To John Riordan
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In spite of the large number of existing mathematical tables, until now there has been no table of sequences of integers. Thus someone coming across the sequence 1, 2, 5, 15, 52, 203, 877, 4140, ... would have had difficulty in finding out that these are the Bell numbers, and that they have been extensively studied. This handbook remedies this situation. The main table contains a list of some 2300 sequences of integers, collected from all branches of mathematics and science. The sequences are arranged in numerical order, and for each one a brief description and a reference is given.

The first part of the book describes how to use the table, gives methods for analyzing unknown sequences, and contains an illustrated description of the most important sequences.

Who will use this handbook? Anyone who has ever been confronted with a strange sequence, whether in an intelligence test in high school, e.g.,

1, 8, 11, 69, 88, 96, 101, 111, 181, 609, ...

(guess!!), or in solving a mathematical problem, e.g.,

1, 2, 5, 14, 42, 132, 429, 1430, ...

(the Catalan numbers), or from a counting problem, e.g.,

1, 1, 2, 4, 9, 20, 48, 115, 286, 719, ...

(the number of rooted trees with n points), or in physics, e.g.,

1, 0, 3, 22, 192, 2046, 24853, ...

For many more terms and the explanation, see the main table.
(coefficients of the partition function for a cubic lattice), or in chemistry, e.g.,

$$1, 1, 1, 2, 3, 5, 9, 18, 35, 75, 159, \ldots$$

(the number of distinct hydrocarbons of the methane series), or in electrical engineering, e.g.,

$$3, 7, 46, 4336, 134281216, \ldots$$

(the number of Boolean functions of $n$ variables), will find this handbook useful.

Besides identifying sequences, the handbook will serve as an index to the literature for locating references on a particular problem, and for quickly finding numbers like $7^{12}$, the number of partitions of 30, the 18th Catalan number, or the expansion of $\pi$ to 60 decimal places. It might also be useful to have around when the first signals arrive from Betelgeuse (sequence 2311 for example would be a friendly beginning).
ACKNOWLEDGMENTS

This book was begun at Cornell University in the years 1965–1969, and finished at Bell Telephone Laboratories from 1969 to 1972. During that time I have been sustained by the support and encouragement of Richard Guy of the University of Alberta, Ron Graham and Henry Pollak of Bell Telephone Laboratories, John Riordan of Rockefeller University, and Ann Snitow of Rutgers University. Most of the sequences were found by searching through the stacks of the libraries of Cornell University, Brown University, and Bell Telephone Laboratories, and I thank the staffs of these excellent libraries for their patience and cooperation. Other sequences were suggested by friends and correspondents, to all of whom I am most grateful. E. R. Berlekamp, J. J. Cannon, D. G. Cantor, B. Ganter, F. Harary, D. E. Knuth, Shen Lin, W. F. Lunnun, R. C. Read, P. R. Stein, and J. W. Wrench, Jr. have been especially helpful. Finally I thank Eleanor Potter and Herman P. Robinson for a thorough reading of the manuscript.

The table was produced by first recording the sequences on punched cards, and (except when the sequence was generated by the author) comparing a listing of the cards with the original tables. These cards were then stored on magnetic tape, and the table has been typeset automatically from this tape. My thanks are due to the staff of the Bell Laboratories computation center at Murray Hill, especially the keypunch operators, for their untiring assistance.
ABBREVIATIONS

Abbreviations of the references are listed in the bibliography

\[ [a] \quad \text{the largest integer} \leq a \]
\[ a ** b \quad a^b \]
\[ C(i, j) \quad \text{the binomial coefficient} \binom{i}{j} \]
\[ \exp(a) \quad e^a \]
\[ \text{gf} \quad \text{generating function} \]
\[ \text{LCM} \quad \text{least common multiple} \]
\[ \text{REF} \quad \text{reference(s)} \]
\[ \text{seq.} \quad \text{sequence} \]
CHAPTER

DESCRIPTION OF THE BOOK

It is the fate of those who toil at the lower employments of life, to be driven rather by the fear of evil, than attracted by the prospect of good; to be exposed to censure, without hope of praise; to be disgraced by miscarriage, or punished for neglect, where success would have been without applause, and diligence without reward.

Among these unhappy mortals is the writer of dictionaries; whom mankind have considered, not as the pupil, but the slave of science, the pioneer of literature, doomed only to remove rubbish and clear obstructions from the paths of Learning and Genius, who press forward to conquest and glory, without bestowing a smile on the humble drudge that facilitates their progress. Every other author may aspire to praise; the lexicographer can only hope to escape reproach, and even this negative recompense has yet been granted to very few.

Samuel Johnson, Preface to the “Dictionary,” 1755

1.1 DESCRIPTION OF A TYPICAL ENTRY

The main table is a list of about 2300 sequences of integers. A typical entry is:

256 1, 2, 3, 5, 8, 13, 21, 34, 55, 89, 144, 233, 377, 610, 987, 1597, 2584, 4181, 6765, 10946, 17711, 28657, 46368, 75025, 121393, 196418, 317811, 514229, 832040, 1346269

and consists of the following items:

256
1, 2, 3, 5, 8, 13, 21, ... the sequence identification number
FIBONACCI NUMBERS the sequence itself
A(N) = A(N - 1) + A(N - 2) a name or descriptive phrase (in this case a recurrence)
REF for the sequence
the references
I DESCRIPTION OF THE BOOK


1.2 ARRANGEMENT

The entries are arranged in lexicographic order, so that sequences beginning 1, 2, 1 come before those beginning 1, 2, 2, and so on.

1.3 NUMBER OF TERMS GIVEN

Whenever possible enough terms are given to fill two lines. If fewer terms are given, it is because they have never been calculated so far as the author knows. (He would be very pleased to be corrected.) Finding the next term in the following sequences is known to be difficult (others of a similar type can be located via the index):


These sequences all represent unsolved problems.

1.4 REFERENCES

To conserve space, journal references are extremely abbreviated. They usually give the exact page on which the sequence may be found, but neither the author nor the title of the article. To find out more the reader must go to a library; this book is meant to used in conjunction with a library. Quite a small one will do. A considerable fraction of the sequences will be found in the following nine great works:

Indexes  Dickson [D12]
          Lehmer [LE1]
          Fletcher. Miller. Rosenhead. and Comrie [FMR]

Tables  Davis [DA2]
         Abramowitz and Stegun [AS1]
         David, Kendall, and Barton [DKB]

Combinatorics  Riordan [R1]
                David and Barton [DB1]
                Comet [CO1]
and in four journals:

American Mathematical Monthly [AMM]
Fibonacci Quarterly [FQ]
Journal of Combinatorial Theory [JCT]
Mathematics of Computation [MTAC]

Unusual sequences may send the reader to more exotic sources, but in any case he should first check Chapter III where additional information about some of the commoner sequences is given, and the index to see if other sequences (and hence references) of a similar type are listed.

Journal references usually give volume, page, and year, in that order. (See the example at beginning of this chapter.) Years after 1899 are abbreviated, by dropping the 19. Earlier years are not abbreviated. Sometimes to avoid ambiguity we use the more expanded form of: journal name (series number), volume number (issue number), page number, year.

References to books give volume (if any) and page. (See the example at the beginning of this chapter.)

The references do not attempt to give the discoverer of a sequence, but rather the most extensive table of the sequence that has been published.

1.5 WHAT SEQUENCES ARE INCLUDED?

Rule 1 The sequence must consist of nonnegative integers. (Sequences alternating in sign have been replaced by their absolute values. Interesting sequences of fractions have been entered by numerators and denominators separately. Some sequences of real numbers have been replaced by their integer parts, others by the nearest integers.)

