# Pictures from 50 Years of the OEIS

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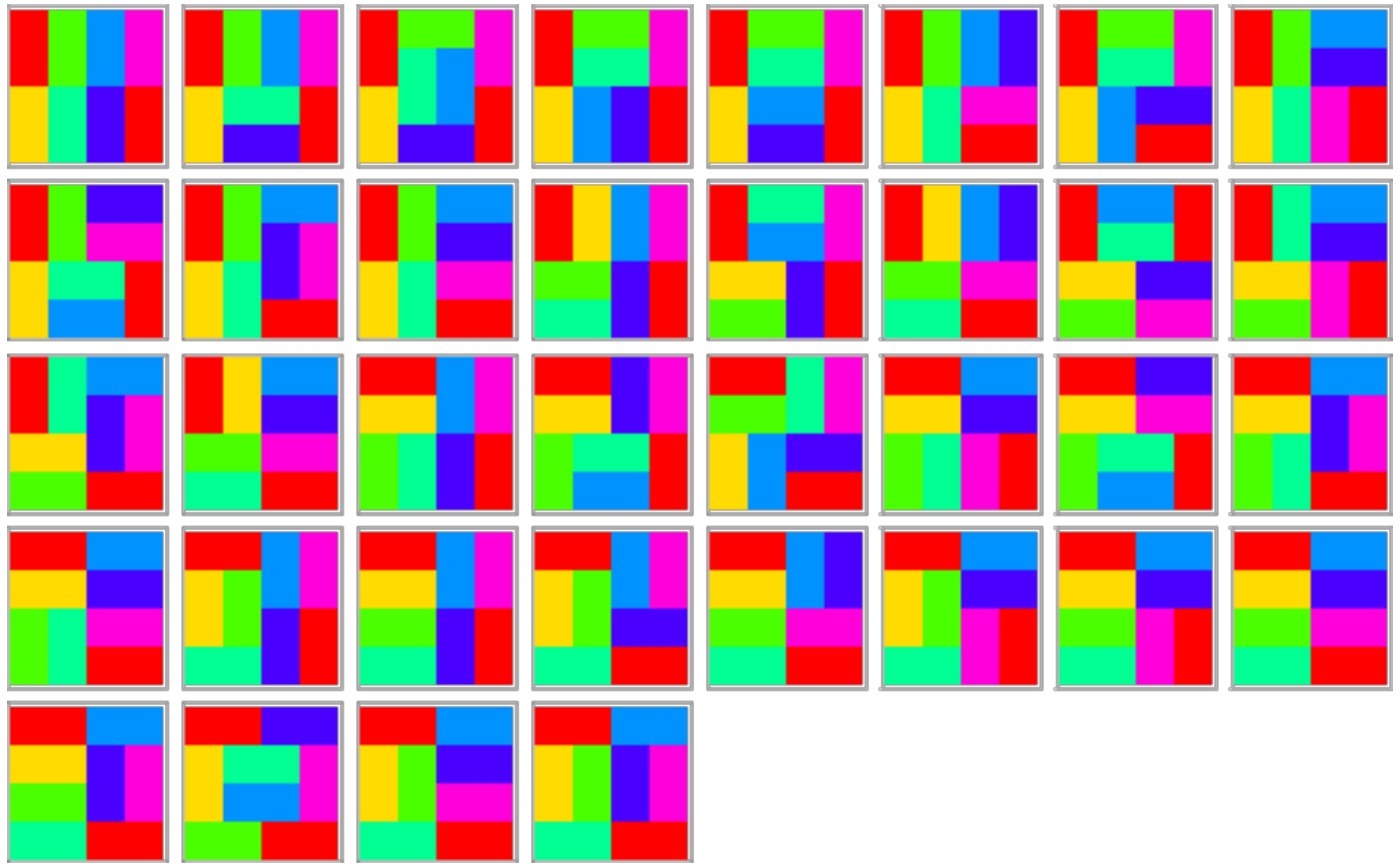
Art and Math Seminar, Department of Mathematics,
Kansas State University, Nov. 2 2023

# Tiling a Square with Dominoes

36 ways to tile a 4X4 square



a(2)=36

1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000, ...

(A4003)

$$a(n) = \prod_{j=1}^{n}\prod_{k=1}^{n}\left(4\cos^2\frac{j\pi}{2n+1} + 4\cos^2\frac{k\pi}{2n+1}\right)$$

(Kastelyn, 1961)

# In 2015, almost the same sequence arose:

Laura Florescu, Daniela Morar, David Perkinson, Nicholas Salter, Tianyuan Xu, Sandpiles and Dominoes, 2015

1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000/5, ... !!

(A256043)

