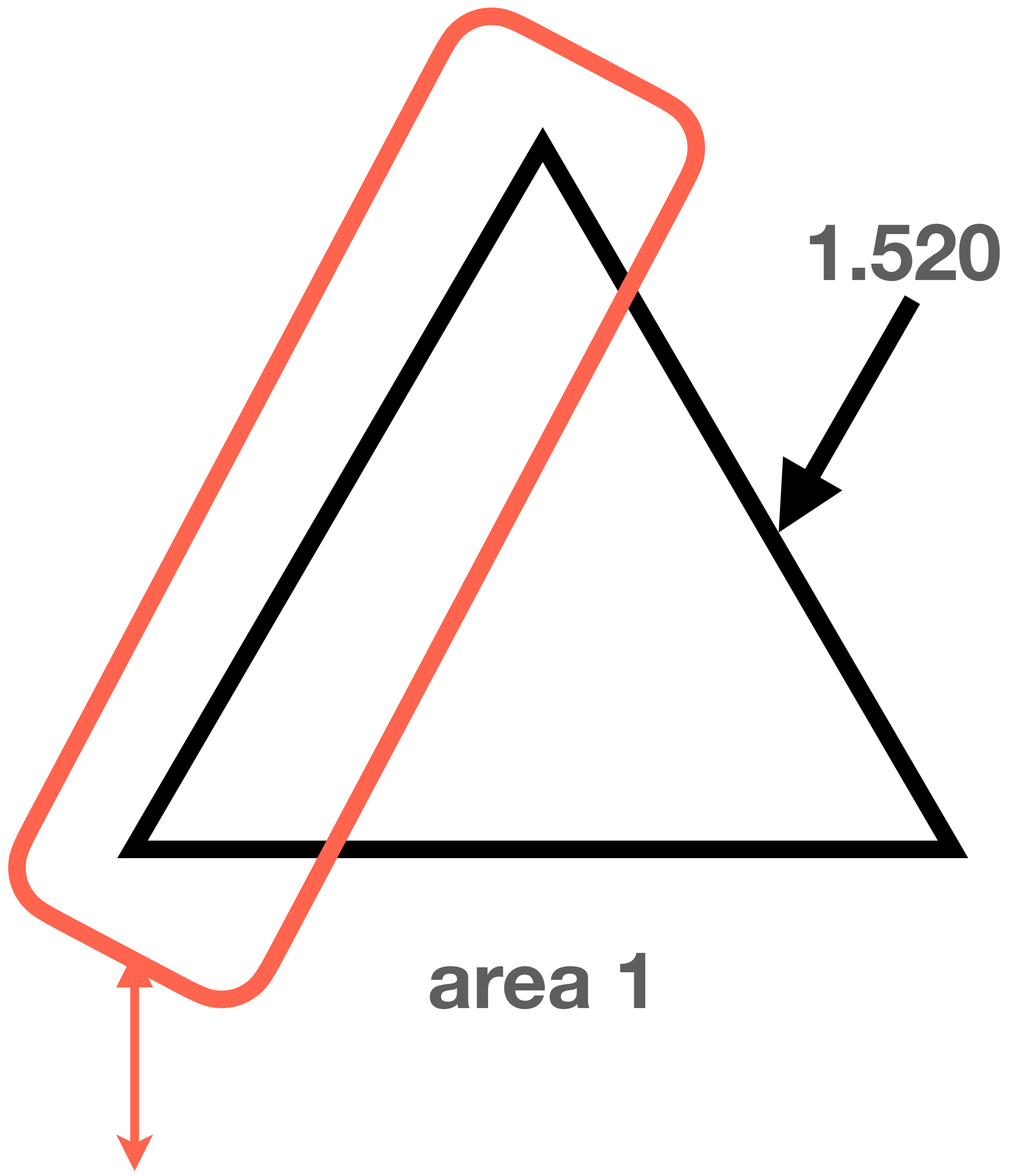


Theorem $s(3) > 2$



Oh, we're not bouncers. We just can't fit through the door.

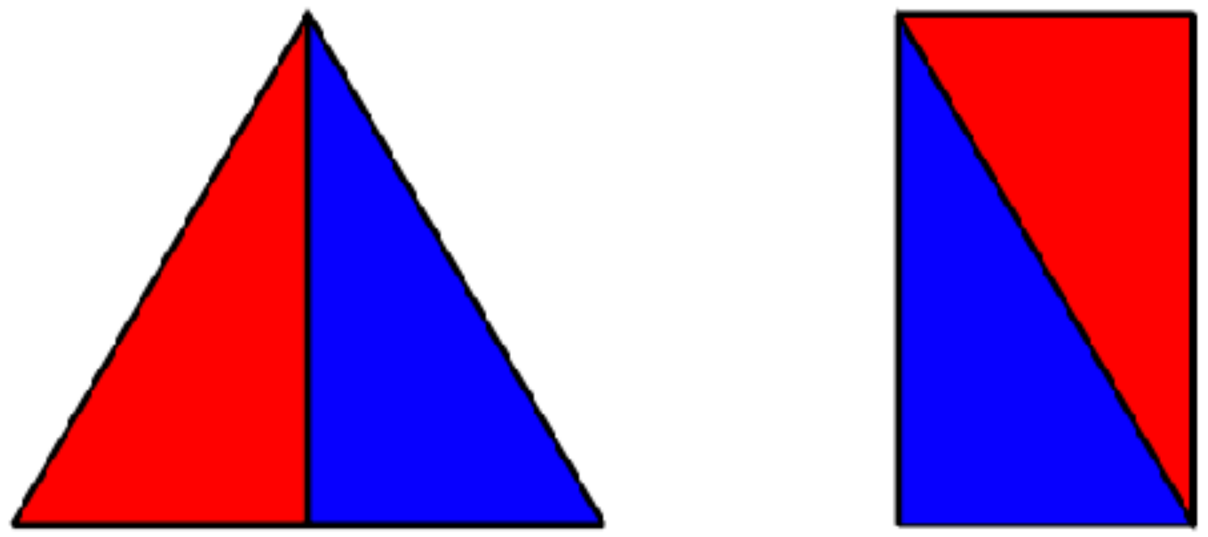




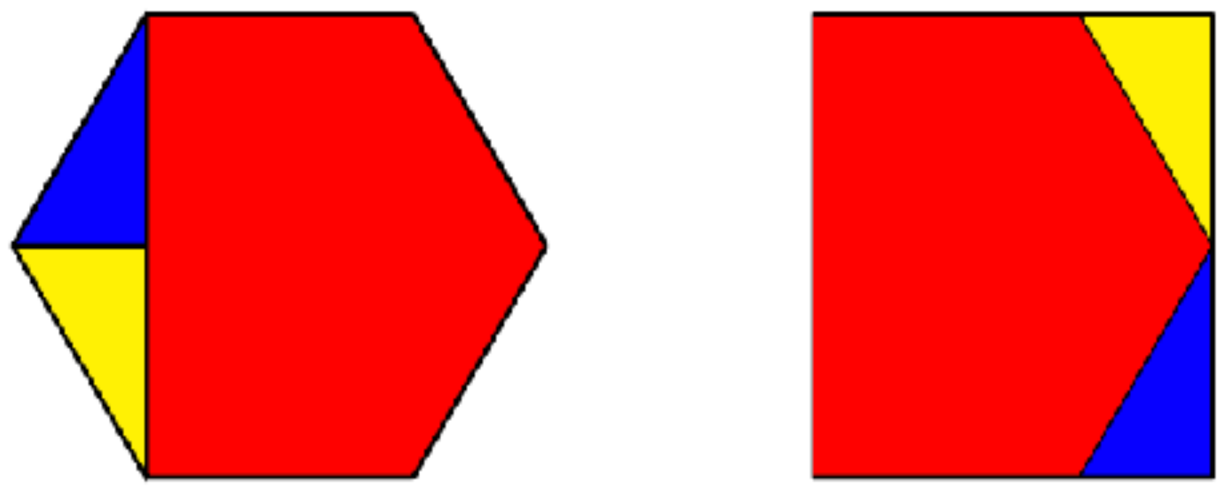
Each piece can contain at most one vertex, so at least 3 pieces!

New problem: r(n) = min number of pieces needed to dissect regular n-gon to a **rectangle** (any rectangle will do)

Rules same for s(n) and r(n); cuts are simple curves, turning over is allowed



r(3) = 2



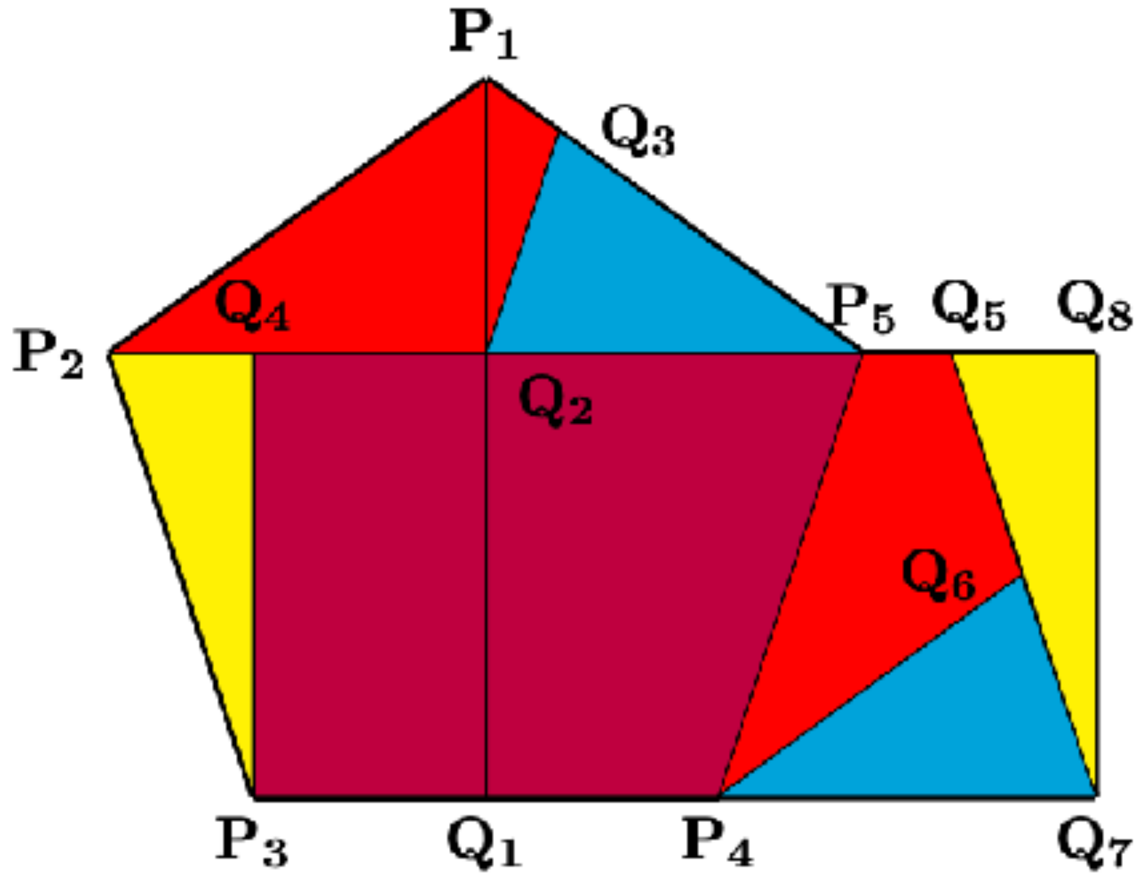
$r(6) \leq 3$ (surely r(6) = 2 is impossible?)