Rule 2 The sequence must be infinite.

A few, like the Mersenne primes, have been given the benefit of the doubt.

Rule 3 The first two terms must be 1, n, where n is between 2 and 999.

An initial 1 has been silently inserted before the first term if this is greater than 1, and extra 1's and 0's at the beginning have been silently deleted. (See the beginning of Chapter II for examples.)

Rule 3 Enough terms must be known to separate the sequence from its neighbors in the table.

Rule 4 The sequence should have appeared in the scientific literature, and must be well-defined and interesting.

The selection has inevitably been subjective, but the goal has been to
include a broad variety of sequences and as many as possible.

1.6 HOW ARE ARRAYS OF NUMBERS TREATED?

Arrays of numbers (binomial coefficients, Stirling numbers of the first kind, etc.) have been entered by rows, columns, or diagonals, whichever seemed appropriate.

1.7 SUPPLEMENTS

It is planned to issue supplements to the Handbook from time to time, containing new sequences and corrections and extensions to the original sequences. Readers wishing to receive these supplements should notify the author.
CHAPTER II

HOW TO HANDLE A STRANGE SEQUENCE

2.1 HOW TO SEE IF A SEQUENCE IS IN THE TABLE

Obtain as many terms of the sequence as possible. The initial terms are handled as follows: Recall that the sequence must begin 1, n, where n is between 2 and 999. Find the first term in the sequence that is greater than 1, and replace all the terms that come before it by a single 1. Then look it up in the table. The initial 1 is just a marker, and need not be in the original sequence. For example, if the sequence begins

1, 2, 3, 5, 8, 13, \ldots 
2, 3, 5, 8, 13, \ldots 
-1, 0, 1, 1, 2, 3, 5, 8, \ldots 
1, 0, 2, 24, 552, 21280, \ldots 

see under 1, 2, 3, 5, 8, 13, \ldots 
see under 1, 2, 3, 5, 8, 13, \ldots 
see under 1, 2, 3, 5, 8, \ldots 
see under 1, 2, 24, 522, 21280, \ldots 

2.2 IF THE SEQUENCE IS NOT IN THE TABLE

(i) Try changing or redefining the sequence. Some typical changes are inserting or deleting an initial term (e.g., seq. 46 occurs as both 1, 2, 1, 2, 3, 6, 9, 18, \ldots and 1, 2, 3, 6, 9, 18, \ldots); adding or subtracting 1 or 2 from all the terms (e.g., seq. 309 occurs as both 1, 2, 3, 6, 20, 168, \ldots and 1, 4, 18, 166, \ldots); and multiplying all the terms by 2 or dividing by any common factor.

(ii) If all these methods fail, and it seems certain that the sequence is not in this handbook, please send the sequence and anything that is known about it, including appropriate references, to the author for possible inclusion in later editions.¹

¹Address: Mathematics Research Center, Bell Telephone Laboratories, Inc., Murray Hill, New Jersey 07974.
2.3 FINDING THE NEXT TERM

Suppose the beginning of a sequence is given as

\[
\begin{array}{cccccccc}
0 & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\
a_0 & a_1 & a_2 & a_3 & a_4 & a_5 & a_6 & a_7
\end{array}
\]

and a rule or explanation for it is desired. If nothing is known about the history of the sequence or if it is an arbitrary sequence, nothing can be said and any continuation is possible. (Any \(n+1\) points can be fitted by an \(n\)th degree polynomial.)

But the sequences normally encountered, and those in this handbook, are distinguished in that they have been produced in some intelligent and systematic way. Occasionally such sequences have a simple explanation, and if so, the methods given below may help to find it. These methods can be divided roughly into two classes: those which look for a systematic way of generating the \(n\)th term \(a_n\) from the terms \(a_0, \ldots, a_{n-1}\) before it, e.g., \(a_n = a_{n-1} + a_{n-2}\), i.e., methods which seek an internal explanation; and those which look for a systematic way of going from \(n\) to \(a_n\), e.g., \(a_n\) is the number of divisors of \(n\), or the number of trees with \(n\) nodes, or the \(n\)th prime number, i.e., methods which seek an external explanation. The former methods are described in the rest of this chapter, the latter in Chapter III.

In practice it is usually clear for one reason or another when a correct explanation for a sequence has been found.

(For the related problems of defining the complexity of a sequence, and extrapolating a sequence of real numbers, see the interesting work of Martin-Löf [IC 9 602 66] and Fine [IC 16 331 70 and F11].)

2.4 LOOK FOR A RECURRENCE

Let the sequence be \(a_0, a_1, a_2, a_3, \ldots\). Is there a systematic way of getting the \(n\)th term \(a_n\) from the preceding terms \(a_{n-1}, a_{n-2}, \ldots\)? A rule for doing this, such as \(a_n = a_{n-1}^2 - a_{n-2}\), is called a recurrence, and of course provides a method for getting as many terms of the sequence as desired.

In studying sequences and recurrences it is useful to define a generating function (gf) associated with the sequence, usually an ordinary gf:

\[
A(x) = a_0 + a_1 x + a_2 x^2 + a_3 x^3 + \cdots,
\]

but sometimes an exponential gf:

\[
E(x) = a_0 \frac{x}{1!} + a_2 \frac{x^2}{2!} + a_3 \frac{x^3}{3!} + \cdots.
\]
2.4 LOOK FOR A RECURRENCE

(These are formal power series having the sequence as coefficients; questions of convergence do not arise.)

Once a recurrence has been found for the sequence, techniques for solving it will be found in the works by Riordan [R1 19], Batchelder [BAT], and Levy and Lessman [LE2].

For example, consider seq. 256, the Fibonacci numbers: 1, 1, 2, 3, 5, 8, 13, 21, 34, . . . . These are generated by the recurrence \( a_n = a_{n-1} + a_{n-2} \), and from this it is not difficult to obtain the generating function

\[
1 + x + 2x^2 + 3x^3 + 5x^4 + \ldots = \frac{1}{1 - x - x^2},
\]

and the explicit formula for the \( n \)th term

\[
a_n = \frac{1}{\sqrt{5}} \left[ \left( \frac{1 + \sqrt{5}}{2} \right)^{n+1} - \left( \frac{1 - \sqrt{5}}{2} \right)^{n+1} \right].
\]

2.4.1 METHOD OF DIFFERENCES

This is the standard method for finding recurrences. In simple cases, it will even find an explicit formula for the \( n \)th term of a sequence, e.g., if this is a polynomial (such as \( a_n = n^2 + 1 \)) or a simple exponential (such as \( a_n = 2^n + n + 1 \)).