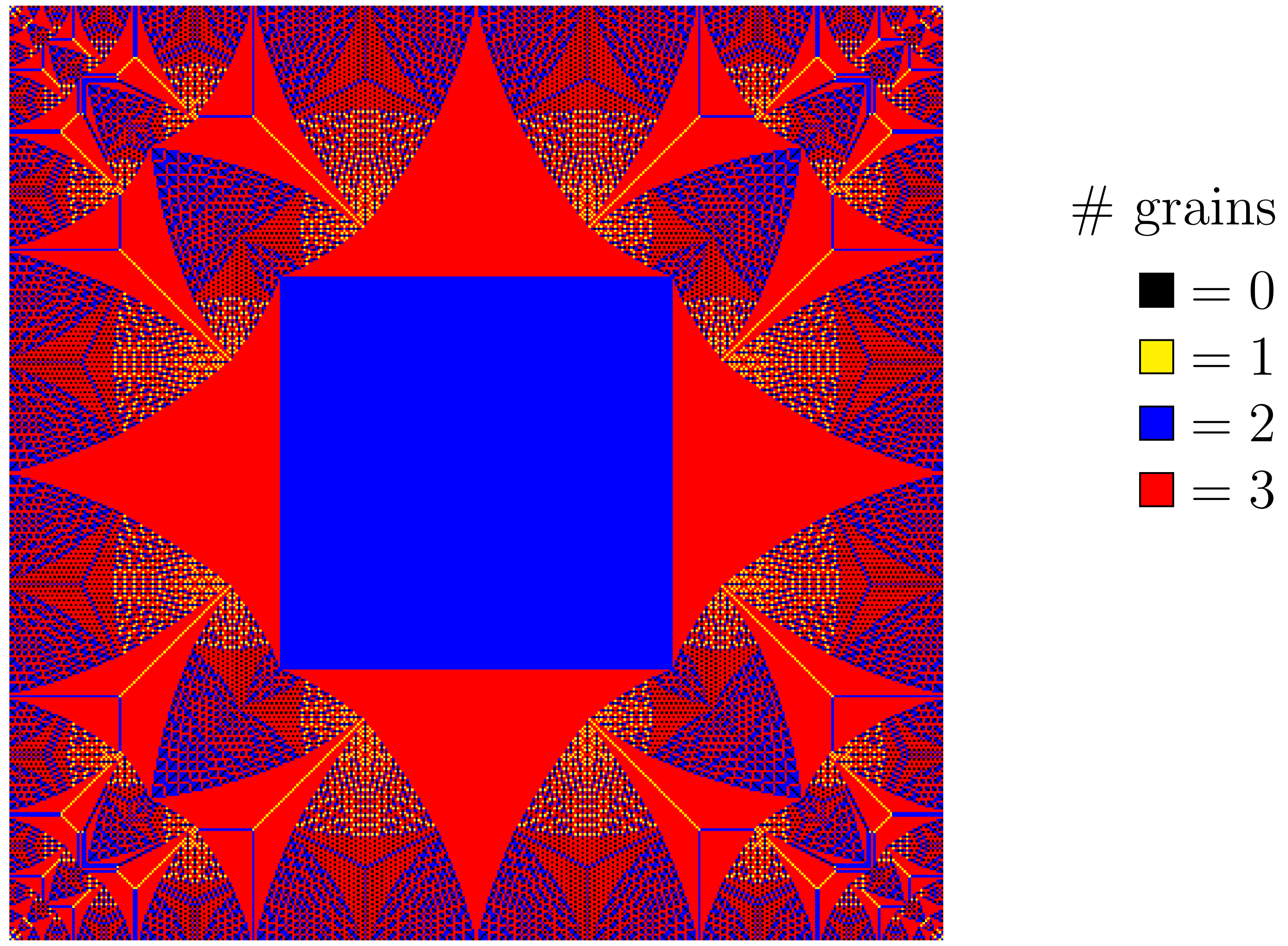



Figure 1: Identity element for the sandpile group of the 400 × 400 sandpile grid graph.

# Outline of talk

- Triangular numbers, Recamán’s sequence, graphs
- Cellular automata (O. Pol, D. Applegate)
- Coordination sequence (C. Goodman-Strauss)
- Ways to draw n circles (J. Wild)
- Cutting polygons into squares, monotiles (G.Theobald)
- Stained glass windows (L. Blomberg, S. Shannon)
- The scariest sequence in the world (S. Shannon)

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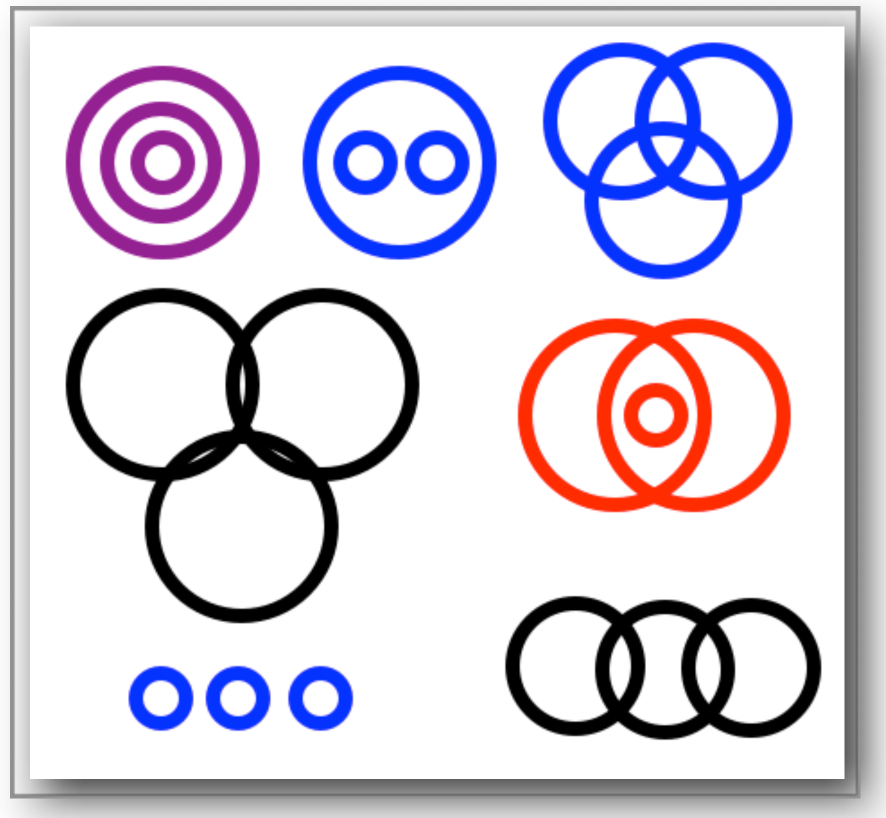
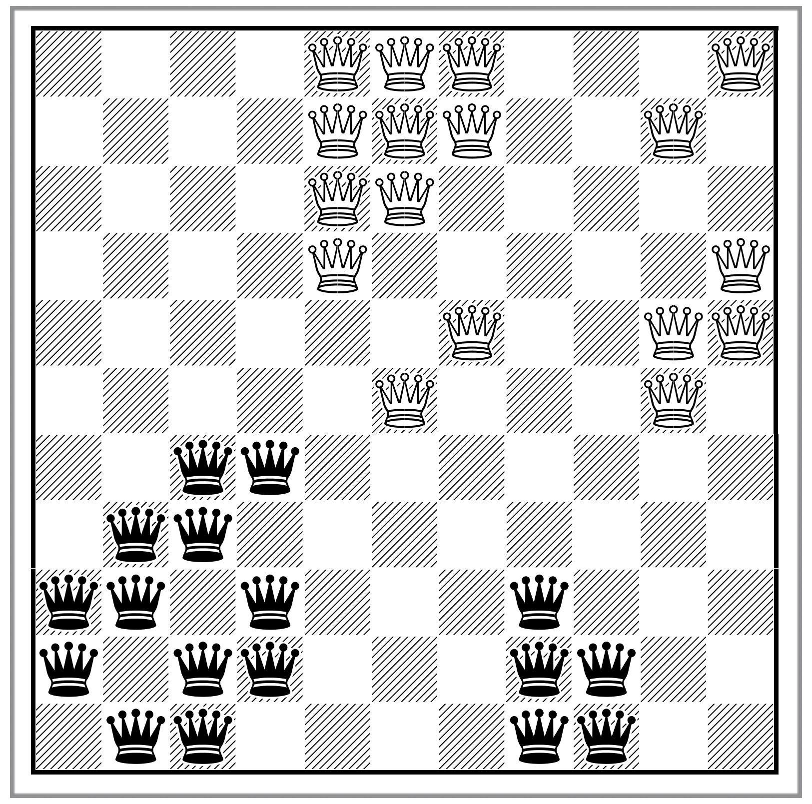
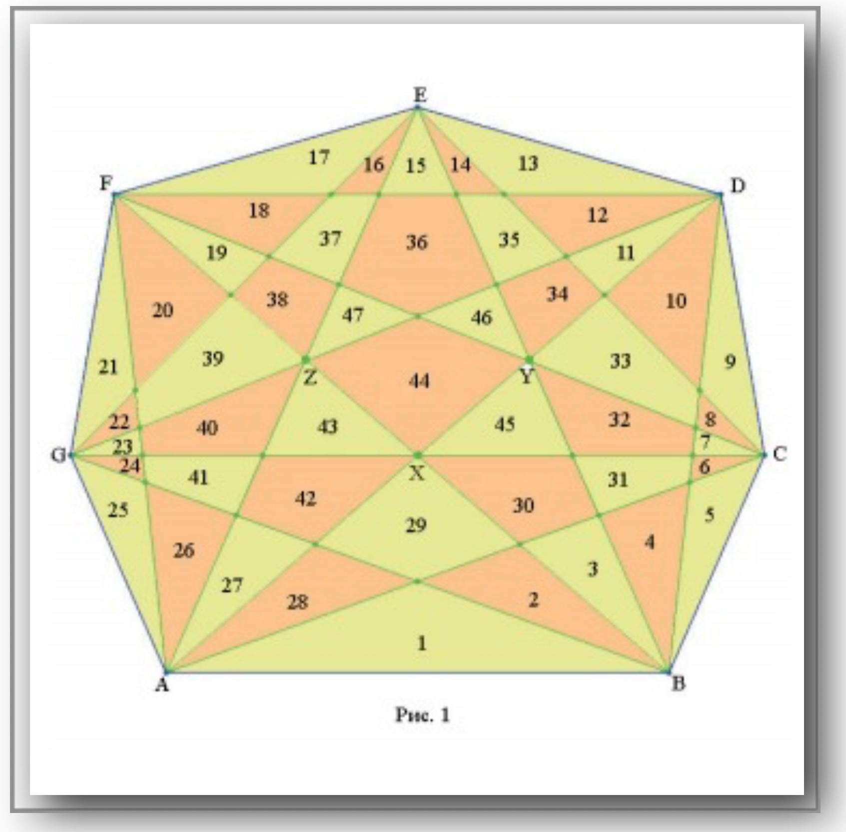
The poster, on the OEIS Foundation web site, http://oeisf.org