n =	3	4	5	6	7	8	9	10	11	12
s(n) <=	4	1	6	5	7	5	9	7	10	6
r(n) <=	2	1	4	3	5	4	7	4	9	5

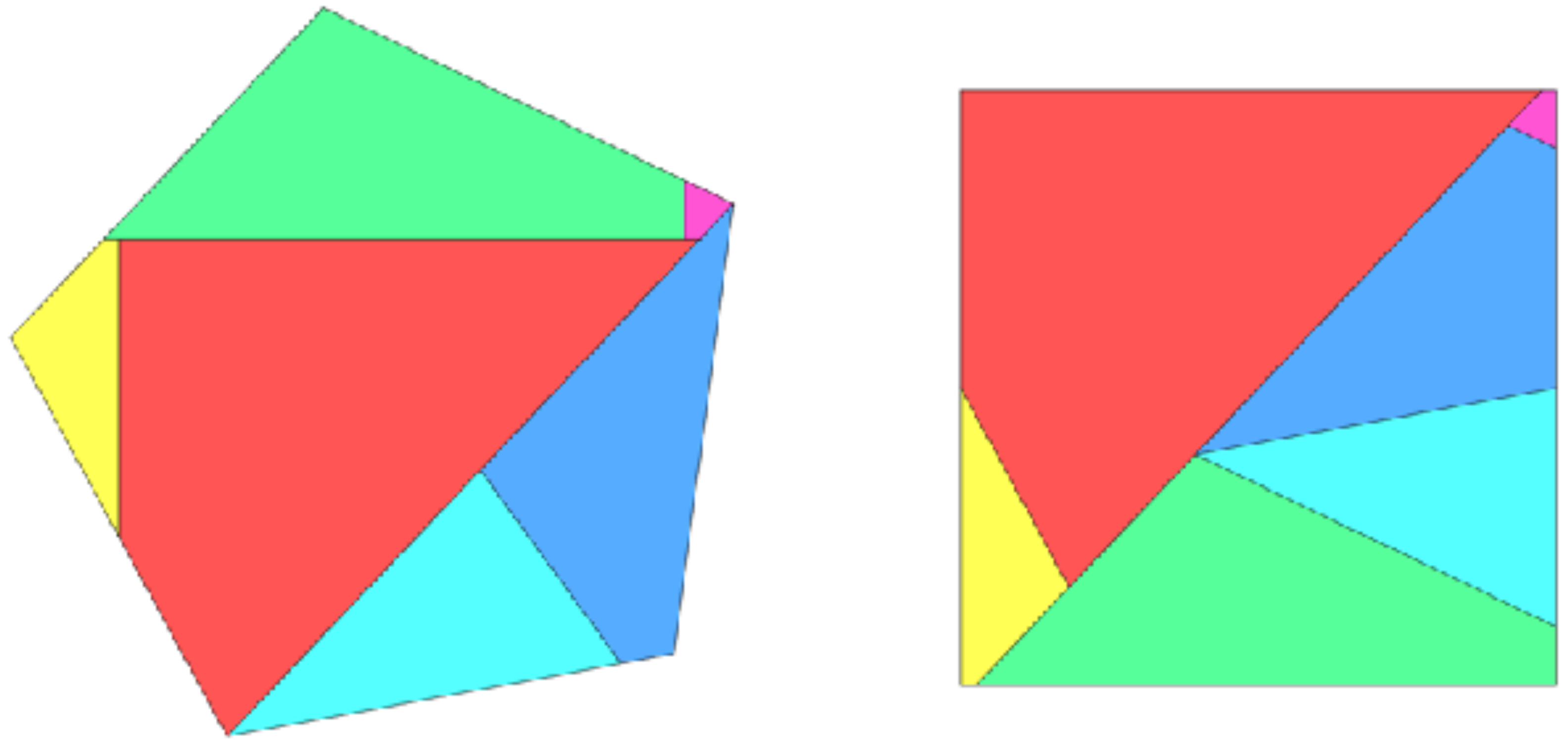
A110312 in OEIS

A362939 in OEIS

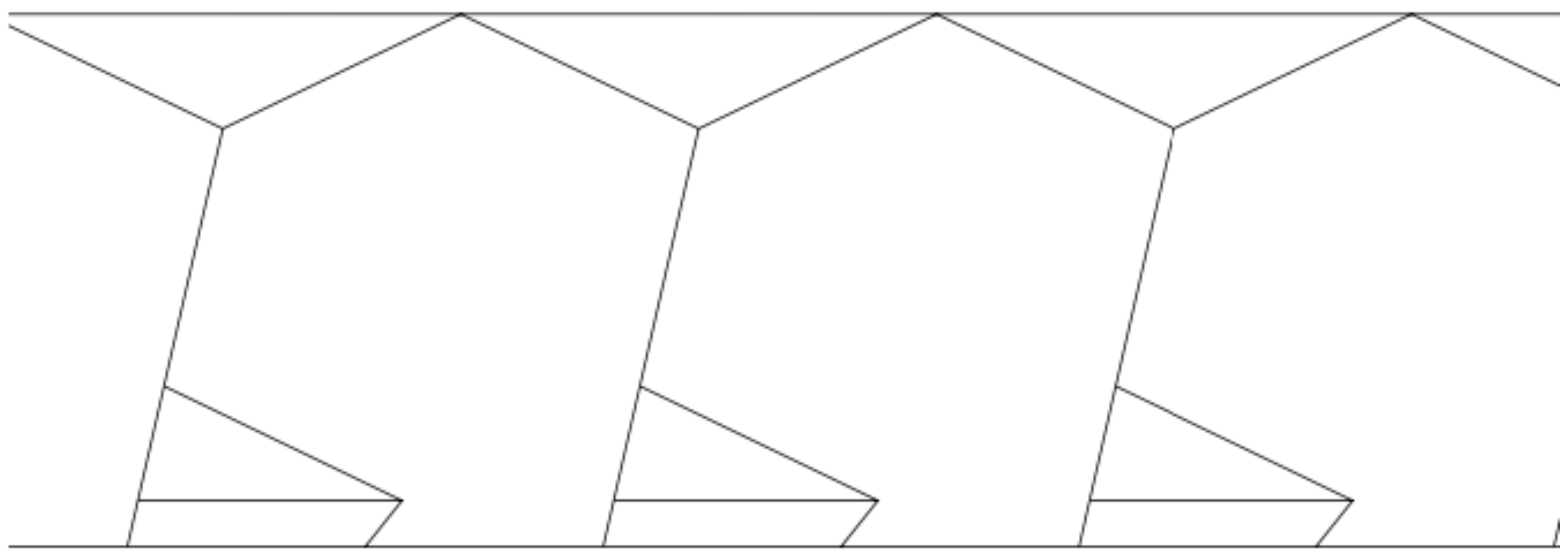
Best dissections known for 5-gon and 7-gon to rectangle