If the sequence is

\[ a_0, a_1, a_2, a_3, a_4, \ldots, \]

its first differences are the numbers

\[ \Delta a_0 = a_1 - a_0, \quad \Delta a_1 = a_2 - a_1, \quad \Delta a_2 = a_3 - a_2, \quad \ldots, \]

its second differences are

\[ \Delta^2 a_0 = \Delta a_1 - \Delta a_0, \quad \Delta^2 a_1 = \Delta a_2 - \Delta a_1, \quad \Delta^2 a_2 = \Delta a_3 - \Delta a_2, \quad \ldots, \]

and so on. The 0th differences are the original sequence: \( \Delta^0 a_0 = a_0 \), \( \Delta^0 a_1 = a_1 \), \( \Delta^0 a_2 = a_2 \), \ldots; and the \( m \)th differences are

\[ \Delta^m a_n = \Delta^{m-1} a_{n+1} - \Delta^{m-1} a_n \]

or, in terms of the original sequence,

\[ \Delta^m a_n = \sum_{i=0}^{m} (-1)^i \binom{m}{i} a_{m+n-i} . \quad (1) \]

Therefore if the differences of some order can be identified, Eq. (1) gives a recurrence for the sequence.
Furthermore, if the differences \( a_k, \Delta a_k, \Delta^2 a_k, \Delta^3 a_k, \ldots \) are known for some fixed value of \( k \), then a formula for the \( n \)th term is given by

\[
a_{n+k} = \sum_{m=0}^{n} \binom{n}{m} \Delta^m a_k.
\]

(2)

**Example (i)** Seq. 1562

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1</td>
<td>4</td>
<td>12</td>
<td>22</td>
<td>35</td>
<td>51</td>
<td>70</td>
<td>92</td>
</tr>
<tr>
<td>( \Delta a_n )</td>
<td>3</td>
<td>8</td>
<td>10</td>
<td>13</td>
<td>16</td>
<td>19</td>
<td>22</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 a_n )</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>3</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
<tr>
<td>( \Delta^3 a_n )</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td></td>
</tr>
</tbody>
</table>

Since \( \Delta^2 a_3 = 3, \Delta a_{n+3} - \Delta a_n = 3 \), or \( a_{n+2} - 2a_n + a_{n-1} = 3 \), a recurrence for the sequence. An explicit formula is obtained from Eq. (2) with \( k = 1 \):

\[
a_{n+1} = 1 + 4 \binom{n}{1} + 3 \binom{n}{2} = 1 + 4n + \frac{3}{2}n(n-1) = \frac{1}{2} (n+1)(3n+2).
\]

In general, if the \( m \)th differences are zero, \( a_n \) is a polynomial in \( n \) of degree \( m - 1 \).

**Example (ii)** Seq. 1382

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1</td>
<td>4</td>
<td>11</td>
<td>26</td>
<td>57</td>
<td>120</td>
<td>247</td>
</tr>
<tr>
<td>( \Delta a_n )</td>
<td>3</td>
<td>7</td>
<td>15</td>
<td>31</td>
<td>63</td>
<td>127</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 a_n )</td>
<td>4</td>
<td>8</td>
<td>16</td>
<td>32</td>
<td>64</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Here \( \Delta^2 a_n = 2^{n+1}, \Delta a_n = 2^{n+1} - 1 \), and \( a_n = 2^{n+1} - n - 2 \). Equation (2) gives the same answer.

**Example (iii)** Seq. 552 (the Pell numbers)

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td>70</td>
<td>169</td>
</tr>
<tr>
<td>( \Delta a_n )</td>
<td>1</td>
<td>3</td>
<td>7</td>
<td>17</td>
<td>41</td>
<td>99</td>
<td></td>
</tr>
<tr>
<td>( \Delta^2 a_n )</td>
<td>2</td>
<td>4</td>
<td>10</td>
<td>24</td>
<td>58</td>
<td></td>
<td></td>
</tr>
<tr>
<td>( \frac{1}{2} \Delta^2 a_n )</td>
<td>1</td>
<td>2</td>
<td>5</td>
<td>12</td>
<td>29</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Since \( \frac{1}{2} \Delta^2 a_n = a_n \), Eq. (1) gives the recurrence \( a_{n+2} - 2a_{n+1} - a_n = 0 \). Calculating further differences shows that \( \Delta^m a_1 = 2^{[m/2]} \) and so Eq. (2) gives the formula

\[
a_{n+1} = \sum_{m=0}^{n} \binom{n}{m} 2^{[m/2]}.
\]
**Example (iv) Seq. 469**

<table>
<thead>
<tr>
<th>( n )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
</tr>
</thead>
<tbody>
<tr>
<td>( a_n )</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>10</td>
<td>26</td>
<td>76</td>
<td>232</td>
<td>764</td>
</tr>
<tr>
<td>( \Delta a_n )</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>16</td>
<td>50</td>
<td>156</td>
<td>532</td>
<td></td>
</tr>
<tr>
<td>( n^{-1} \Delta a_n )</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>4</td>
<td>10</td>
<td>26</td>
<td>76</td>
<td></td>
</tr>
</tbody>
</table>

Notice that \( \Delta a_n \) is divisible by \( n \), and in fact \( n^{-1} \Delta a_n = a_{n-1} \), so that \( a_{n+1} = a_n + na_{n-1} \). Again Eq. (2) gives a formula for \( a_n \).

### 2.4.2 Other Methods of Attack

Is the sequence close to a known sequence, such as the powers of 2? If so, try subtracting off the known sequence. For example, seq. 1382 (again): 1, 4, 11, 26, 57, 120, 247, 502, 1013, 2036, 4083, \ldots The last four numbers are close to powers of 2: 512, 1024, 2048, 4096; and then it is easy to find \( a_n = 2^n - n - 1 \).

Is a simple recurrence such as \( a_n = a_{n-1} + \beta a_{n-2} \) likely? For this to happen, the ratio \( \rho_n = a_{n+1}/a_n \) of successive terms must approach a constant as \( n \) increases. Use the values \( a_2 \) to \( a_5 \) to determine \( \alpha \) and \( \beta \) and then see if \( a_6, a_7, \ldots \) are generated correctly.

If the ratio \( \rho_n \) has first differences which are approximately constant, this suggests a recurrence of the type \( a_n = \alpha a_{n-1} + \cdots \). For example, seq. 704: 1, 2, 7, 30, 157, 972, 6961, 56660, 516901, \ldots has successive ratios 2, 3.5, 4.29, 5.23, 6.19, 7.16, 8.14, 9.12, \ldots with differences approaching 1, suggesting \( a_n = na_{n-1} + \cdots \). Subtracting \( na_{n-1} \) from \( a_n \), we obtain the original sequence 0, 1, 2, 7, 30, 157, 972, \ldots again, so \( a_n = na_{n-1} + a_{n-2} \).

This example illustrates the principle that whenever \( \rho_n = a_{n+1}/a_n \) seems to be close to a recognizable sequence \( r_n \), one should try to analyze the sequence \( b_n = a_{n+1} - r_n a_n \).

A recurrence of the form \( a_n = na_{n-1} + (\text{small term}) \) can be identified by the fact that the 10th term is approximately 10 times the 9th. For example, seq. 766: 0, 1, 2, 9, 44, 265, 1854, 14833, 133496, 1334961, \ldots, \( a_n = na_{n-1} + (-1)^n \).

The recurrence \( a_n = a_{n-1}^2 + \cdots \) is characterized by the fact that each term is about twice as long as the one before. For example, seq. 331: 1, 2, 3, 7, 43, 1807, 3263443, 10650056950807, \ldots, and \( a_n = a_{n-1}^2 - a_{n-1} + 1 \).

### 2.4.3 Factorizing

Does the sequence, or one obtained from it by some simple operation, have many factors?

**Example (i) Seq. 1614: 1, 5, 23, 119, 719, 5039, 40319, \ldots** As it stands, the sequence cannot be factored, since 719 is prime, but the addi-
tion of 1 to all the terms gives the highly composite sequence $2, 6 = 2 \cdot 3, 24 = 2 \cdot 3 \cdot 4, 120 = 2 \cdot 3 \cdot 4 \cdot 5, \ldots$, which are the factorial numbers (see Section 3.13).