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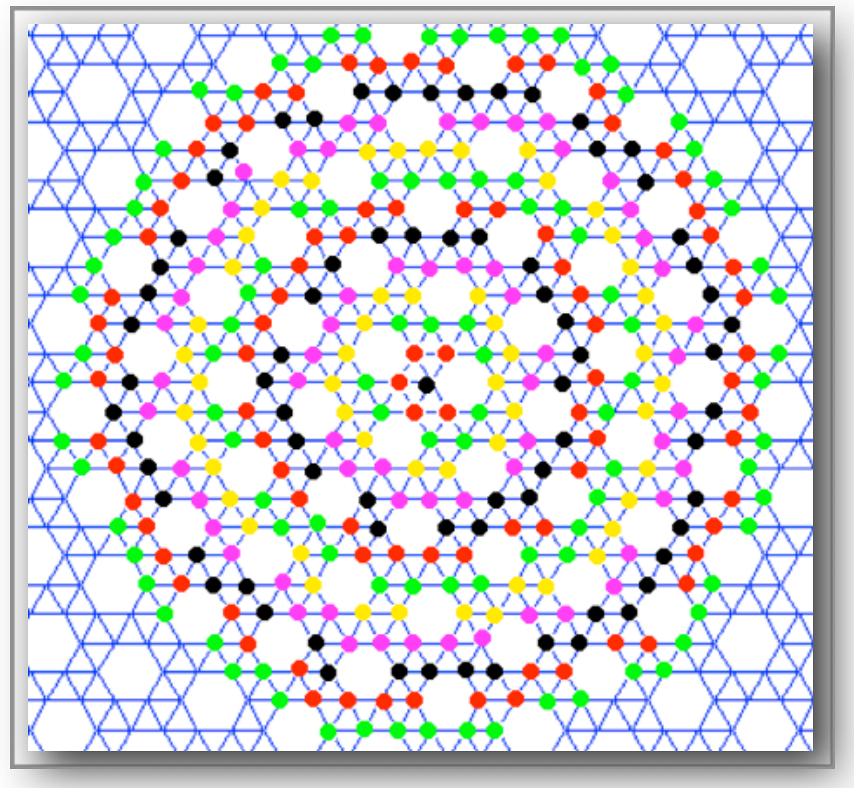
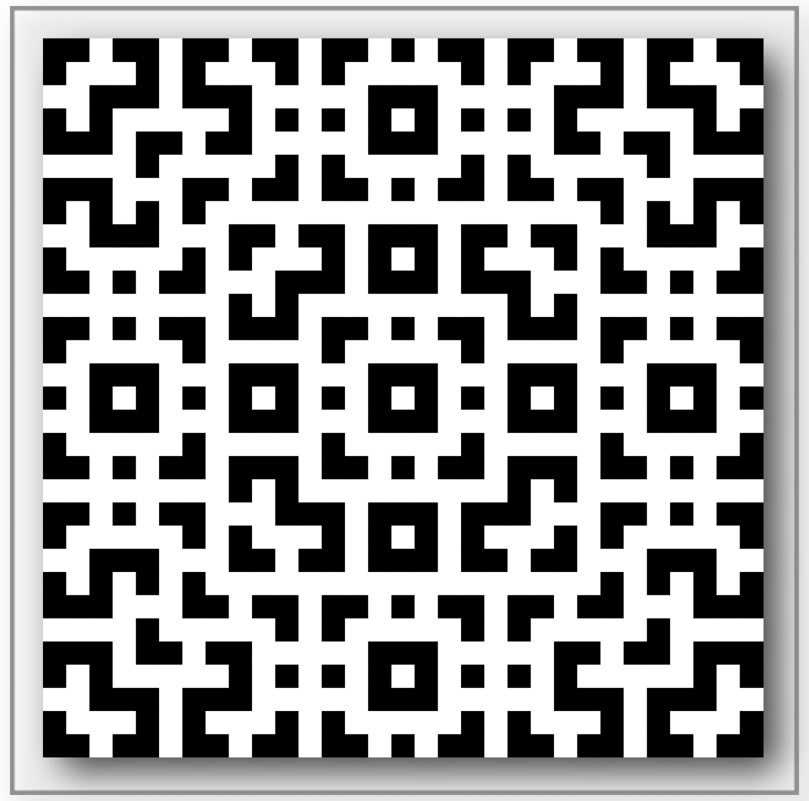
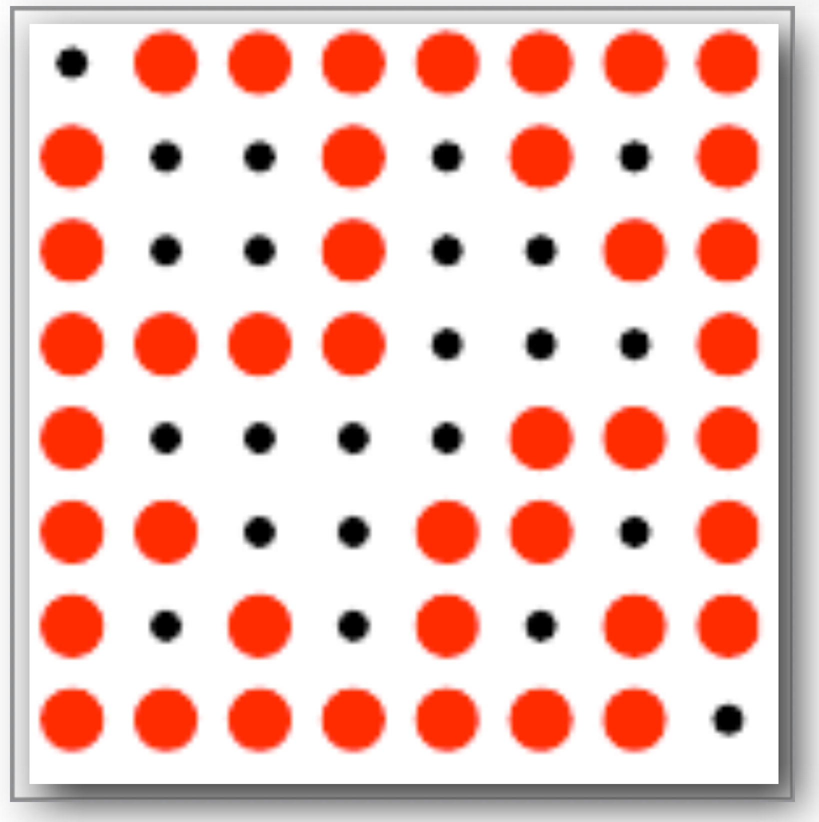
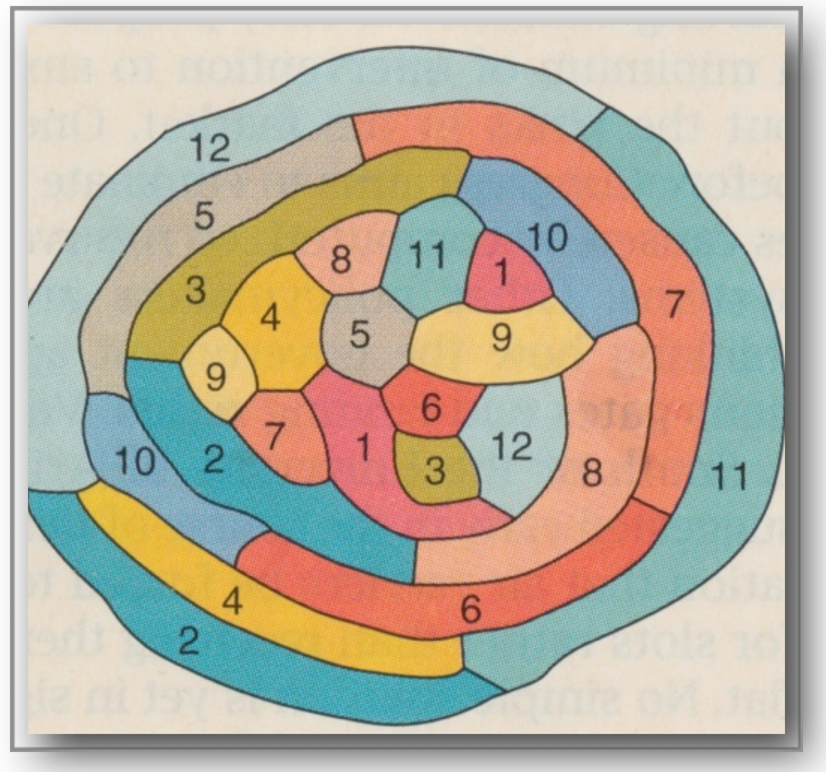
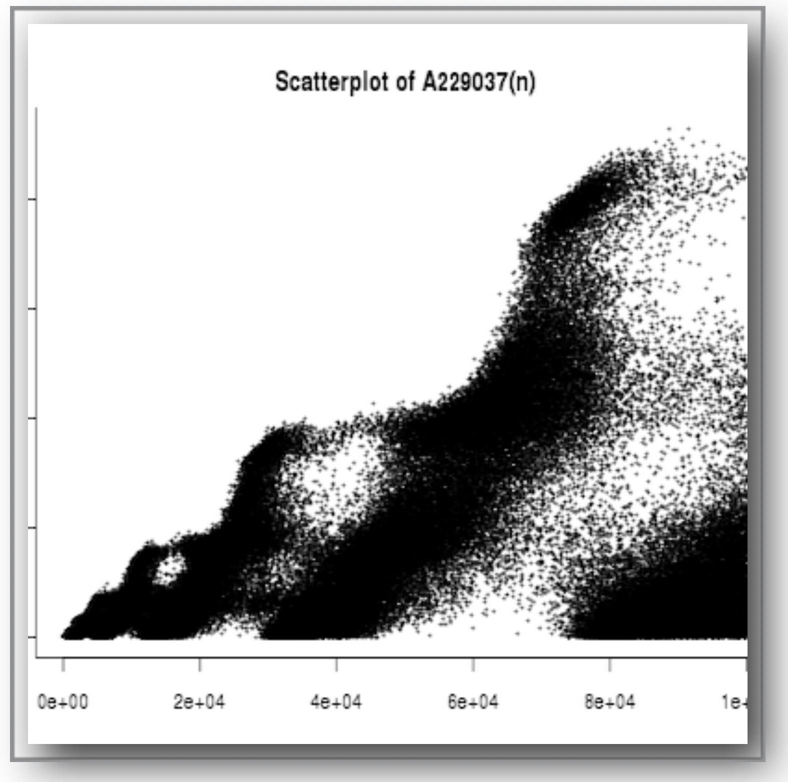