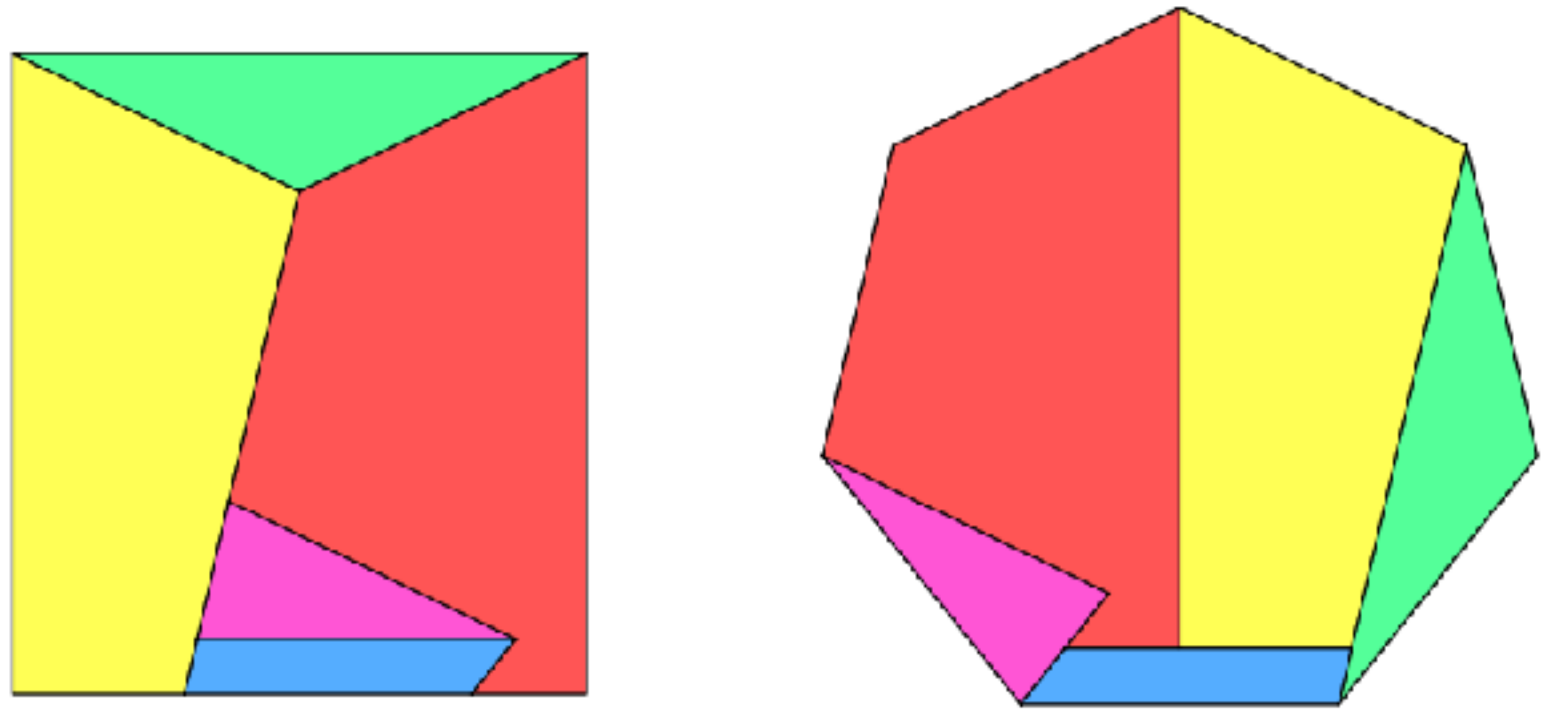


r(5) <= 4



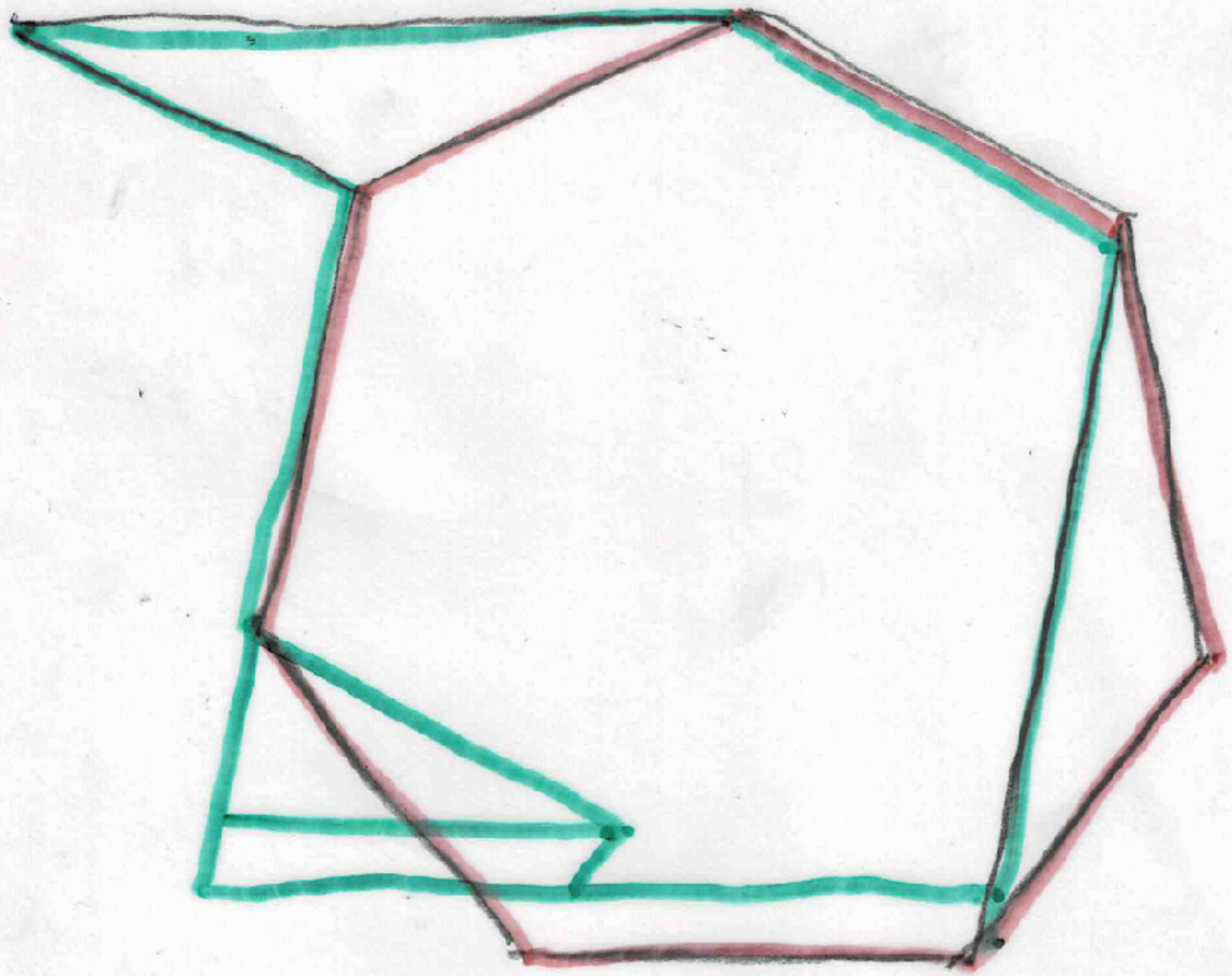
Pentagon to square: s(5) <= 6





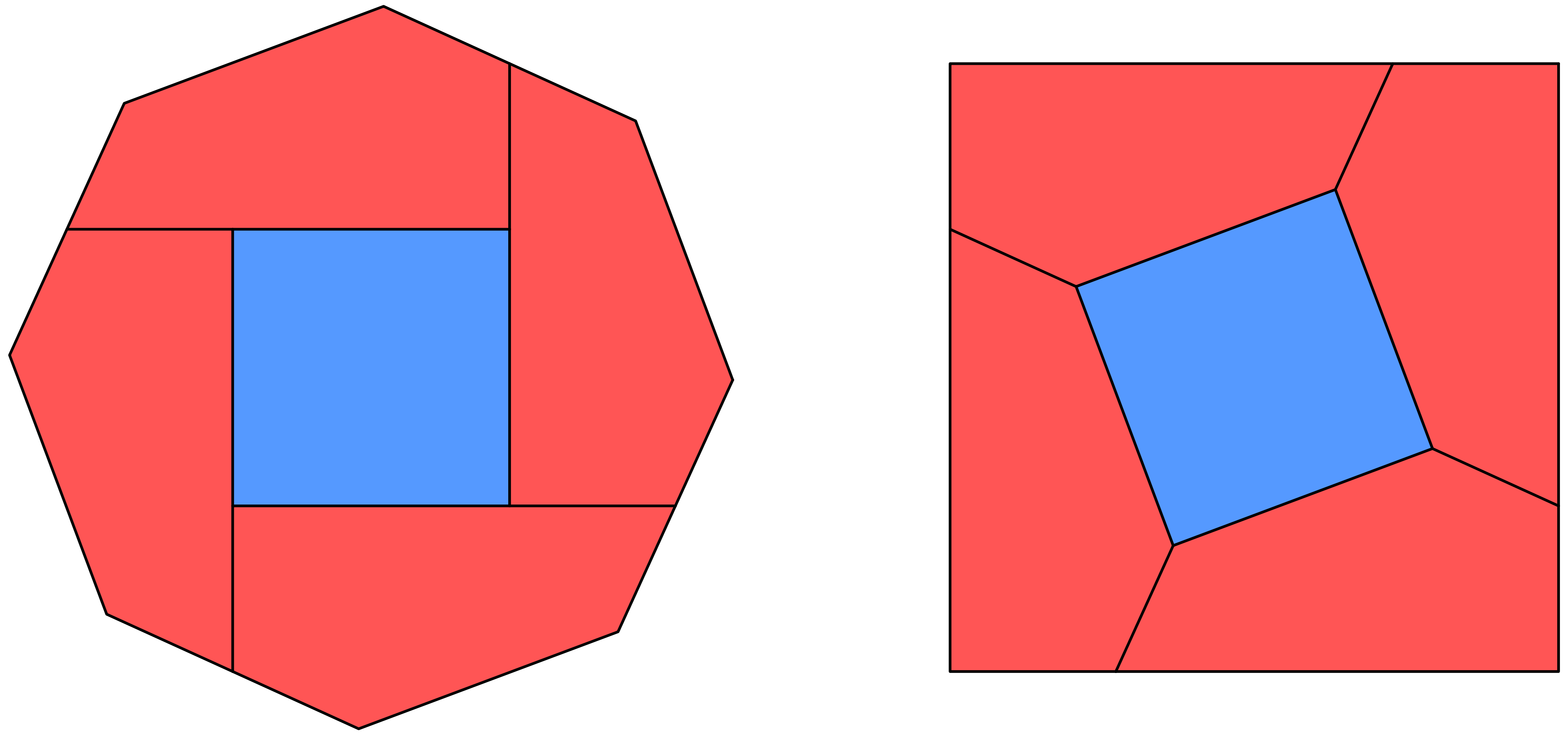
r(7) <= 5

Proof: By giving explicit straightedge and compass consruction starting with the 7-gon.



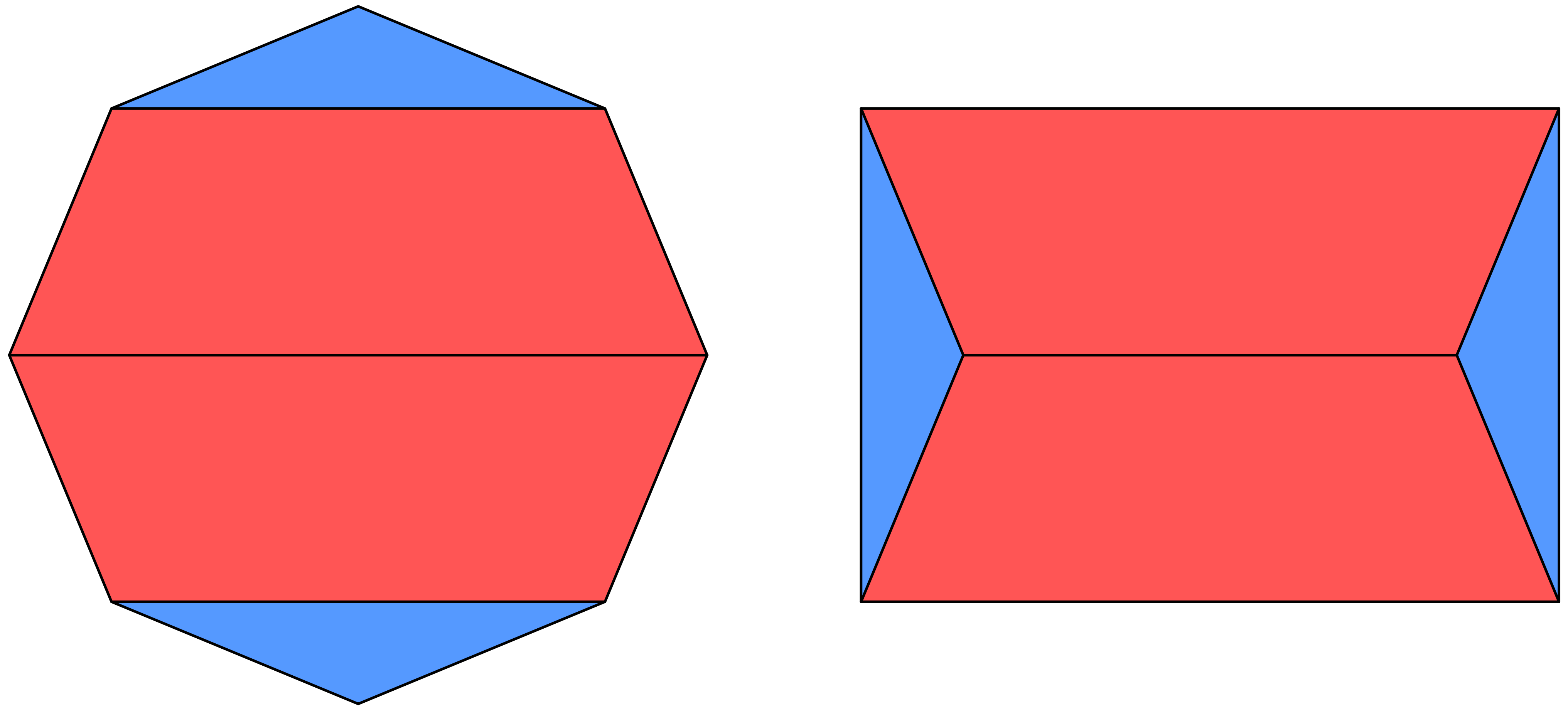
7gon (red) dissected
into strip element
(green)