The presence of only small primes may also suggest binomial coefficients:

Example (ii) Seq. 577 (the Catalan numbers): $1, 2, 5, 14 = 2 \cdot 7, 42 = 2 \cdot 3 \cdot 7, 132 = 4 \cdot 3 \cdot 11, 429 = 3 \cdot 11 \cdot 13, 1430 = 2 \cdot 5 \cdot 11 \cdot 13, 4862 = 2 \cdot 11 \cdot 13 \cdot 17, \ldots$ and

$$a_n = \frac{1}{n+1} \binom{2n}{n}$$

(see Section 3.5).

Sequences arising in number theory are sometimes multiplicative, i.e., have the property that $a_{mn} = a_m a_n$ whenever $m$ and $n$ have no common factor. For example, seq. 86: $1, 2, 2, 3, 2, 4, 2, 4, \ldots$, the number of divisors of $n$.

2.4.4 Self-Generating Sequences

This section describes some recurrences of a simple yet unusual type. They have been called (rather arbitrarily) self-generating.

In the first two examples let $A = \{a_0 = 1, a_1, a_2, \ldots\}$ be a sequence of 1's and 2's.

(i) If every 1 in $A$ is replaced by 1, 2 and every 2 by 2, 1 a new sequence $A'$ is obtained. Imposing the condition that $A = A'$ forces $A$ to be seq. 71: $1, 2, 2, 1, 2, 1, \ldots$. Sequences 21 and 36 are of the same type.

(ii) Let $A'' = \{b_0, b_1, b_2, \ldots\}$, where $b_n$ is the length of the $n$th run in $A$. (A run is a maximal string of identical symbols.) The condition $A = A''$ forces $A$ to be seq. 70: $1, 2, 2, 1, 1, 2, 1, 2, 2, 1, \ldots$.

In the remaining examples, $A = \{a_0 = 1, a_1, a_2, \ldots\}$ is a nondecreasing sequence of integers.

(iii) Let $c_n$ be the number of times $n$ occurs in $A$, for $n = 1, 2, \ldots$. If $c_n = n, A$ is seq. 89: $1, 2, 2, 3, 3, 3, 4, 4, 4, 4, \ldots$. If $c_n = a_{n-1}, A$ is seq. 91: $1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 6, \ldots$ (Seq. 965 is related to the latter sequence.)

(iv) The condition that $a_{n+1} - a_n$ be the smallest positive integer not equal to $a_i - a_j$ for any $i, j \leq n$ forces $a$ to be seq. 416: $1, 2, 4, 8, 13, 21, \ldots$. The conditions $a_0 = 1, a_2 = 2$, and that $a_n$ be the smallest integer which can be written uniquely as the sum of two distinct preceding terms force $A$ to be seq. 201: $1, 2, 3, 4, 6, 8, 11, 13, \ldots$. Sequences 231, 254, 425, and 909 have similar explanations.
CHAPTER III

ILLUSTRATED DESCRIPTION OF SOME IMPORTANT SEQUENCES

While Chapter II studied ways of getting the $n$th term of a sequence from the preceding terms, this chapter considers externally generated sequences, such as the sequences in which the $n$th term is the number of graphs with $n$ nodes or the $n$th triangular number. An informal and illustrated description is given of some of the most important such sequences.

3.1 GRAPHS AND TREES

Stated informally, a graph consists of a finite set of points (or nodes) some of which are joined by lines (or edges). Figure 1 illustrates seq. 479, the number of graphs with $n$ nodes.

![Graphs with 1, 2, 4, and 11 nodes](image)

Fig. 1. Seq. 479, graphs or reflexive symmetric relations.

A digraph, or directed graph, is a graph with arrows on the edges (Fig. 2, seq. 1229). Figure 3 shows seq. 1069, digraphs of functions, i.e., digraphs with exactly one arrow directed out of each node.
III SOME IMPORTANT SEQUENCES

\[ d_1 = 1 \quad \quad d_2 = 3 \quad \quad d_3 = 16 \]

Fig. 2. Seq. 1229, digraphs or reflexive relations.

\[ f_1 = 1 \quad \quad f_2 = 3 \quad \quad f_3 = 7 \]

Fig. 3. Seq. 1069, functional digraphs.

A tree is a connected graph containing not closed paths (Fig. 4, seq. 299). A rooted tree is a tree with a distinguished node called Eve, or the root. Figure 5 illustrates seq. 454, the number of rooted trees with \( n \) nodes. The generating function (gf) of this sequence is

\[ r(x) = x + x^2 + 2x^3 + 4x^4 + 9x^5 + \cdots \]

and satisfies

\[ r(x) = x \exp[r(x) + \frac{1}{2}r(x^2) + \frac{1}{3}r(x^3) + \cdots] \]

The generating function for trees,

\[ t(x) = x + x^2 + x^3 + 2x^4 + 3x^5 + 6x^6 + \cdots \]

is then given by

\[ t(x) = r(x) - \frac{1}{2}r^2(x) + \frac{1}{3}r(x^2) \]
3.1 GRAPHS AND TREES

Fig. 4. Seq. 299, trees.

Fig. 5. Seq. 454, rooted trees. (The numbers in parentheses give seq. 771, labeled rooted trees.)

Any of these graphs may be labeled by (if there are $n$ nodes) attaching the numbers from 1 to $n$ to the nodes. For example in Fig. 5, the numbers in parentheses give the number of ways of labeling each tree, and then the total number of labeled rooted trees with $n$ nodes is $n^{n-1}$, seq. 771. Usually when graphs are mentioned in the main table they are unlabeled unless stated otherwise.
III SOME IMPORTANT SEQUENCES

The degree of a node is the number of edges meeting it. Figure 6 shows seq. 118, series-reduced trees, or trees without nodes of degree 2.

For further information about the preceding sequences and for the enumeration of other kinds of graphs, see Riordan [R1] and Harary [HA5].

\[ a_4 = 1 \]

\[ a_5 = 1 \]

\[ a_6 = 2 \]

\[ a_7 = 2 \]

\[ a_8 = 4 \]

\[ a_9 = 5 \]

Fig. 6. Seq. 118, series-reduced trees.

3.2 RELATIONS

A relation \( R \) on a set \( S \) is any subset of \( S \times S \), and \( xRy \) means \( (x, y) \in R \) or "\( x \) is related to \( y \)." A relation is reflexive if \( xRx \) for all \( x \) in \( S \), symmetric if \( xRy \Rightarrow yRx \), antisymmetric if \( xRy \) and \( yRx \Rightarrow x = y \), and transitive if \( xRy \) and \( yRz \Rightarrow xRz \).

The most important types of relations are:

1. unrestricted, or digraphs with loops of length 1 allowed (seq. 784: 2, 10, 104, 3044, 291968, . . . );
2. symmetric, or graphs with loops of length 1 allowed (seq. 646: 2, 6, 20, 90, 544, 5096, 79264, . . . );
3. reflexive, or digraphs (Fig. 2, seq. 1229 again);
4. reflexive symmetric, or graphs (Fig. 1, seq. 479 again);
3.2 RELATIONS

(5) reflexive transitive, or topologies (Fig. 7, seq. 1133: 1, 3, 9, 33, 139, 718, 4535, ?). For the connection between digraphs and topologies, see Birkhoff [B11 117]);

(6) reflexive symmetric transitive, or partitions (Fig. 20, p. 24, seq. 244);

(7) reflexive antisymmetric transitive, or partially ordered sets (Fig. 8, seq. 588: 1, 2, 5, 16, 63, 318, 2045, ?).

Fig. 7. Seq. 1133, topologies.