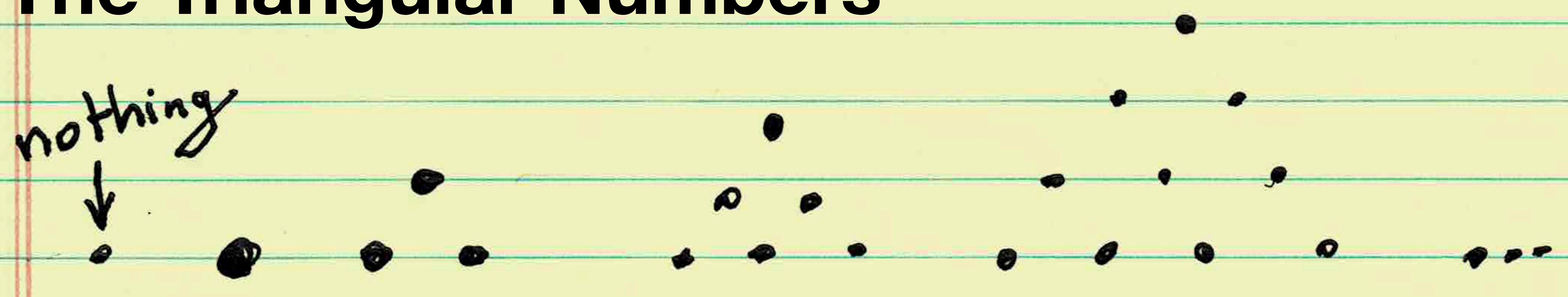
# Graphs

Triangular numbers,

Recaman’s sequence and others,

A Facial Recognition Project

# The Triangular Numbers



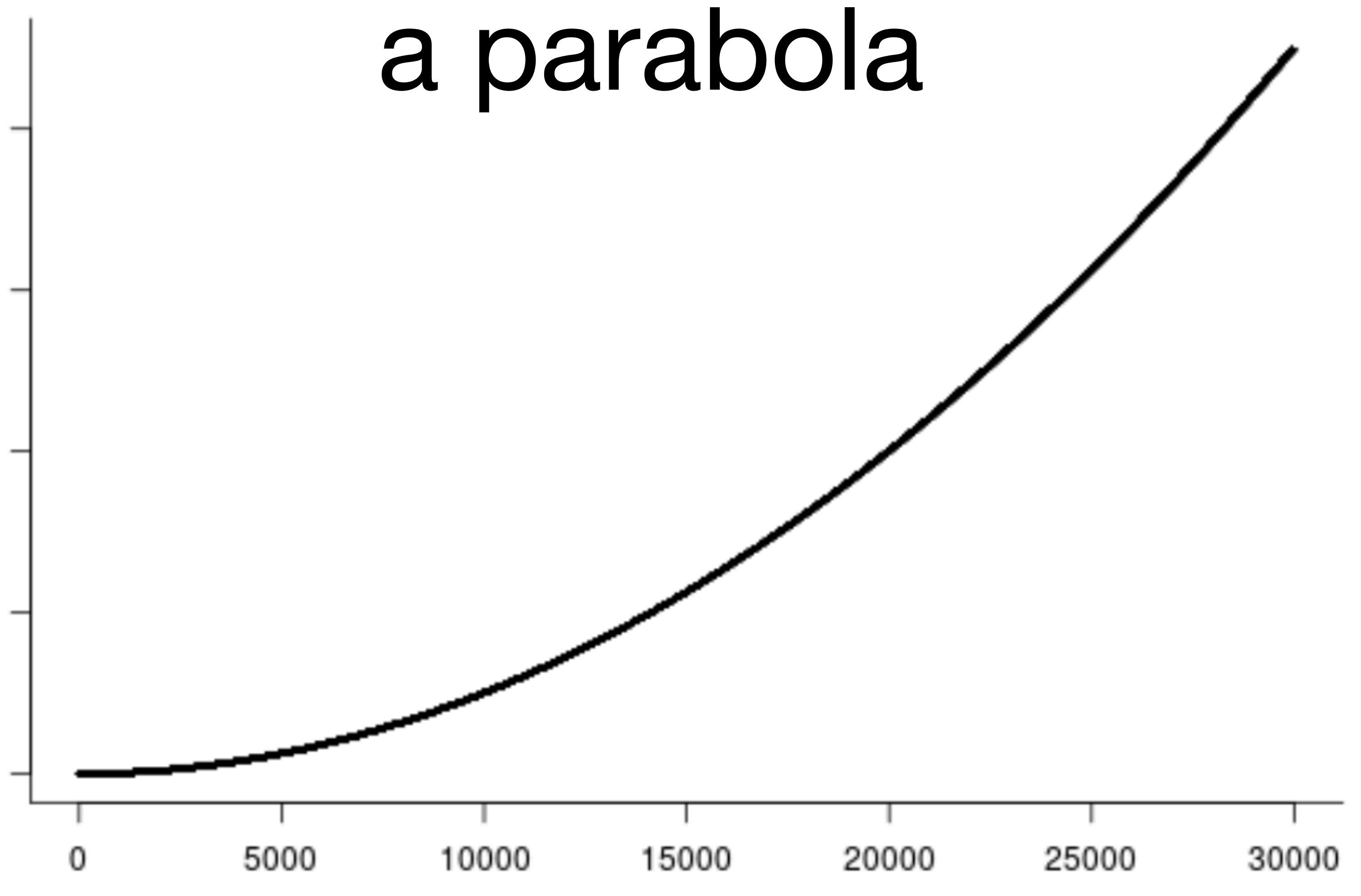

0 1 2 3 4

0 1 1+2=3 1+2+3=6 1+2+3+4 = 10

The triangular numbers

0, 1, 3, 6, 10, 15, 21, 28, ...

$$a_n = \frac{n(n+1)}{2}$$




A217

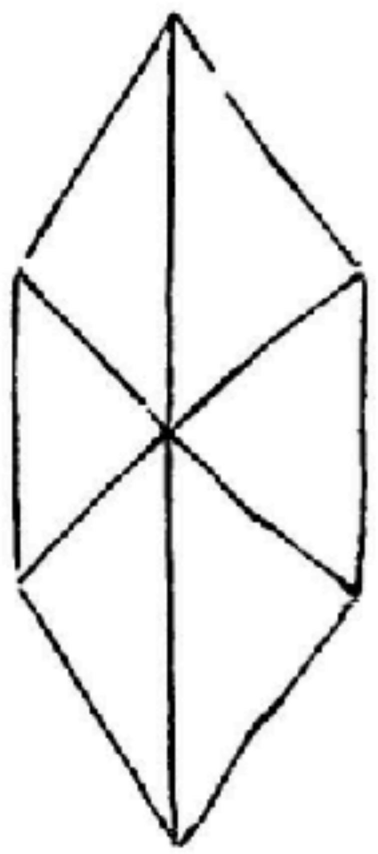
# Triangular Numbers

## Anicius Manlius Severinus Boethius,

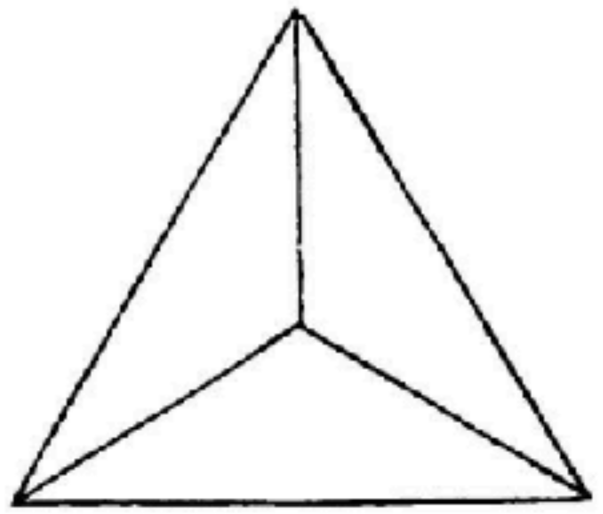
**Boethius (480-524 AD),**
**[De institutione arithmetica]()**



exagonus in sex triangulos divisus.⌉ At vero triangula figura, cum eam quis ita diviserit, in alias figuras non resolvitur, nisi in se ipsam. In tria enim triangula dissipatur.