Best dissections known for octagon to square and rectangle



s(8) <= 5

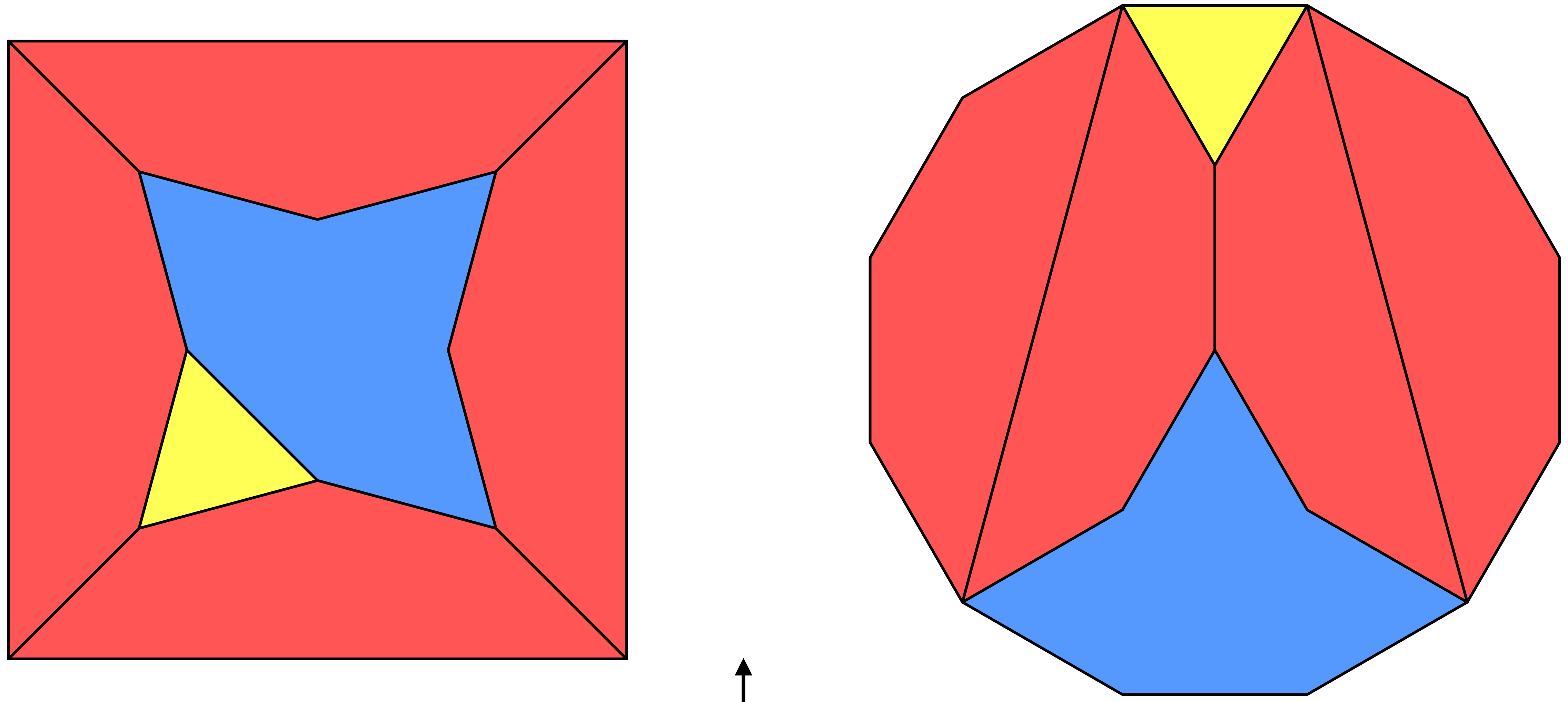
circa 1400 AD Persian MS



r(8) <= 4

Beautiful! But can be improved if only need a rectangle.