Fig. 8. Seq. 588, partially ordered sets.
This assumes that the graphs are unlabeled, i.e., that the elements of the set $S$ are indistinguishable. If the elements of $S$ are labeled 1 through $n$, the corresponding numbers are:

1. $2^{n^2}$;
2. $2^{n(n+1)/2}$;
3. $2^{n(n-1)}$;
4. $2^{2^n}$

(These four [(1)–(4)] are not in the table, but the sequences of their exponents are):

5. seq. 1476: 1, 4, 29, 355, 6942, 209527, 9535241, ?;
6. seq. 585: the Bell numbers or the number of equivalence relations on a set of $n$ objects (see Fig. 22, p. 25);
7. seq. 1244: 1, 3, 19, 219, 4231, 130023, 6129859, ?.

### 3.3 Geometries

The numbers of topologies were shown in Fig. 7; the following are also basic geometrical sequences:

A **linear space** is a system of (abstract) points and lines such that every two points lie on a unique line, and every line contains at least two points. A **geometry** is a system of points, lines, planes, ... with an analogous definition. Figure 9 shows seq. 462: 1, 1, 2, 4, 9, 26, 101, 950, ?, the number of geometries with $n$ points. (See Crapo and Rota [JM2 49 127 70]. The * denotes 5 points in general position in 4-dimensional space.) The planar figures in Fig. 9 form seq. 271: 1, 1, 2, 3, 5, 10, 24, 69, 384, ?, the number of linear spaces (Doyen [BSM 19 424 67]).

\[ g_2 = 1 \]
\[ g_3 = 2 \]
\[ g_4 = 4 \]
\[ g_5 = 9 \]

*Fig. 9. Seq. 462, geometries (for * see text).*
3.4 COMBINATIONS AND FIGURATE NUMBERS

The most basic combinatorial number is the *binomial coefficient*

\[
\binom{n}{r} = \frac{n(n-1)(n-2) \cdots (n-r+1)}{1 \cdot 2 \cdot 3 \cdots r} = \frac{n!}{r!(n-r)!},
\]

which is the number of selections, or combinations, of \(n\) unlike things taken \(r\) at a time, has the form

\[
(1 + x)^n = \sum_{r=0}^{n} \binom{n}{r} x^r,
\]

and is the \((r+1)\)th term in the \((n+1)\)th row of Pascal’s triangle

\[
\begin{array}{cccccc}
1 \\
1 & 1 \\
1 & 2 & 1 \\
1 & 3 & 3 & 1 \\
1 & 4 & 6 & 4 & 1 \\
1 & 5 & 10 & 10 & 5 & 1 \\
1 & 6 & 15 & 20 & 15 & 6 & 1 \\
\vdots & & & & & & \\
\end{array}
\]

These are also called *figurate numbers* since they are the numbers of points in certain figures. For example, \(\binom{n}{2}\) and \(\binom{n}{3}\) are the *triangular* and *tetrahedral* numbers (Fig. 10, seqs. 1002, 1363).

Fig. 10. Seqs. 1002 and 1363, the triangular and tetrahedral numbers.
Other examples of figurate numbers are the polygonal numbers \( P(r, s) = \frac{1}{2}r(rs - s + 2) \). Figure 11 shows seq. 1350, the square numbers \( P(r, 2) = r^2 \); and seq. 1562, the pentagonal numbers \( P(r, 3) = \frac{1}{2}r(3r - 1) \).

Many other figurate numbers, including cubes, fourth powers, etc., will be found in the table. For further pictures see Hogben [HO3].

![Fig. 11. Seqs. 1350 and 1562, the square and pentagonal numbers.](image)

### 3.5 CATALAN NUMBERS AND DISSECTIONS

Next to the figurate numbers, the Catalan numbers are the most frequently occurring combinatorial numbers. (Gould [GO4] lists over 240 references.) They are defined by

\[
c_n = \frac{1}{n+1} \binom{2n}{n},
\]

and form seq. 557: 1, 2, 5, 14, 42, 132, 429, 1430, 4862, 16796, \ldots A gf is

\[
1 + x + 2x^2 + 5x^3 + \cdots = (2x)^{-1}[1 - (1 - 4x)^{1/2}].
\]

Some of the interpretations of \( c_n \) are:

1. The number of ways of dissecting a convex polygon of \( n + 2 \) sides into \( n \) triangles by drawing nonintersecting diagonals (Fig. 12a).

2. The number of ways of completely parenthesizing a product of \( n + 1 \) letters (so that there are two factors inside each set of parentheses). The examples for \( n = 1, 2, 3 \) (arranged to show the correspondence with the dissections of Fig. 12a) are:

\[
n = 1 \quad (ab);
\]

\[
n = 2 \quad a(bc), \quad (ab)c;
\]

\[
n = 3 \quad (ab)(cd), \quad a((bc)d), \quad ((ab)c)d, \quad a(b(cd)), \quad (a(bc))d.
\]
(3) The number of bifurcated rooted planar trees with \( n + 1 \) endpoints. (A planar tree is one which has been drawn on a plane, and bifurcated means that each edge splits in two at each node. See Fig. 12b. The trees are drawn to show the correspondence with the dissections and the parentheses.)

(4) In an election with two candidates A and B, each receiving \( n \) votes, \( c_n \) is the number of ways the votes can come in so that A is never behind B (Feller [FE1 171] and Comtet [COI 194]).

\[
\begin{align*}
(a) \quad c_1 &= 1 \quad \triangle \\
& \quad c_2 = 2 \\
& \quad c_3 = 5 \\
& \quad c_4 = 14
\end{align*}
\]

\[
\begin{align*}
(b) \quad c_1 &= 1 \\
& \quad c_2 = 2 \\
& \quad c_3 = 5
\end{align*}
\]

Fig. 12. Seq. 577, the Catalan numbers.

Figure 13 illustrates seq. 942, the number of different dissections of a polygon when two dissections are considered to be the same if a rotation or reflection sends one into the other.

Figure 14 illustrates seq. 391, giving \( \frac{1}{2}n(n + 1) + 1 \), the maximum number of pieces obtained by slicing a pancake with \( n \) slices. The numbers
III SOME IMPORTANT SEQUENCES

of \( n \)-sided polygons in the \( n \)th diagram of Fig. 14 form seq. \textit{1181}: 0, 0, 1, 3, 12, 70, 465, 3507, 30016, \ldots\) (Robinson [AMM 58 462 51]). Seq. \textit{491}: 2, 4, 8, 15, 26, 42, 64, 93, 130, 176, \ldots\) gives \( (n + 2)(n + 3)/6 \), the maximum number of pieces obtained with \( n \) slices of a cake.

\[
\begin{align*}
\text{Fig. 13. Seq. 942, dissections of a polygon.}
\end{align*}
\]

\[
\begin{align*}
\text{Fig. 14. Seq. 391, slicing a pancake.}
\end{align*}
\]
3.6 NECKLACES AND IRREDUCIBLE POLYNOMIALS

Figure 15 illustrates seq. 203, $T_n$, the number of different necklaces that can be made from beads of two colors, when the necklaces can be rotated but not turned over. This is also the number of irreducible binary polynomials whose degree divides $n$, an important sequence in digital circuitry; and has the formula $T_n = \sum \phi(d) 2^{n/d}$, where $\phi(d)$ is the Euler totient function (seq. III, Section 3.14) and the sum is over all divisors $d$ of $n$. (See Berlekamp [BE2 70] and Golomb [CMA 1 358 69].) If turning over is allowed, the number of different necklaces is given by seq. 202: 2, 3, 4, 6, 8, 13, 18, 30, 46, 78, ... (See Gilbert and Riordan [IJM 5 657 61].)

![Fig. 15. Seq. 203, necklaces.](image)

3.7 KNOTS

Figure 16 shows seq. 322: 0, 0, 1, 1, 2, 3, 7, 18, 41, 123, 367, ?, the number of knots with $n$ crossings, in which the crossings alternate. (See Tait [TA1 1 334] and Conway [JL2 343].)