Triangulus in tres triangulos divisus:

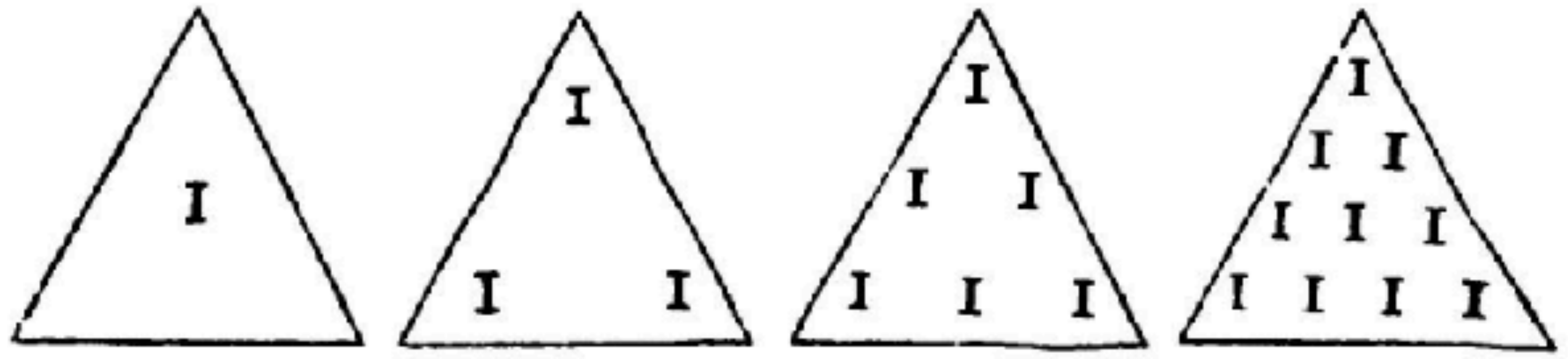


Adeo haec figura princeps est latitudinis, ut ceterae omnes superficies in hanc resolvantur, ipsa vero, quoniam nullis est principiis obnoxia neque ab alia latitudine sumpsit initium, in sese ipsam solvatur. Idem autem et in numeris fieri, sequens operis ordo monstrabit.

*Dispositio triangulorum numerorum.*

VII. Est igitur primus triangulus numerus, qui in solis tribus unitatibus dissipatur secundum superficiei positionem, triangula scilicet descriptione, et post hunc quicunque aequalitatem laterum in trina laterum spatia segregant.

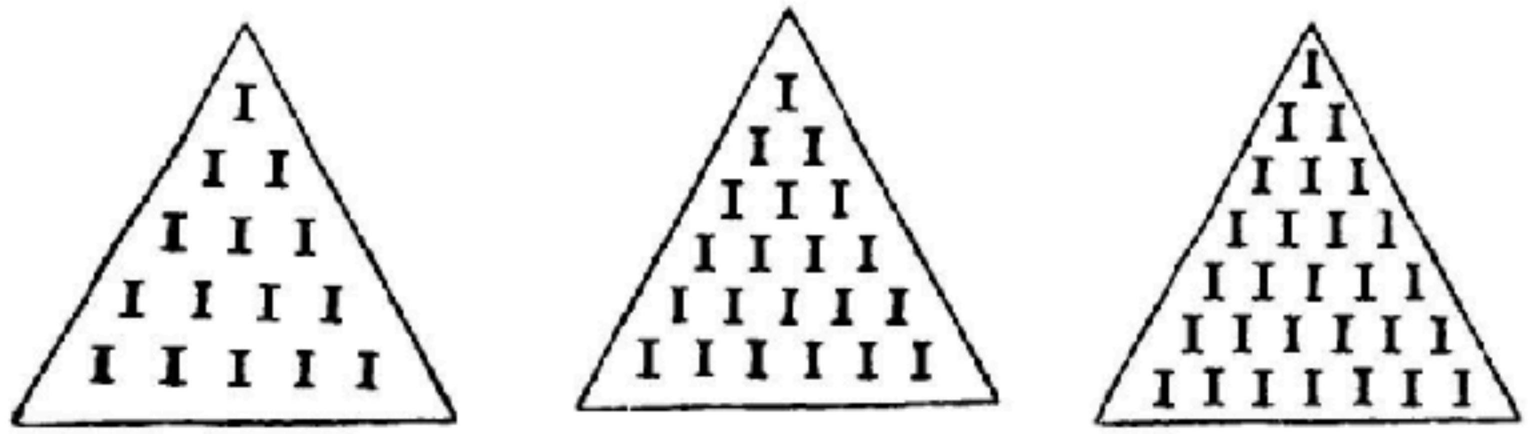
I. III. VI. X.



1 exagonus] *Figuram exagoni regularis habent* d, s. 3 in se *om.* c. 5 Triangulus etc.] *Hanc inscriptionem om.* c, d, f, l. 6 Ideo f. 8 est *om.* c. || obnoxia principiis c. 9 ipsa a, b, c, d, f, l, s; *vide supra versum quartum et septimum.* 10 operis *om.* c. || ordo operis s. 11 *Titulum om.* c, d. 12 numerus *om.* d. 15 *Post* segregant *addunt:* iuxta subiectas discriptionis formulas r, s. 16 *Numeros om.* c, d, f.