Best dissections known for 12-gon to square and rectangle

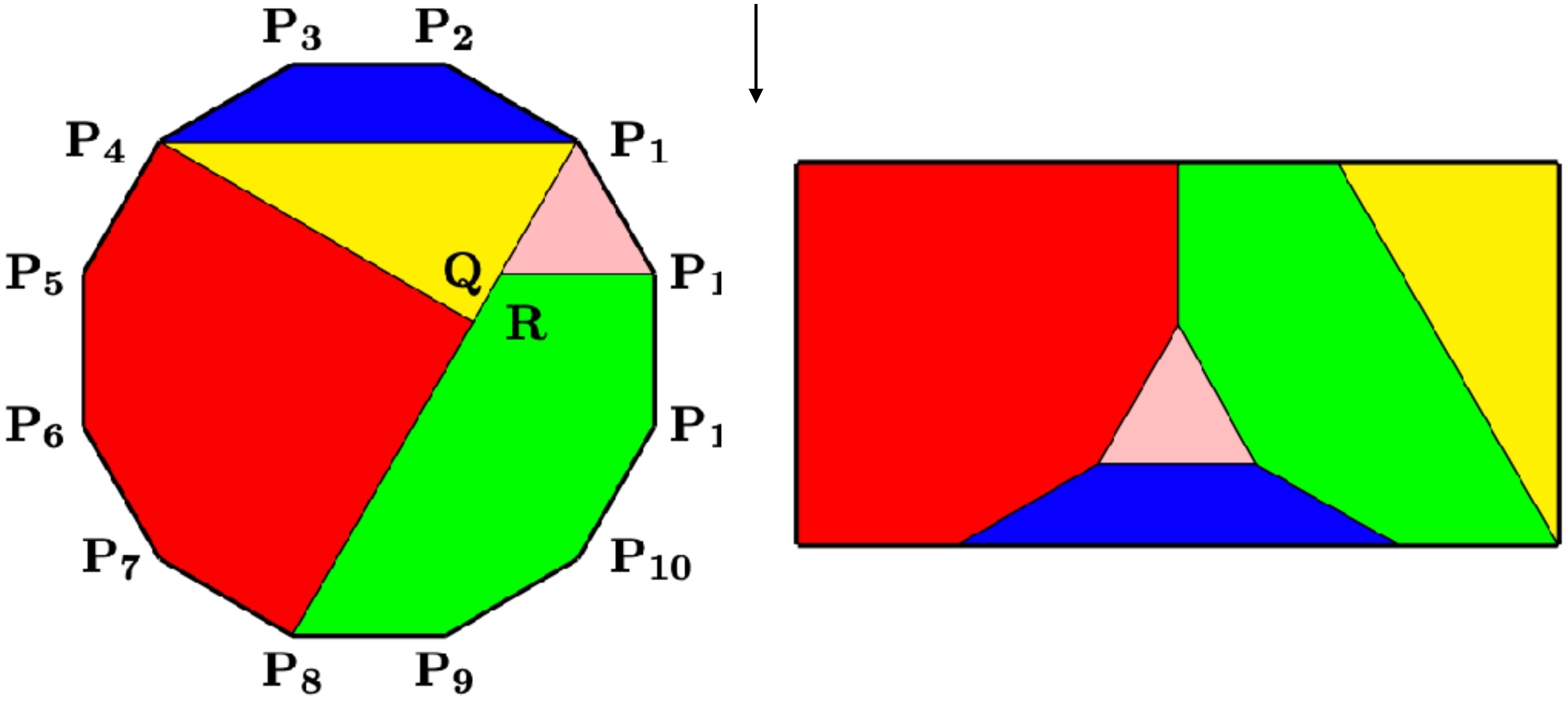


Lindgren: s(12) <= 6

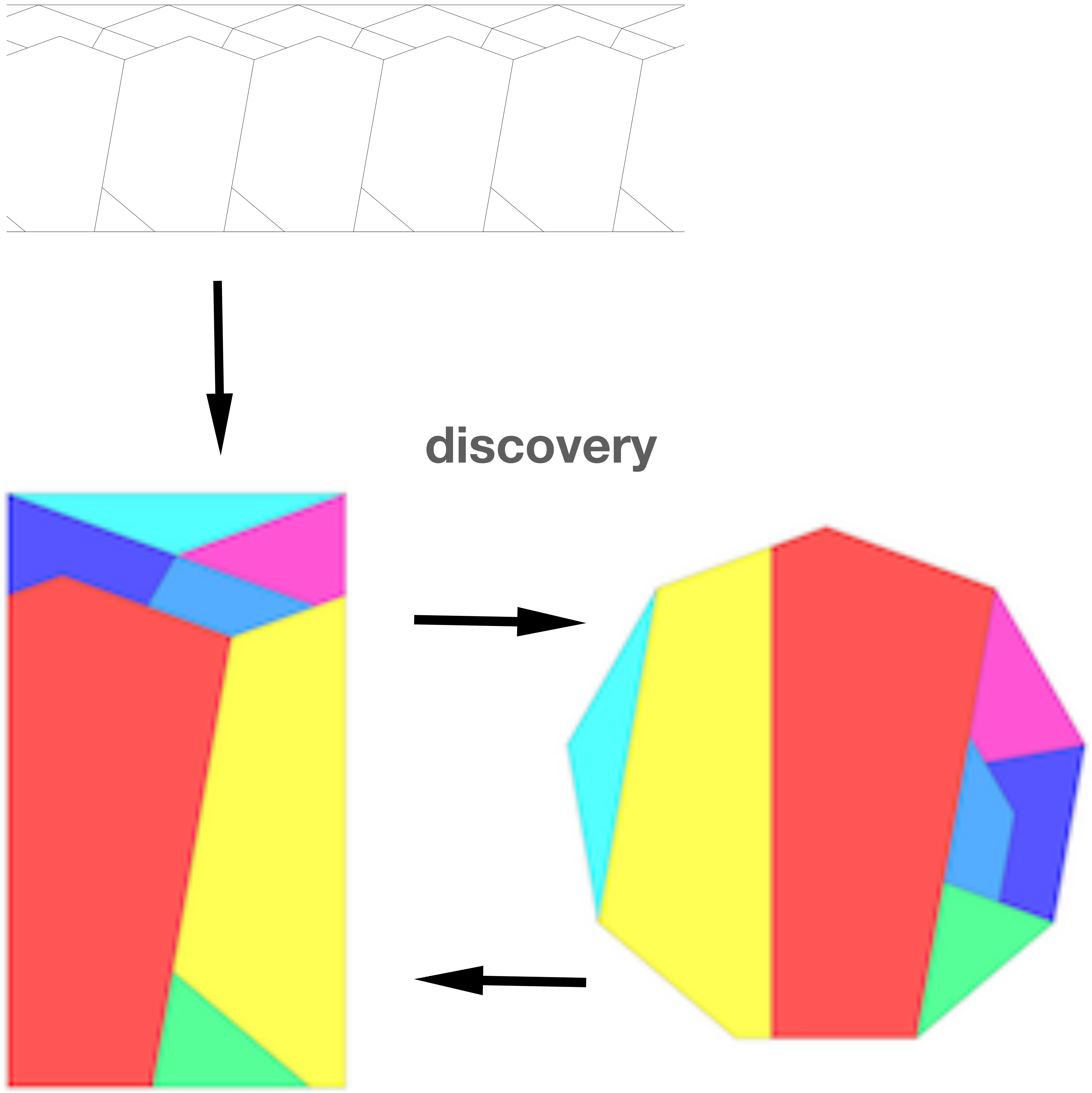
Astonishing! But can be improved

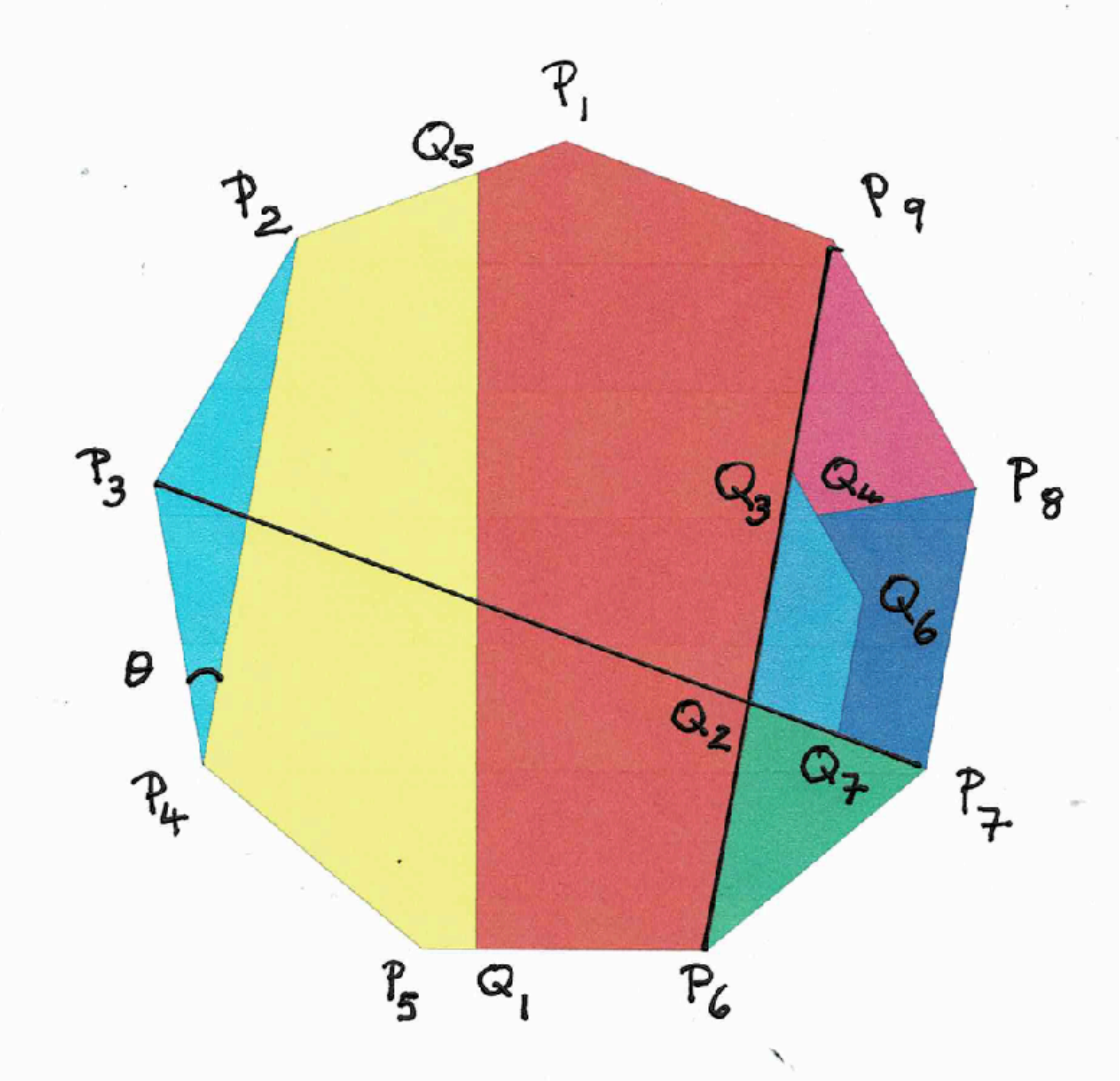
if only need a rectangle.

r(12) <= 5



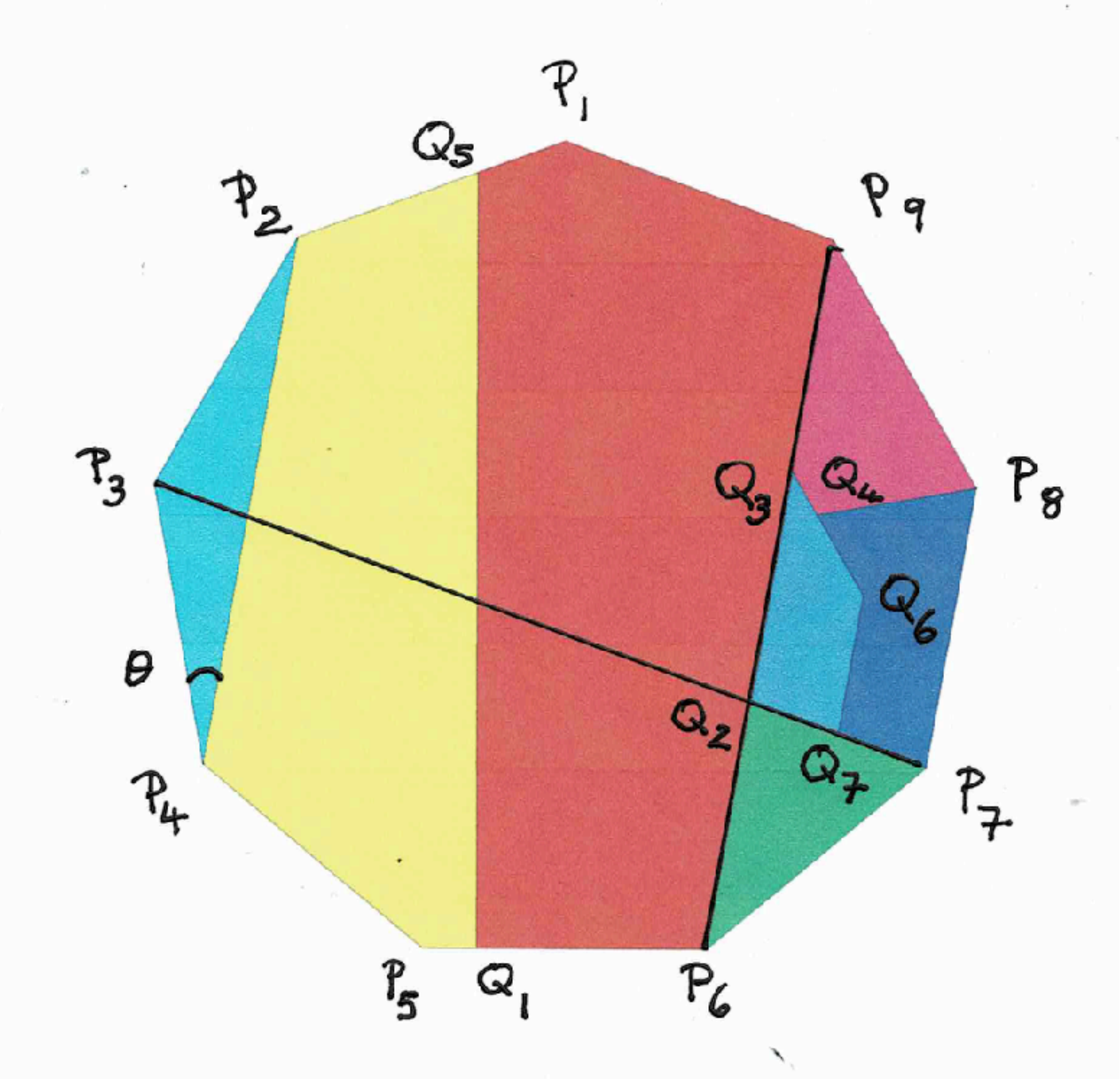
7-Piece dissection of 9-gon to rectangle





Straightedge and compass construction

7-Piece dissection of 9-gon to rectangle (continued)



This is a straightedge and compass construction.
Start with 9-gon P1 P2 ... P9, edge length 1.

Draw chords P2 P4, P3 P7, P4 P9.
Q2 = intersection, Q3 = midpoint, Q7 = midpoint,
Q7 Q6 length 1/2,
Q4 = intersection Q3 Q6 and perpendicular from P8 to midpoint of P3 P4.
Finally Q5 is on P1 P2 at distance |Q4Q6| from P1, and Q5 Q1 is perpendicular to P5 P6.

7-Piece dissection of 9-gon to rectangle (continued)

$\theta = \pi / 11$

C1 = cos θ satisfies
8x^3-6x-1 = 0.

The amazing coincidence:

|Q2 P7| = |Q4 Q6 | = |P1 Q5| = |Q2 Q3| - 1/2
= cos θ / (2 cos θ + 1) = 0.3263 ..... (*)

No easy geometric proof. Need to use minimal polynomial for cos theta.

To prove (*). Straightedge and compass gives exact expressions and 20 dec places.
E.g. Q2 = (0.65270364466613930216, -0.50771330594287249271)

Ask WolframAlpha to express each number in terms of C1 = cos θ and S1 = sin θ: The result:

$$Q_2 = \left(\frac{2C_1}{2C_1+1},\ -(2S_1 - \frac{1}{S_1} + \sqrt{3})\right)$$

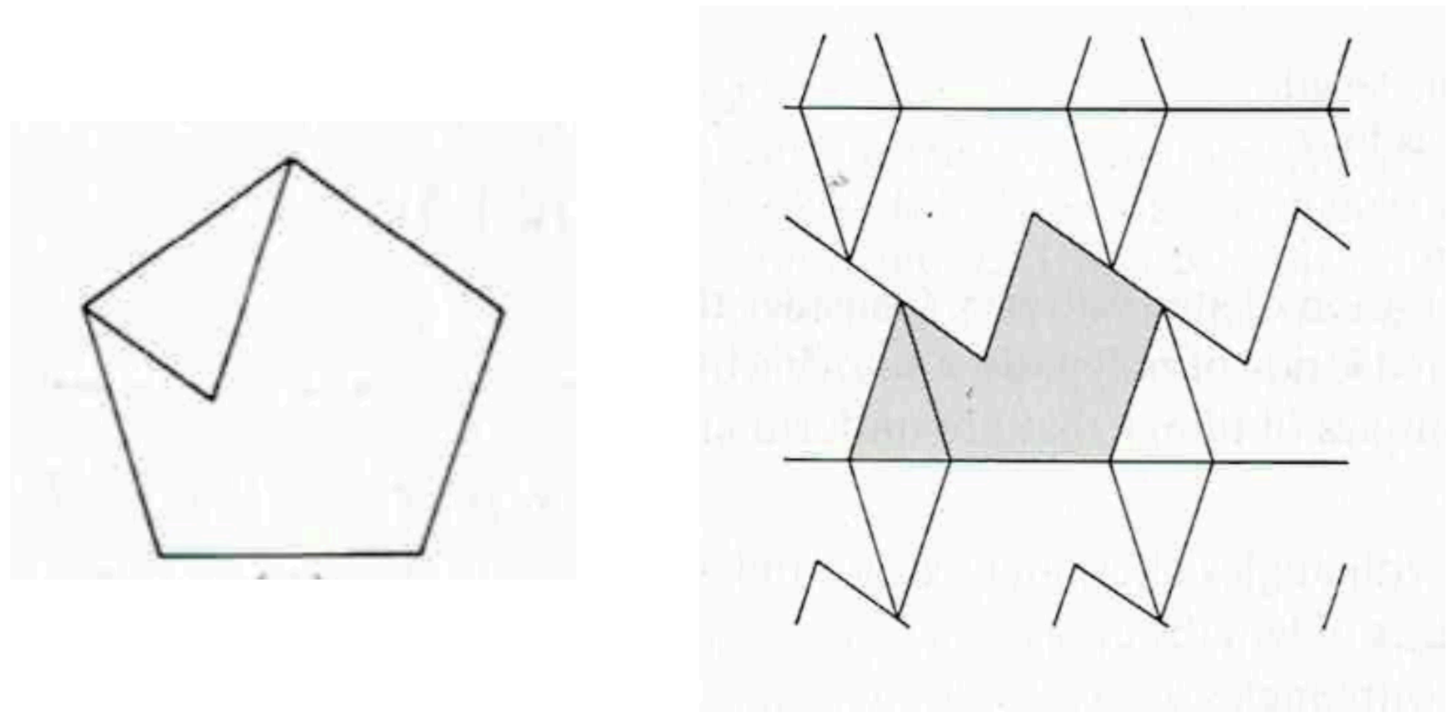
Given exact expressions for points, and using minimal polynomial for C1, (*) follows easily

The rectangle has width 2 cos θ, height 9 / (8 sin θ)