![Fig. 16. Seq. 322, knots.](image)
### 3.8 STAMPS

Figure 17 shows seq. 576: 1, 1, 2, 5, 14, 39, 120, 358, 1176, 3527, ... (six more terms are known), the number of ways of folding a strip of stamps.

\[
\begin{align*}
\alpha_1 &= 1 & \alpha_2 &= 1 & \alpha_3 &= 2 \\
\alpha_4 &= 5 & & \\
\alpha_5 &= 14 & &
\end{align*}
\]

Fig. 17. Seq. 576, folding a strip of stamps.

### 3.9 POLYOMINOES

A polyomino with \( p \) squares is a connected set of \( p \) squares from a chessboard pattern. Polyominoes are free if they can be rotated and turned over (Fig. 18), and fixed otherwise. Unless otherwise stated, all polyominoes are free. Polyominoes may also be formed from triangles, rectangles, cubes (Fig. 19), etc. In no case is a formula known for the general term. (See Golomb [GO2].)

\[
\begin{align*}
\alpha_1 &= 1 & \alpha_2 &= 1 \\
\alpha_3 &= 2 & & \\
\alpha_4 &= 5 & & \\
\alpha_5 &= 12 & &
\end{align*}
\]

Fig. 18. Seq. 561, square polyominoes.
3.11 Pólya Counting Theory

Fig. 19. Seq. 731, polyominoes made from cubes.

3.10 Boolean Functions

A Boolean (or switching) function is a function \( f(x_1, \ldots, x_n) \), where each variable \( x_i \) is 0 or 1, and \( f \) takes on the values 0 or 1.

These arise in the design of logical circuits, when the names of the variables do not matter. So it is natural to say that two such functions are equivalent if they differ only in the names of the variables (so that \( x_1 + x_2 x_3 \) is equivalent to \( x_2 + x_1 x_3 \)), and to ask for the number of inequivalent functions. The answers to this (which is seq. 1405: 4, 12, 80, 3984, \ldots) and to many similar questions (allowing complementation of the variables, etc.) are given by the Pólya counting theory (Section 3.11).

Two generalizations that will be found in the table are (i) Post functions, which are functions \( f(x_1, \ldots, x_n) \), where each \( x_i \) and \( f \) can take any value from 0 to \( m - 1 \); and (ii) switching networks, which are \( n \)-input, \( k \)-output networks such that each of the outputs is a Boolean function of the \( n \) inputs. For details see Harrison [HA2, MU3 85].

3.11 Pólya Counting Theory

A large number of counting problems involving graphs, necklaces, Boolean functions, and patterns of various kinds have been solved by the
Theorems of Redfield, Pólya, and De Bruijn. (See Riordan [R1 131], De Bruijn [BE6 144], Harrison [HA2 127, MU3 85], and Harary [HA5 178].)

3.12 PARTITIONS

The following are the most important sequences of partitions.

The main such sequence is number 244: 1, 2, 3, 5, 7, 11, . . . , giving the number of partitions of \( n \) into integer parts (Fig. 20). A gf is

\[
1 + x + 2x^2 + 3x^3 + 5x^4 + \cdots = \prod_{i=1}^{\infty} (1 - x^i)^{-1}.
\]

(See Gupta [RS2] and David et al. [DKB 273].)

Those partitions of \( n \) in which all parts are distinct form seq. 100: 1, 1, 2, 2, 3, 4, 5, . . . with gf

\[
1 + x + x^2 + 2x^3 + 2x^4 + \cdots = \prod_{i=1}^{\infty} (1 + x^i).
\]

The partitions of the even numbers into parts which are powers of two form the binary partition function \( b(n) \), seq. 378: 1, 2, 4, 6, 10, 14, 20, 26, 36, 46, . . . , with recurrence \( b(n) = b(n - 1) + b([\frac{n}{2}]) \).

\[
\begin{align*}
p(1) &= 1 & 1 \\
p(2) &= 2 & 2, 1^2 \\
p(3) &= 3 & 3, 21, 1^3 \\
p(4) &= 5 & 4, 31, 2^2, 21^2, 1^4 \\
p(5) &= 7 & 5, 41, 32, 31^2, 2^21, 21^3, 1^6 \\
p(6) &= 11 & 6, 51, 42, 41^2, 3^2, 321, 31^3, 2^3, 2^21^2, 21^4, 1^6 \\
p(7) &= 15 & 7, 61, 52, 51^2, 43, 421, 41^3, 3^31, 32^2, 321^2, 31^4, 2^31, 2^21^3, 21^5, 1^7
\end{align*}
\]

Fig. 20. Seq. 244, the number of partitions of \( n \).

Figure 21 illustrates the number of planar partitions of \( n \), seq. 1016, with gf

\[
1 + x + 3x^2 + 6x^3 + \cdots = \prod_{i=1}^{\infty} (1 - x^i)^{-1}.
\]

Figure 22 shows \( S(n, k) \), the Stirling numbers of the second kind, or the number of partitions of a set of \( n \) labeled objects into \( k \) parts.
3.12 PARTITIONS

\[
\begin{array}{c|cccc}
 r(1) & 1 \\
 r(2) & 2 & 1 & 1 & 1 \\
 r(3) & 3 & 2 & 1 & 1 \\
 r(4) & 4 & 3 & 1 & 1 \\
 & 1 & 1 & 1 & 1 \\
\end{array}
\]

Fig. 21. Seq. 1016, planar partitions.

\[
\begin{array}{cccc|c}
 n & k & 1 & 2 & 3 & 4 & \text{Total} \\
\hline
 1 & 1 & 1 & & & & 1 \\
\hline
 2 & 12 & 1 & 2 & & & 2 \\
\hline
 3 & 123 & 1 & 2 & 3 & & 5 \\
 & 2 & 3 & 12 \\
\hline
 4 & 1234 & 1 & 2 & 3 & 4 & 15 \\
 & 1, 234 & 2, 134 \\
 & 3, 124 & 4, 123 \\
 & 12, 34 & 13, 24 \\
 & 14, 23 & & & & \\
\end{array}
\]

Fig. 22. \(S(n, k)\), the Stirling numbers of the second kind, and seq. 585, the Bell numbers.

The numbers continue:

\[
\begin{array}{ccccc}
\text{row sums} & \text{B(n)} \\
1 & 1 \\
1 & 1 & 2 \\
1 & 3 & 1 & 5 \\
1 & 7 & 6 & 1 & 15 \\
1 & 15 & 25 & 10 & 1 & 52 \\
1 & 31 & 90 & 65 & 15 & 1 & 203 \\
1 & 63 & 301 & 350 & 140 & 21 & 1 & 877 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
\end{array}
\]

25
A gf for $S(n, k)$ is

$$x^n = \sum_{k=0}^{n} S(n, k) \ x(x - 1) \cdots (x - k + 1).$$

Both the columns and diagonals of this array will be found in the main table.

The row sums are the Bell numbers $B(n)$, seq. 585. $B(n)$ is also the number of equivalence relations on a set of $n$ objects (Section 3.2) and has gf

$$1 + x + 2 \frac{x^2}{2!} + 5 \frac{x^3}{3!} + \cdots = e^{e^x-1}.$$  

(See Abramowitz and Stegun [AS1 835], David et al. [DKB 223], and Comtet [CO1 2 38].)