XV. XXI. XXVIII.



*De lateribus triangulorum numerorum.*

VIII. Ad hunc modum infinita progressio est, omnesque ex ordine trianguli aequilateri procreabuntur, primum omnium ponenti quod ex unitate nascitur ut haec vi sua triangulus sit, non tamen etiam opere atque actu. Nam si cunctorum mater est numerorum, quicquid in his, quae ab ea nascuntur, numeris invenitur, necesse est ut ipsa naturali quadam potestate contineat.⌉ Et huius trianguli latus est unitas. Ternarius vero, qui primus est opere et actu ipso triangulus, crescente unitate binarium numerum latus habebit. Vi enim et potestate primi trianguli, id est unitatis, unitas latus est, actu vero et opere trianguli primi, id est ternarii, dualitas, quam Graeci dyada vocant. Secundi vero trianguli, qui opere atque actu secundus est, id est senarii, crescente naturali numero in lateribus ternarius invenitur; tertii vero, id est denarii, quaternarius latus continet;

1 *Descriptionem numeri ·XXVIII· om.* d. 2 *Titulum om.* d. 3 VIII. *om.* f. || in infinita a, b, c, d; in infinitum l. || omnisque f. 4 ex *om.* a, d, f; *supra versum* r. 5 ponenti] ponent f, s; *in* s *tamen* i *rasura est deletum. Supra versum:* In al. ponent id b; tibi d. || ·VI· sua a. 6 sua *om.* c. 8 quae] *Coniicias* qui *esse legendum.* 11 ipse l; ipso, o *in* a *mutato* s; in c. || triangulos c, s. 13 unitas *supra versum* a.