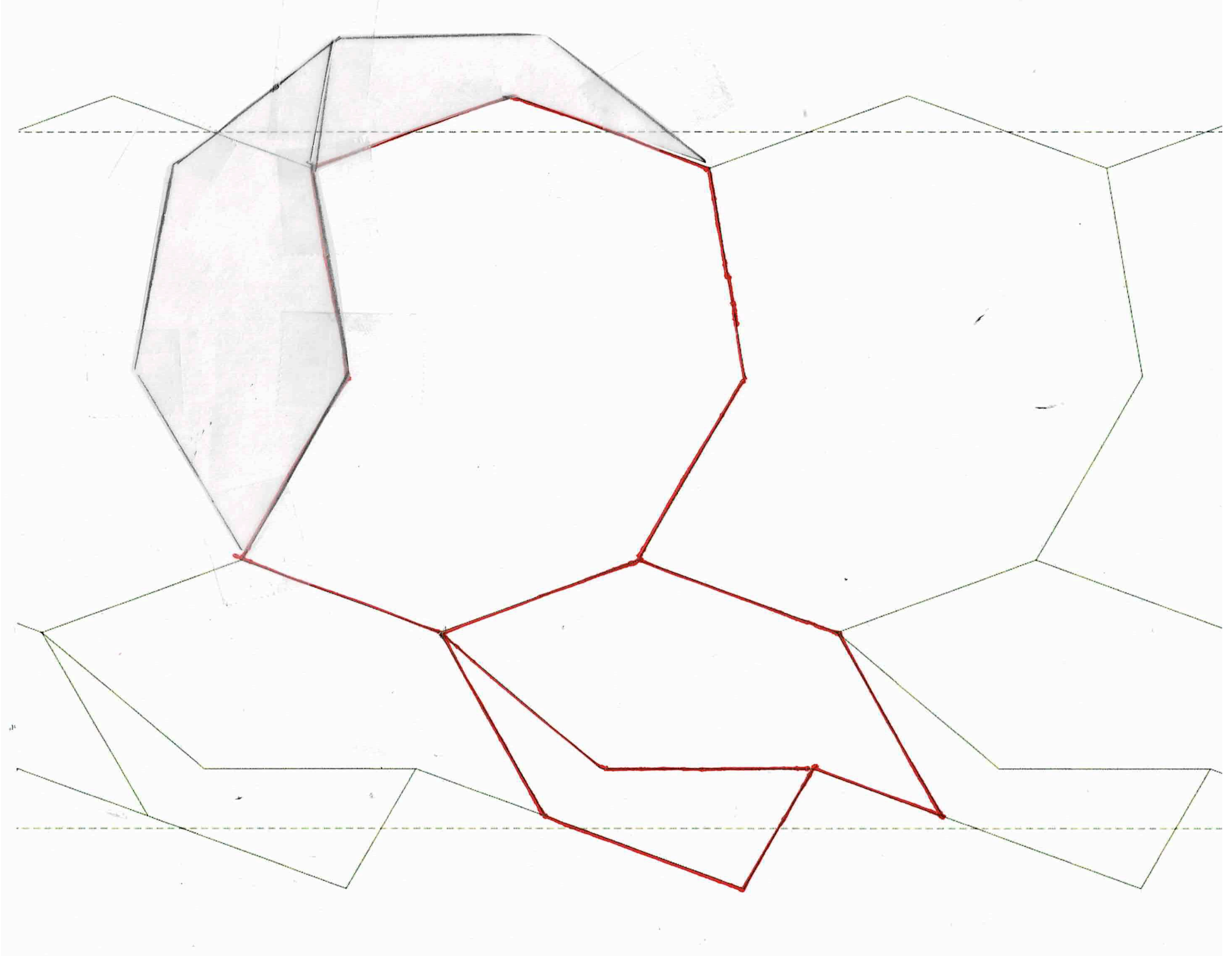
Another fundamental sequence!

q(n) = min number of pieces needed to dissect regular n-gon to a monotile

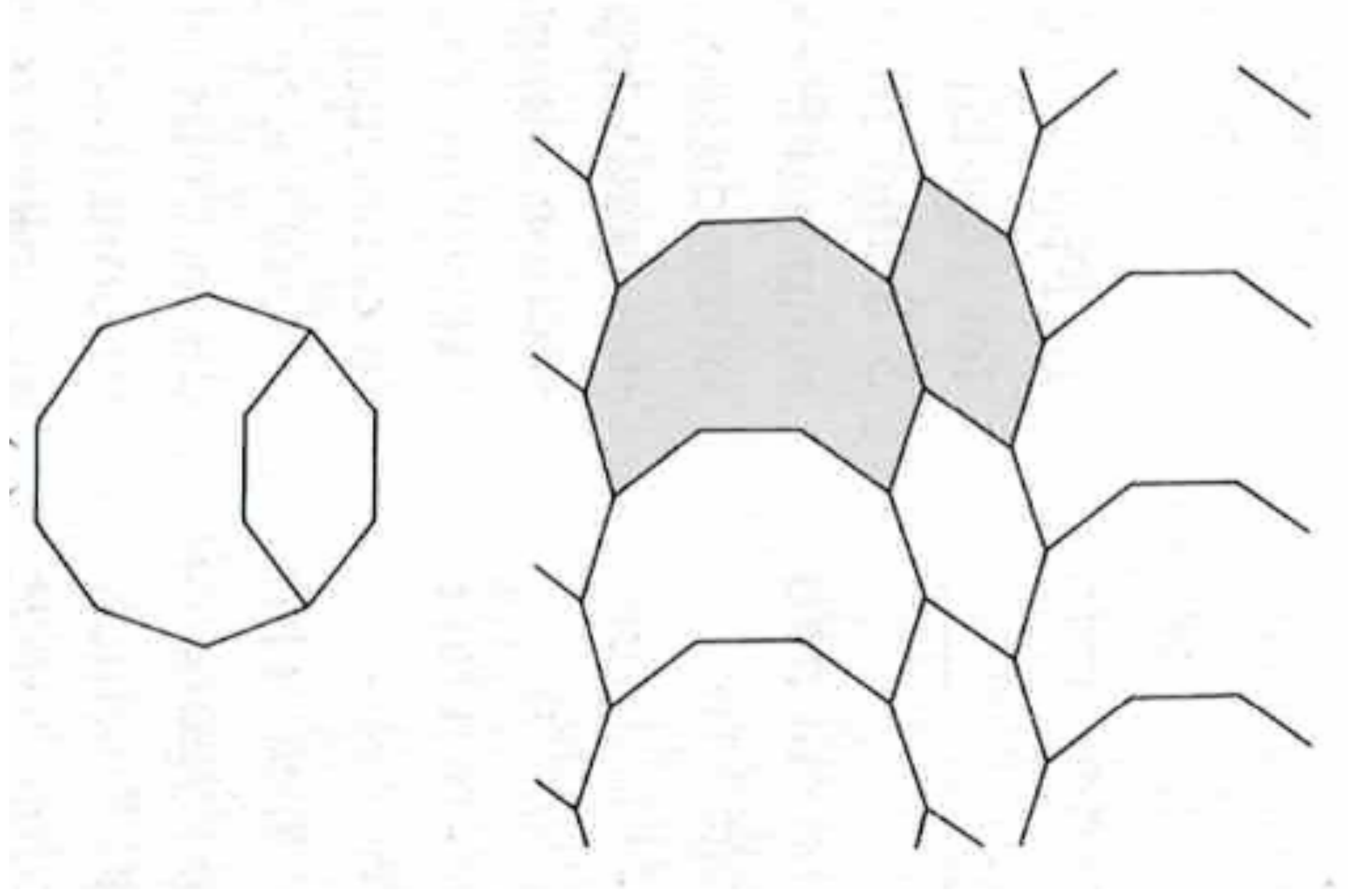
	n =	3	4	5	6	7	8	9	10	11	12	OEIS
n-gon to square	s(n) <=	4	1	6	5	7	5	9	7	10	6	A110312
n-gon to rectangle	r(n) <=	2	1	4	3	5	4	7	4	9	5	A362939
n-gon to monotile	q(n) <=	1	1	2	1	3	2	3	2	4	3	A362938



q(5) = 2



$q(9) \leq 3$ (Gavin Theobald)



q(10) = 2

Grunbaum & Shephard, Patterns and Tilings, 1996, p. 92 give q(9)=5.

New Gilbreath Conjectures

New Gilbreath Conjectures

Norman L. Gilbreath, Magician and Mathematician, 1958: **(François Proth, 1878)**

2	**3**	**5**	**7**	**11**	**13**	**17**	**19**	**23**	**29**	**31**	**37**	**41**	**43**	**47**	**53**	**59**	**61**	**67**	**71**	**73**	**79**	**83**	**89**	**97**	**101**	Primes
1	2	2	4	2	4	2	4	6	2	6	4	2	4	6	6	2	6	4	2	6	4	6	8	4	2	Absolute values of differences of prev. row:
1	0	2	2	2	2	2	2	4	4	2	2	2	2	0	4	4	2	2	4	2	2	2	4	2	2	Absolute values of differences of prev. row:
1	2	0	0	0	0	0	2	0	2	0	0	0	2	4	0	2	0	2	2	0	0	2	2	0	0	Absolute values of differences of prev. row:
1	2	0	0	0	0	2	2	2	2	0	0	2	2	4	2	2	2	0	2	0	2	0	2	0	0	Repeat
1	2	0	0	0	2	0	0	0	2	0	2	0	2	2	0	0	2	2	2	2	2	2	2	0	8	

↑

A036262, A362463

(The array read by antidiagonals)

Conjecture: Leading entries are always 1! Astonishing. Still unproved.

30 years ago, Andrew Odlyzko checked 10^11 primes, 635 rows of table. Could now be checked a lot further.

New Gilbreath Conjectures (2)

Call the first column the "Gilbreath Transform"

Change "prime(n)" to "tau(n)" = number of divisors of n.

1	**2**	**2**	**3**	**2**	**4**	**2**	**4**	**3**	**4**	**2**	**6**	**2**	**4**	**4**	**5**	**2**	**6**	**2**	**6**
1	0	1	1	2	2	2	1	1	2	4	4	2	0	1	3	4	4	4	2
1	1	0	1	0	0	1	0	1	2	0	2	2	1	2	1	0	0	2	2
0	1	1	1	0	1	1	1	1	2	2	0	1	1	1	1	0	2	0	0
1	0	0	1	1	0	0	0	1	0	2	1	0	0	0	1	2	2	0	2

tau(n) = A000005

March 2023: Wayman Eduardo Luy and Robert G. Wilson V submitted A361897, the Gilbreath Transform of tau(n):

1, 1, 1, 0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, ... A361897

and conjectured that it is a 0,1 sequence!