### 3.13 PERMUTATIONS

A permutation of $n$ objects is any rearrangement of them, and is specified either by a table:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1</td>
<td>3</td>
<td>4</td>
<td>1</td>
<td>2</td>
</tr>
</tbody>
</table>

or by a product of cycles: $(134)(25)$, both of which mean replace 1 by 3, 3 by 4, 4 by 1, 2 by 5, and 5 by 2.

Figure 23 shows $s(n, k)$, the Stirling numbers of the first kind, or the numbers of permutations of $n$ objects containing $k$ cycles. The numbers continue:

<table>
<thead>
<tr>
<th></th>
<th>1</th>
<th>1</th>
<th>6</th>
<th>24</th>
<th>120</th>
<th>720</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>1</td>
<td>2</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
</tr>
<tr>
<td>2</td>
<td>3</td>
<td>1</td>
<td>6</td>
<td>24</td>
<td>120</td>
<td>720</td>
</tr>
<tr>
<td>6</td>
<td>11</td>
<td>6</td>
<td>1</td>
<td>24</td>
<td>120</td>
<td>720</td>
</tr>
<tr>
<td>24</td>
<td>50</td>
<td>35</td>
<td>10</td>
<td>1</td>
<td>120</td>
<td>720</td>
</tr>
<tr>
<td>120</td>
<td>274</td>
<td>225</td>
<td>85</td>
<td>15</td>
<td>1</td>
<td>720</td>
</tr>
<tr>
<td>720</td>
<td>1764</td>
<td>1624</td>
<td>735</td>
<td>175</td>
<td>21</td>
<td>5040</td>
</tr>
</tbody>
</table>

A gf for $s(n, k)$ is

$$x(x - 1) \cdots (x - n + 1) = \sum_{k=0}^{n} (-1)^{n-k}s(n, k)x^k.$$  

Both the columns and diagonals of this array will be found in the main table. The row sums are the factorial numbers $n!$, seq. 659, the total num-
number of permutations of \( n \) objects. References are as given above for the Stirling numbers of the second kind.

Factorial \( n \) is the product \( 1 \cdot 2 \cdot 3 \cdot \ldots \cdot n \) of the first \( n \) numbers. The products of the first \( n \) even numbers, \((2n)! = 2 \cdot 4 \cdot 6 \cdots (2n) = 2^n \cdot n!\), seq. 742: 2, 8, 48, 384, 3840, 46080, \ldots, and of the first \( n \) odd numbers, \((2n-1)! = 1 \cdot 3 \cdot 5 \cdots (2n-1) = (2n)!/(2^n \cdot n!)\), seq. 1217: 1, 3, 15, 105, 945, 10395, \ldots, are called double factorials.

<table>
<thead>
<tr>
<th>( \frac{n}{k} )</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>Total</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>(1)</td>
<td></td>
<td></td>
<td></td>
<td>1</td>
</tr>
<tr>
<td>2</td>
<td>(12)</td>
<td>(1)(2)</td>
<td></td>
<td></td>
<td>2</td>
</tr>
<tr>
<td>3</td>
<td>(123)</td>
<td>(1)(23)</td>
<td>(1)(2)(3)</td>
<td></td>
<td>6</td>
</tr>
<tr>
<td></td>
<td>(132)</td>
<td>(2)(13)</td>
<td>(3)(12)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1243)</td>
<td>(2)(134)</td>
<td>(2)(143)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1324)</td>
<td>(3)(124)</td>
<td>(3)(142)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1342)</td>
<td>(4)(123)</td>
<td>(4)(132)</td>
<td></td>
<td></td>
</tr>
<tr>
<td></td>
<td>(1423)</td>
<td>(12)(34)</td>
<td>(13)(24)</td>
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<tr>
<td></td>
<td>(1432)</td>
<td>(14)(23)</td>
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</tr>
</tbody>
</table>

Fig. 23. \( s(n, k) \), the Stirling numbers of the first kind; and seq. 659, the factorial numbers.

\[
\begin{align*}
D_2 &= 1 & 1 & 2 \\
    &   & 2 & 1 \\
D_3 &= 2 & 1 & 2 & 3 & 1 & 2 & 3 \\
    &   & 2 & 3 & 1 & 3 & 1 & 2 \\
D_4 &= 9 & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
    &   & 2 & 1 & 4 & 3 & 2 & 3 & 4 & 1 & 3 \\
    &   & 1 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
    &   & 3 & 1 & 4 & 2 & 3 & 4 & 1 & 2 & 3 & 4 \\
    &   & 4 & 1 & 2 & 3 & 4 & 3 & 1 & 2 & 4 & 3 & 2 & 1
\end{align*}
\]

Fig. 24. Seq. 766, derangements.

Figure 24 shows \( D_n \), the number of derangements of \( n \) objects, or the permutations in which every object is moved from its original position (seq. 766). These are also called subfactorial or rencontres numbers, and have the recurrence \( D_n = nD_{n-1} + (-1)^n \). (See Riordan [R1 57].)
III SOME IMPORTANT SEQUENCES

Figure 25 illustrates seq. 587, the Euler numbers \( E_n \), or the number of permutations of \( n \) objects which first rise and then alternately fall and rise. (Only the second rows of the permutations are shown.)

The even numbered Euler numbers form seq. 1667: 1, 5, 61, 1385, 50521, \ldots, and have gf

\[
1 + \frac{x^2}{2!} + \frac{5x^4}{4!} + \frac{61x^6}{6!} + \cdots = \sec x.
\]

(Often these are called the Euler numbers instead of seq. 587.)

The odd numbered Euler numbers form seq. 829: 1, 2, 16, 272, 7936, 353792, \ldots, and are called the tangent numbers \( T_n = E_{2n-1} \). They have gf

\[
x + \frac{2x^3}{3!} + \frac{16x^5}{5!} + \cdots = \tan x.
\]

\[
E_1 = 1 \\
E_2 = 1 \hspace{1cm} 12 \\
E_3 = 2 \hspace{1cm} 132 \hspace{1cm} 231 \\
E_4 = 5 \hspace{1cm} 1324 \hspace{1cm} 1423 \hspace{1cm} 2314 \hspace{1cm} 2413 \hspace{1cm} 3412 \\
E_5 = 16 \hspace{1cm} 13254 \hspace{1cm} 14253 \hspace{1cm} 14352 \hspace{1cm} 15243 \hspace{1cm} 15342 \hspace{1cm} 23154 \hspace{1cm} 24153 \hspace{1cm} 24351 \hspace{1cm} 25143 \hspace{1cm} 25341 \hspace{1cm} 34152 \hspace{1cm} 34251 \hspace{1cm} 35142 \hspace{1cm} 35241 \hspace{1cm} 45132 \hspace{1cm} 45231
\]

Fig. 25. Seq. 587, the Euler numbers.

The Bernoulli numbers \( B_n \) are defined by

\[
B_n = \frac{2nE_{2n-1}}{2^{2n}(2^{2n} - 1)},
\]

and form the sequence

\[
\frac{1}{6}, \frac{1}{30}, \frac{1}{42}, \frac{1}{30}, \frac{5}{66}, \frac{691}{2730}, \frac{7}{6}, \frac{3617}{510}, \ldots
\]

with gf

\[
1 - \frac{x}{2} + \frac{1}{6} \frac{x^2}{2!} - \frac{1}{30} \frac{x^4}{4!} + \frac{1}{42} \frac{x^6}{6!} - \cdots = \frac{x}{e^x - 1}.
\]

The numerators and denominators form seqs. 1677 and 1746.
Finally the Genocchi numbers are defined by \( G_n = 2^{2-2n} n E_{2n-1} \), and form seq. 1233: 1, 1, 3, 17, 155, 2073, 38227, \ldots, with gf

\[
\frac{x}{2!} + 1 \frac{x^3}{4!} + 3 \frac{x^5}{6!} + 17 \frac{x^7}{8!} + \cdots = \tan \frac{1}{2} x.
\]

The Euler, tangent, Bernoulli, and Genocchi numbers arise in all branches of mathematics. For applications and properties see Jordan [JO2], David and Barton [DB1], Comtet [CO1] and Gould [AMM 79 44 72]; for tables see Fletcher et al. [FMR 1 65] and Knuth and Buckholtz [MTAC 21 663 67].