New Gilbreath Conjectures (3)

Gilbreath Transform of Euler phi function (A000010) appears to be 1,0,1,0,1,0,.... = (1,0)*

See A362913

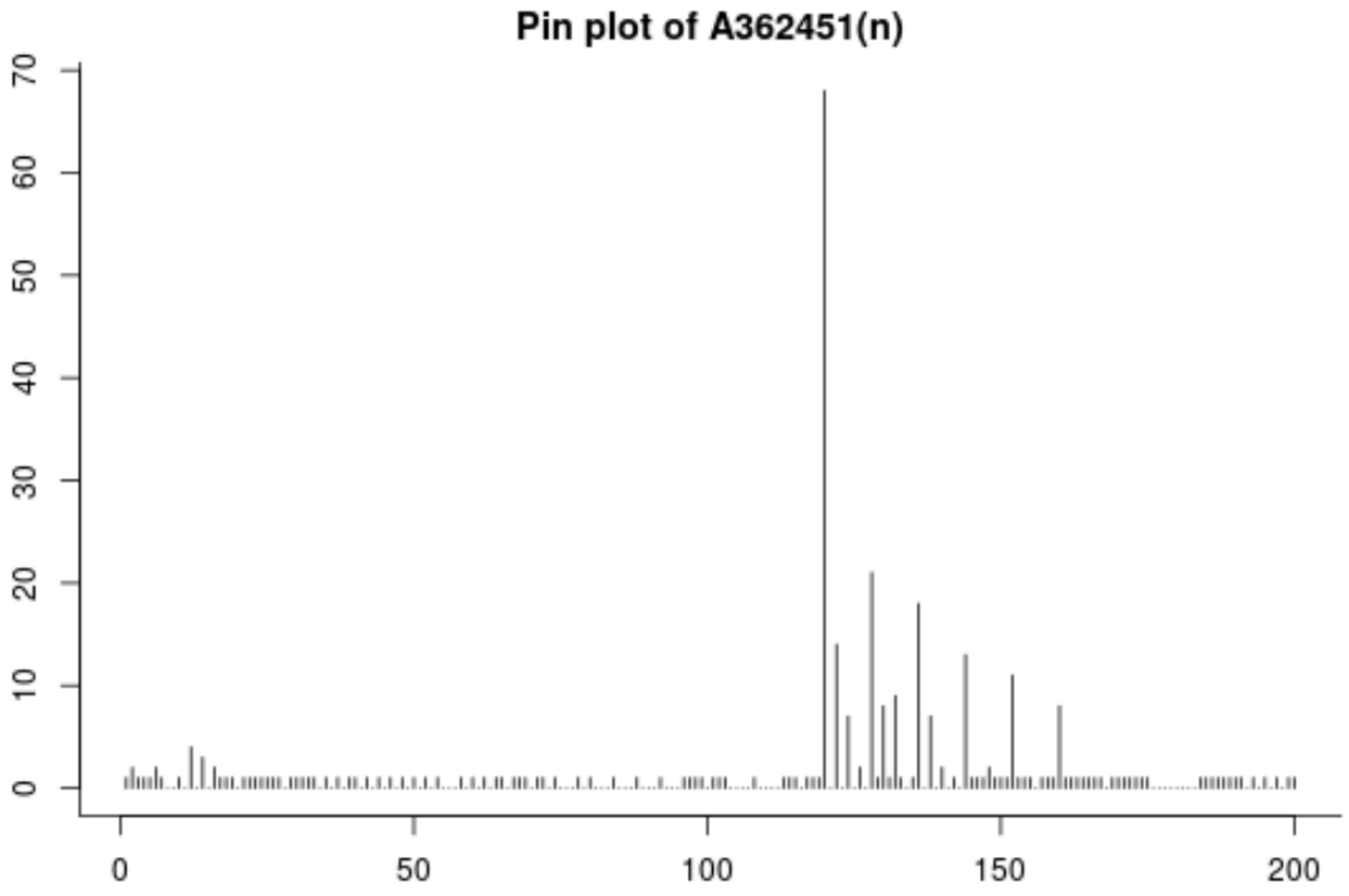
[1	1	2	2	4	2	6	4	6	4	10	4]
[0	1	0	2	2	4	2	2	2	6	6	8]
[1	1	2	0	2	2	0	0	4	0	2	2]
[0	1	2	2	0	2	0	4	4	2	0	2]
[1	1	0	2	2	2	4	0	2	2	2	0]
[0	1	2	0	0	2	4	2	0	0	2	4]
[1	1	2	0	2	2	2	2	0	2	2	0]
[0	1	2	2	0	0	0	2	2	0	2	2]
[1	1	0	2	0	0	2	0	2	2	0	0]
[0	1	2	2	0	2	2	2	0	2	0	0]
[1	1	0	2	2	0	0	2	2	2	0	0]
[0	1	2	0	2	0	2	0	0	2	0	2]

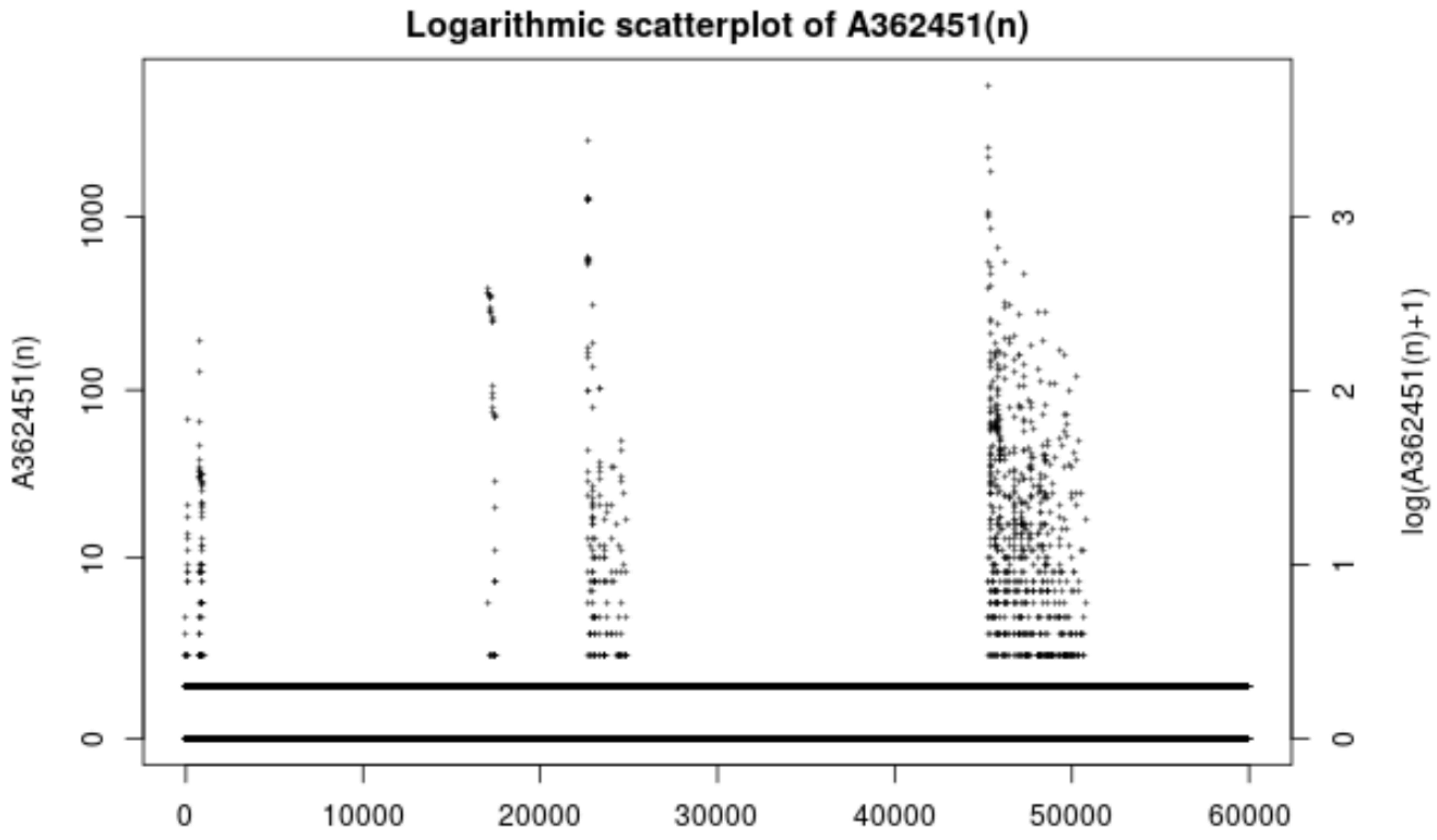
A000010 = phi(n)

New Gilbreath Conjectures (4)

Gilbreath transform of sigma(n) = sum of divisors function A000203 gives A362451:

```
1, 2, 1, 1, 1, 2, 1, 0, 0, 1, 0, 4, 0, 3, 0, 2, 1, 1, 1, 0, 1, 1, 1, 1, 1, 1, 1, 0, 1, 1, 1, 1, 1,
0, 1, 0, 1, 0, 1, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 1, 0, 1, 1, 0,
1, 1, 1, 0, 1, 1, 0, 1, 0, 0, 0, 1, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 0, 0, 0, 1, 1, 1, 1,
0, 1, 1, 1, 0, 0, 0, 0, 1, 0, 0, 0, 0, 1, 1, 1, 0, 1, 1, 1, 68, 0, 14, 0, 7, 0, 2, 0, 21, 1, 8, 1,
9, 1, 0, 1, 18, 0, 7, 0, 2, 0, 1, 0, 13, 1, 1, 1, 2, 1, 1, ...
```





Even more dramatic: sigma(n) - n = sum of aliquot parts: see G.T. A362452

Sum and Erase (Éric Angelini)

Eric Angelini, Does this iteration end? (Sum and erase),
Personal blog "Cinquante Signes", blogspot.com, Jul 26 2022.

Éric Angelini's Sum and Erase Sequence (1)

Take a number n, with initial digit d. Let s = sum of digits of n;
write down t = ns (the concatenation).
If d appears in s, delete all copies of d from t.
The next term is what's left.

n = 318, d = 3, s = 12, t = 31812. 3 is not in 12, so we get 31812
n = 319, d = 3, s = 13, t = 31913. 3 is in 13, so we get 191
n = 1019, d = 1, s = 11, t = 101911 -> 09 which we write as -9

A359142 = what n becomes:

0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 112, 123, 134, 145, ..., 189, 90, 0, 213, ...

A359143 = trajectory of 11:

```
11, 112, 1124, 11248, 2486, 4860, 486018, 48601827,... ?
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```
Comment from Michael S. Branicky, Jul 26 2022:
Starting at 11, this first reaches 0 at step 1399141.
The longest string encountered has length 222:
444444141444444541444441454544551551545455154545647565175555456576766 64\
465677675961617616416561527541551562575592651853254255356658359962263264\
365667368971272273374676377982812823836853869892911922935952968991101010\
121016.
```

Éric Angelini's Sum and Erase Sequence (2)

A cycle of length 49 found by Hans Havermann

Astonishing!

```
A cycle of length 49 found by Hans Havermann:

 0 5464644657500000011711019071641751
 1 5464644657500000011711019071641751109
 2 5464644657500000011711019071641751109119
 3 5464644657500000011711019071641751109119130
.....
22 5464644657500000011711019071641751109119130134142149163173184197214221226236247260268284298317328341
23 5464644657500000011711019071641751109119130134142149163173184197214221226236247260268284298317328341349
24 4646446700000011711019071641711091191301341421491631731841972142212262362472602682842983173283413 4936
25 666700000011711019071617110911913013121916317318197212212262362726026828298317328313936 35
26 700000117110190711711091191301312191317318197212212223272028282983173283139335 30
27 700000117110190711711091191301312191317318197212212223272028282983173283139335 30249
28 700000117110190711711091191301312191317318197212212223272028282983173283139335 30249264
29 000000111101901111091191301312191313181921221222322028282983132831393353024926426
30 000000111101901111091191301312191313181921221222322028282983132831393353024926426228
31 11111911119119131312191313181921221222322282829831328313933532492642622824
32 11111911119119131312191313181921221222322282829831328313933532492642622824246
33 11111911119119131312191313181921221222322282829831328313933532492642622824246258
.....
38 11111911119119131312191313181921221222322282829831328313933532492642622824246258273285300303309
39 99933293389222222322282829833283393353249264262282424625827328530030330932
40 99933293389222222322282829833283393353249264262282424625827328530030330932303
41 3323382222223222828283328333353242642622824246258273285300303303230330
42 282222222228282828524264262282424625827285000020021
43 282222222228282828524264262282424625827285000020021171
44 282222222228282828524264262282424625827285000020021171180
45 282222222228282828524264262282424625827285000020021171180189
46 8888854646844658785000000117118018907
47 8888854646844658785000000117118018907164
48 8888854646844658785000000117118018907164175
49 5464644657500000011711019071641751
```

Other cycles found by Michael Branicky

```
There is a cycle of length 20173 starting at 34674044445.
```

There is a cycle of length 46 that includes
998222898992822922222829202026260298265278295291026

Éric Angelini's Sum and Erase Sequence (4)

Studied by Angelini, Branicky, Hasler, Havermann, ...

25 eventually joins the earliest cycle, which has length 583792 and smallest term 3374

Q: Do most numbers reach 0 or cycle, or do most numbers go to infinity?

If n is large, it lengthens by about log n, then shrinks by a linear factor.
So the chance of n blowing up is tiny.

Can we construct a Chernobyl word w:
100000000000000000w
goes to
100000000000000000w'
and keeps slowly growing ?

(Maybe w only has even digits, but then w' would have an odd digit...?)

Report on Status of OEIS

Report on Status of OEIS (1)

1. The submissions stack: https://oeis.org/draft

If I neglect it for a week it creeps up to 350 or 450, with waiting time 2 months or more.
Board of Trustees of OEIS Foundation has voted to hire a full-time managing editor,
and to raise $3M endowment for salary.
Initial step: get OEIS Foundation classified as a 509(c)(3) supporting organization
for a major university. Almost completed.
Next steps: Raise the money, find good candidate. Must be US resident.

In the meantime, you can help! Go to bottom of submissions stack,
look at submissions, add comments (any registered user can do this).
You can add "Pink Box" comments like:
Definition unclear / Excellent sequence, looks ready for approval /
Second term is wrong, when corrected this is sequence A123456 /
Seems contrived, why is this interesting? /
intger is misspelled / what is k in the definition? / and so on

[However, if sequence is actively being reviewed by an editor, don't touch it]

Report on Status of OEIS (2)

2. 2023 is 50th anniversary of 1973 book "A Handbook of Integer Sequences"

My article, "A Handbook of Integer Sequences" 50 Years Later, appeared in The Mathematical Intelligencer (also on arXiv),

followed by articles in New York Times, Spektrum der Wiss., Pour la Science, Scientific American, Science et Avenir, etc.

See my home page, http://neilsloane.com, for copies. Should help with raising the $3M.

3. Statistics, August 2023

- 365000 sequences
- 750 to 900 new seqs accepted per month
- 45 proposed seqs rejected in July 2023
- 39 Editors in Chief; 133 Associate Editors
- Main file, "cat25" has 7.2 million lines; 512 MB

Sequences with No 3-Term Arithmetic Progressions

No 3-Term AP's

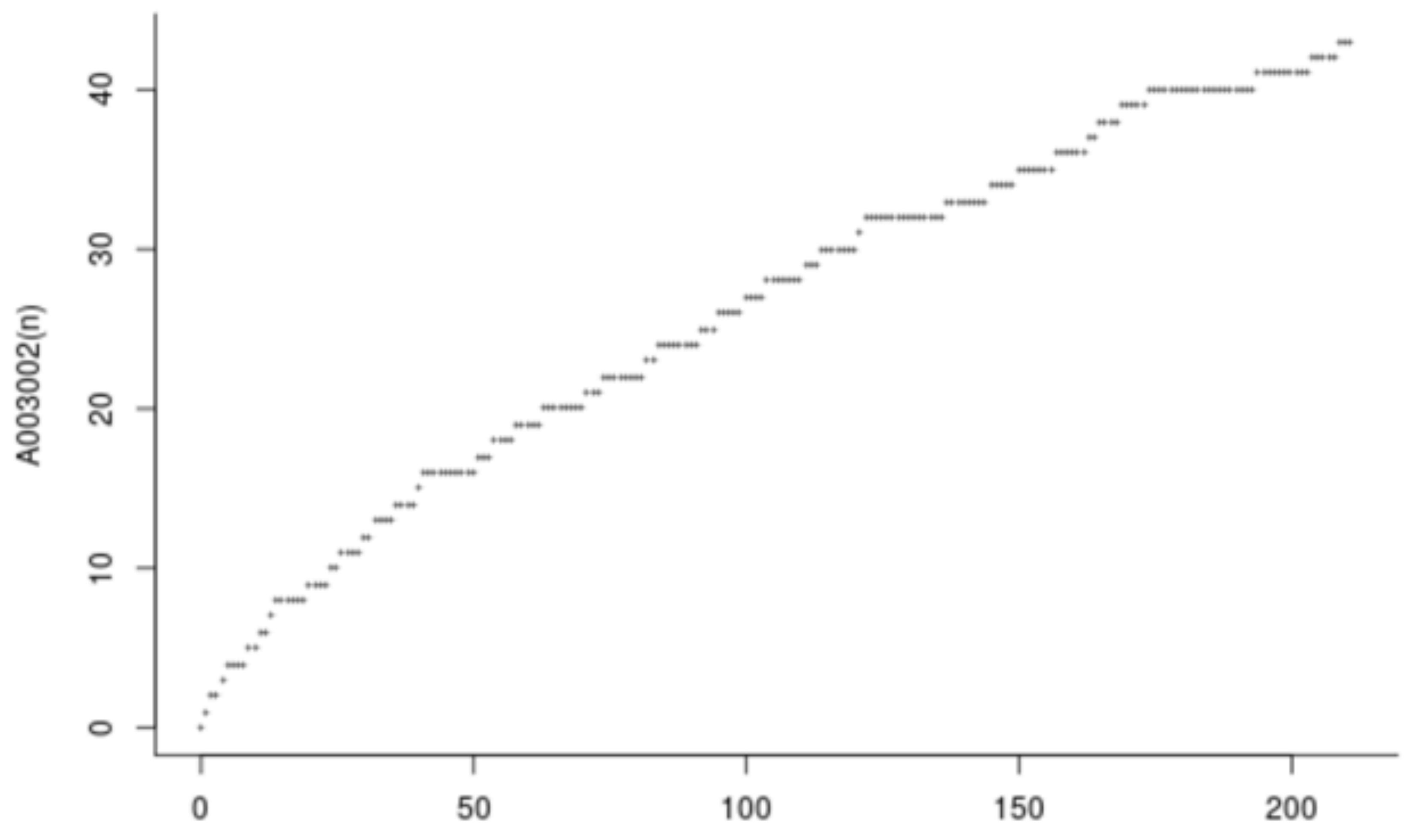
Talk by Thomas Bloom, April 26 2023

R(n) = max subset S of [1...n] containing no 3-term AP: A003002

K. Roth (1953): S positive density implies S contains 3T AP.

F. Behrend (1946): S = numbers written in base 3 with no 2's contain no 3T AP.

Zander Kelly & Raghu Meka reduced upper bound to close to Behrend's lower bound.



A003002

211 terms computed by Fausto Cariboni in 2018

- it would be nice to have more terms!

No 3-Term AP's (2)

Use greedy algorithm to construct

sequence with no 3-Term AP: A229037

with partial sums A362942

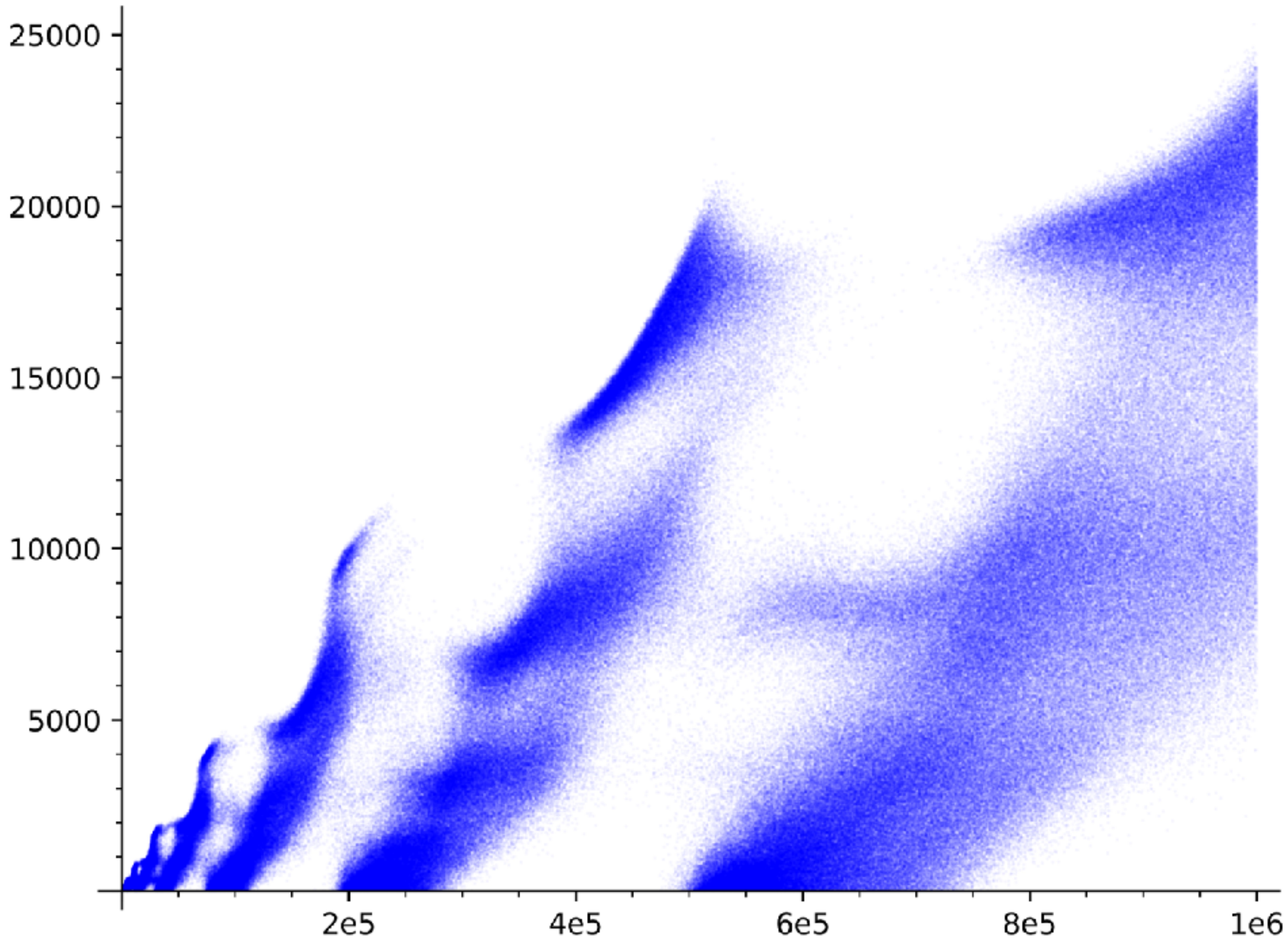
Questions: How does the greedy version

compare with optimal solution (A003002),

with the "base 3" construction

and with the new bounds?

Explain the fractal structure!



Sébastien Palcoux, Plot of first million terms of A229037 (also several other spectacular plots, see A229037)