### 3.14 SEQUENCES FROM NUMBER THEORY

The table contains many number-theoretic sequences, of which the following are typical:

1. The prime numbers, lucky numbers, and other sequences generated by sieves (seqs. 241, 377, 1035, 1048);
2. the Euler totient function \( \phi(n) \): the number of integers not exceeding and relatively prime to \( n \) (seq. 111);
3. from the Goldbach conjecture: the number of ways of writing \( 2n \) as a sum of two primes (various sequences—see index);
4. quadratic partitions of primes: a prime of the form \( 4n + 1 \) has a unique representation as \( a^2 + b^2 \) with \( a \geq b \). Sequences 169 and 33 give \( a \) and \( b \);
5. the number of integers less than or equal to \( 2^n \) expressible in the form \( u^2 + v^2 \), where \( u \) and \( v \) are integers (seq. 265);
6. Mersenne primes: the numbers \( n \) such that \( 2^n - 1 \) is prime (seq. 248);
7. from Euler’s proof that there are an infinity of primes: let \( p_1 = 2 \), \( p_2, \ldots, p_n \) be primes, and define \( p_{n+1} \) to be the smallest (largest) prime factor of \( p_1 p_2 \cdots p_n + 1 \) (seqs. 329, 330);
8. Beatty sequences: if \( \alpha, \beta \) are positive irrational numbers such that \( (1/\alpha) + (1/\beta) = 1 \), then the Beatty sequences

\[
[\alpha], [2\alpha], [3\alpha], \ldots \quad \text{and} \quad [\beta], [2\beta], [3\beta], \ldots
\]

together contain all the positive integers without repetition, where \([x]\) denotes the greatest integer less than or equal to \( x \). (See Honsberger [HO2].) For example, \( \alpha = \frac{1}{2}(1 + \sqrt{5}) = 1.61803\ldots \) gives seqs. 917: 1, 3, 4, 6, 8, 9, \ldots and 509: 2, 5, 7, 10, 13, 15, \ldots.

The following test for Beatty sequences is due to R. L. Graham. If \( a_1, a_2, \ldots \) is a Beatty sequence, then the values of \( a_1, \ldots, a_{n-1} \) determine
III SOME IMPORTANT SEQUENCES

\(a_n\) to within 1. Look at the sums \(a_1 + a_{n-1}, a_2 + a_{n-2}, \ldots, a_{n-1} + a_1\). If all these sums have the same value, \(V\) say, then \(a_n\) must equal \(V\) or \(V + 1\); but if they take on the two values \(V\) and \(V + 1\), and no others, then \(a_n\) must equal \(V + 1\). If anything else happens, it is not a Beatty sequence. For example, in seq. 917, \(a_1 + a_1 = 2\) so \(a_2\) must be 2 or 3 (it is 3); \(a_1 + a_2 = 4\) so \(a_3\) must be 4 or 5 (it is 4); \(a_1 + a_3 = 5\) and \(a_2 + a_2 = 6\), so \(a_4\) must be 6 (it is); and so on.

For further information about number-theoretic sequences see the comprehensive works of Dickson [D12] and Lehmer [LE1].

3.15 PUZZLE SEQUENCES

This section describes some sequences with simple yet unexpected generating principles. They have all been given as puzzles at one time or another. Of course all of the sequences given in Chapters II and III make good puzzles.

1. Sequences related to well-known constants (e.g., seq. 1291: 1, 4, 1, 4, 2, 1, 3, 5, 6, 2, 3, \ldots, the decimal expansion of \(\sqrt{2}\)) or to other common sequences (seq. 2127: 1, 15, 29, 12, 26, 12, 26, 9, \ldots is related to the calendar—guess!). See also seqs. 684, 880, 1679, 1812, etc.

2. Sequences depending on the binary expansions of numbers (e.g., seq. 41: 1, 2, 1, 2, 2, 3, 1, 2, 2, \ldots gives the number of 1's in the binary expansion of \(n + 1\); see also seqs. 360, 388).

3. Sequences depending on the English words or Arabic numerals used to describe them (e.g., seq. 2218: 1, 21, 21000, 101, 121, 1101, \ldots, the smallest number requiring \(n\) words in English; see also seqs. 1818, 1897).

4. The terms not in some well-known sequence (e.g., seq. 1319: 4, 6, 7, 9, 10, 11, 12, 14, 15, \ldots, the non-Fibonacci numbers).

5. Sequences obtained by bisecting (i.e., taking every other term of) well-known sequences (e.g., seq. 1067: 1, 3, 7, 18, 47, 123, 322, \ldots, a bisection of seq. 924, the Lucas numbers; see also seqs. 569, 1101).

6. Sequences obtained by alternating the terms of two sequences (seq. 889: 3, 2, 1, 7, 4, 1, 1, 8, 5, 2, 9, \ldots, mixing \(\pi\) and \(e\), is the only example given).

The following pleasing puzzles are not in the table because they are finite or are not integers.

7. \(\frac{1}{4}, \frac{1}{3}, 1, 3, 6, 12, 24, 30, 120, 240, 1200, 2400,\) English money in 1950.

8. 3, 8, 8, 4, 89, 75, 30, 28, ?, planetary diameters in thousands of statute miles.
(9) \(8, 5, 4, 9, 1, 7, 6, 3, 2, 0; \) or \(8, 8000000000, \ldots, 18, 18000000000, \ldots, 18000000, \ldots, 18000, \ldots, 80, \ldots, 88, \ldots, 85, \ldots, 84, \ldots, 11, \ldots, 15, \ldots, 5, \ldots, 4, \ldots, \) the numbers arranged in alphabetical order (in English).

(10) \(12, 13, 14, 15, 20, 22, 30, 110, 1100, \) the number 12 written to the bases 10, 9, 8, \ldots, 2.


(12) \(1714, 1727, 1760, 1820, 1910, 1936, \) dates of the accessions of the Georges to the English throne.

(13) \(1732, 1735, 1743, 1751, 1758, 1767, 1767, 1782, 1773, 1790, 1795, 1784, 1800, 1804, 1791, 1809, 1808, 1822, 1822, 1831, 1830, 1837, 1833, 1837, 1843, 1858, 1857, 1856, 1865, 1872, 1874, 1882, 1884, 1890, 1917, 1908, 1913, \) dates of birth of presidents of the U.S.A.

(14) The integers 1, 2, 3, \ldots \) drawn next to a mirror. (See Fig. 26.)


\[
\begin{align*}
M & \heartsuit 8 M \\
\end{align*}
\]

Fig. 26. A puzzle.

### 3.16 SEQUENCES FROM LATTICE STUDIES IN PHYSICS

In the last twenty years physicists have studied a number of basic combinatorial problems related to crystal lattices. Typical problems are to find the number of self-avoiding paths of length \(n\) on a given lattice, or the number of ways a particular graph can be drawn on the lattice. A number of such sequences will be found in the main table. For further information see Montroll [BE6 96], Sykes et al. [JMP 7 1557 66], Kasteleyn [HA1 43], Percus [PE3], and Domb [ACP 15 229 69].