# Recamán's Sequence

**Bernardo Recamán Santos, 1991**

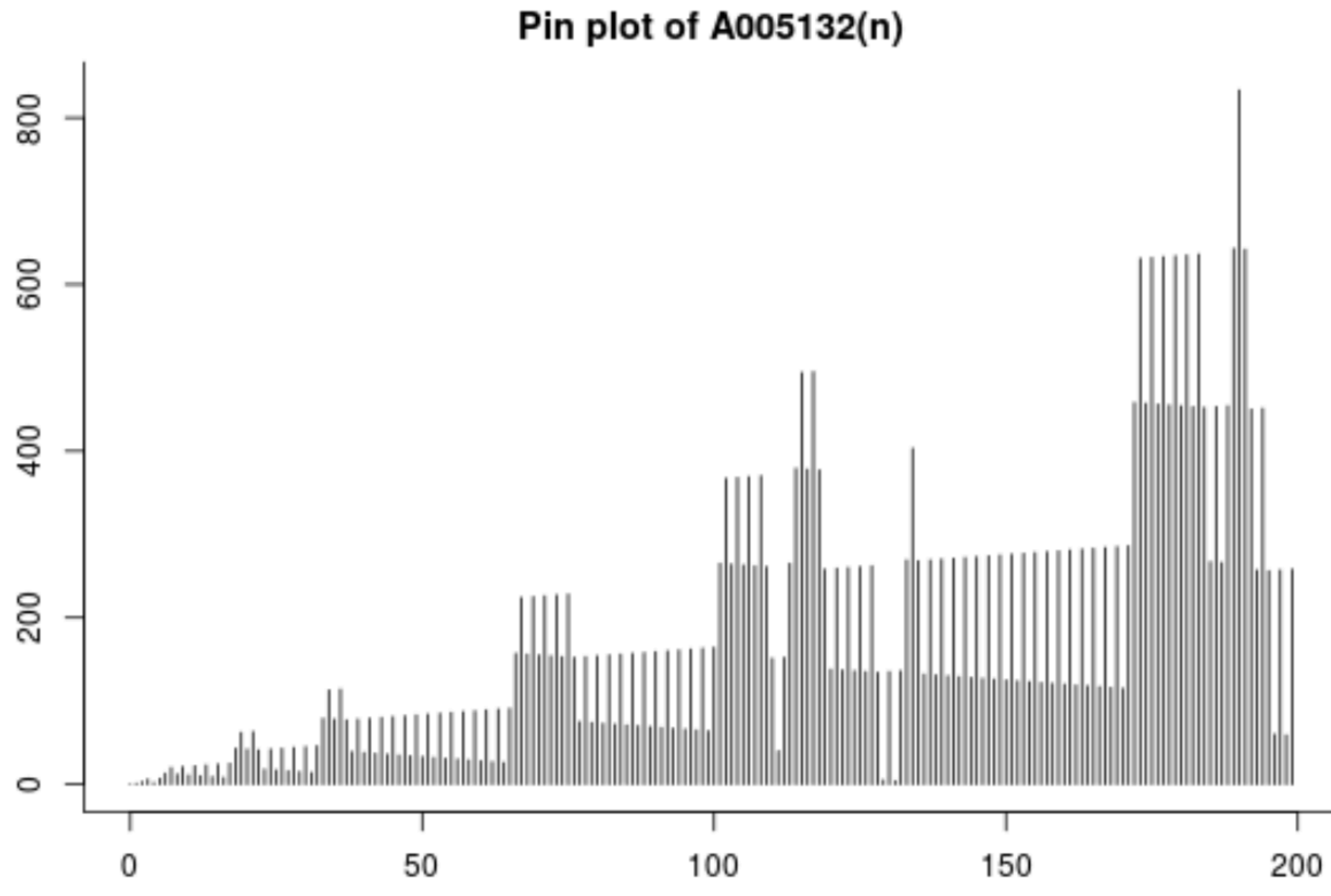
Subtract or add: 1, 2, 3, 4, 5, 6, …

No negative terms, no repeats

0, 1, 3, 6, 2, 7, 13, 20, 12, 21, 11, …

1 2 3 -4 5 6 7 -8 9 -10 …

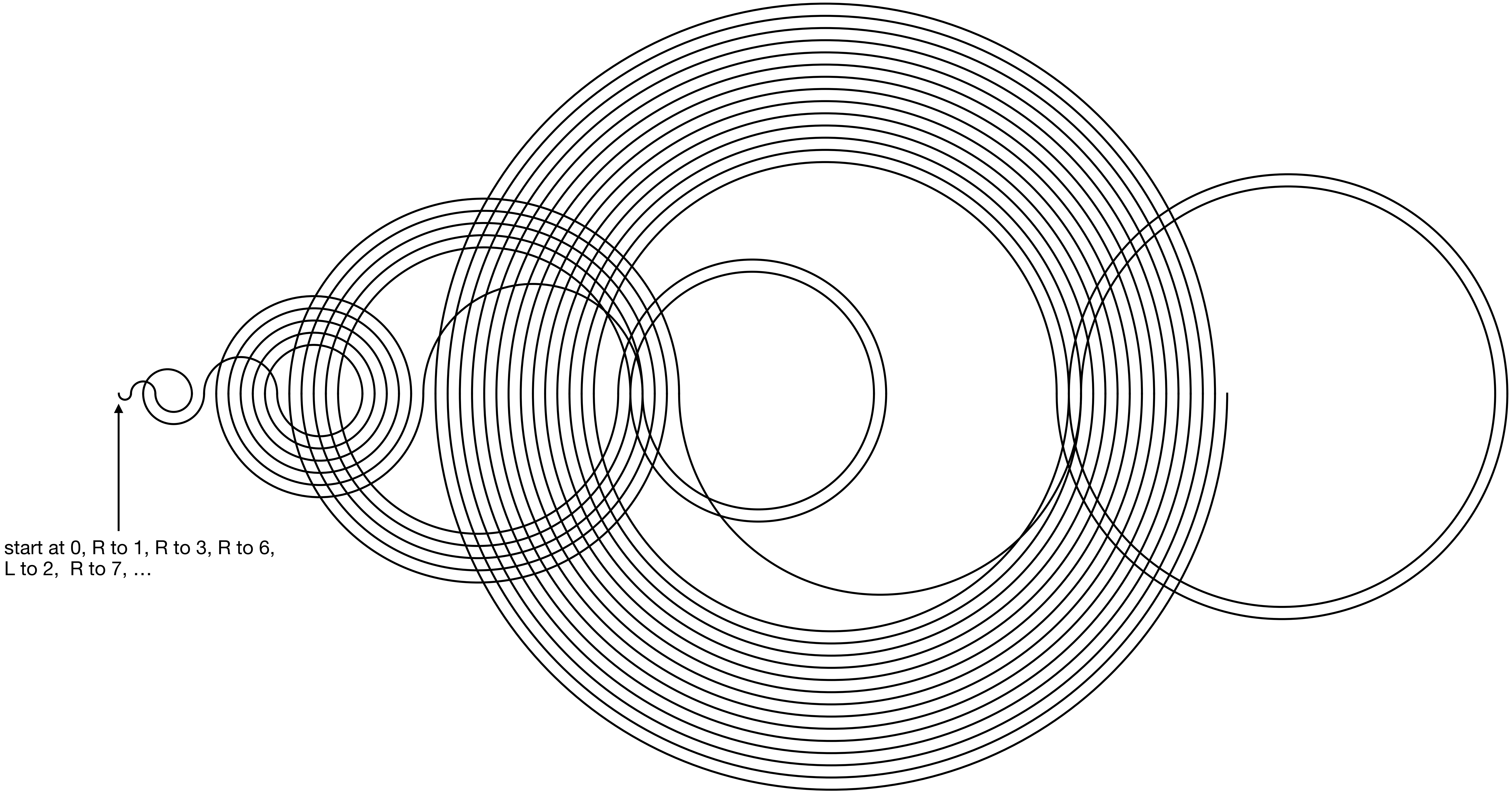
A5132



# Recamán's Sequence (2)

Edmund Harriss,

First 62 terms drawn as a spiral



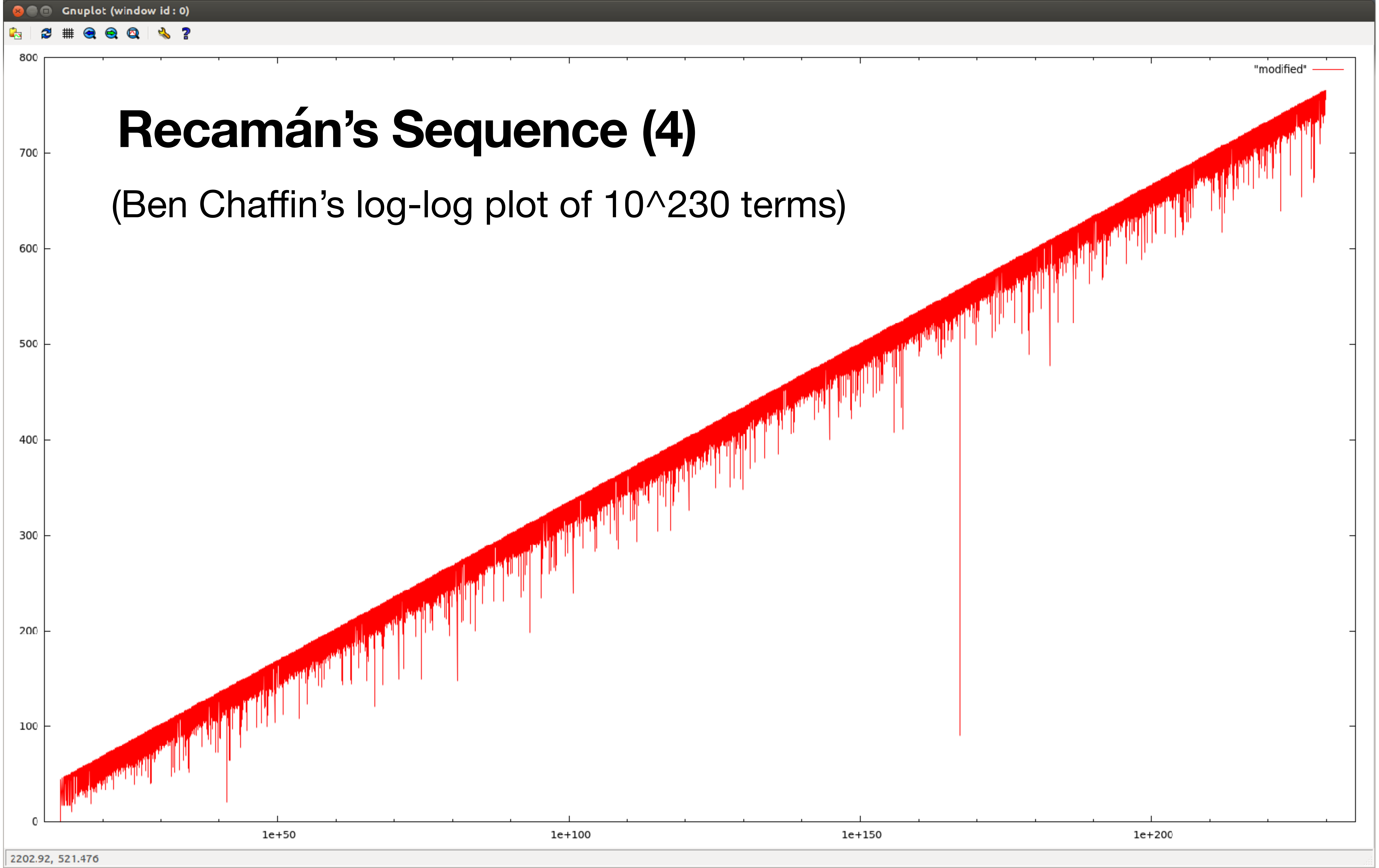
# The Recamán Sequence (3)

**oeis.org/A005132**

Animation: Michael De Vlieger. There is a longer version on YouTube - see https://oeis.org/A005132 for link

# Recamán’s Sequence (4)

(Ben Chaffin’s log-log plot of 10^230 terms)



# Recamán's Sequence (5)

A5132

**The Big Question: Does every number appear?**

After 10^15 terms, 852655 = 5 x 31 x 5501 was missing (Allan Wilks, 2001)

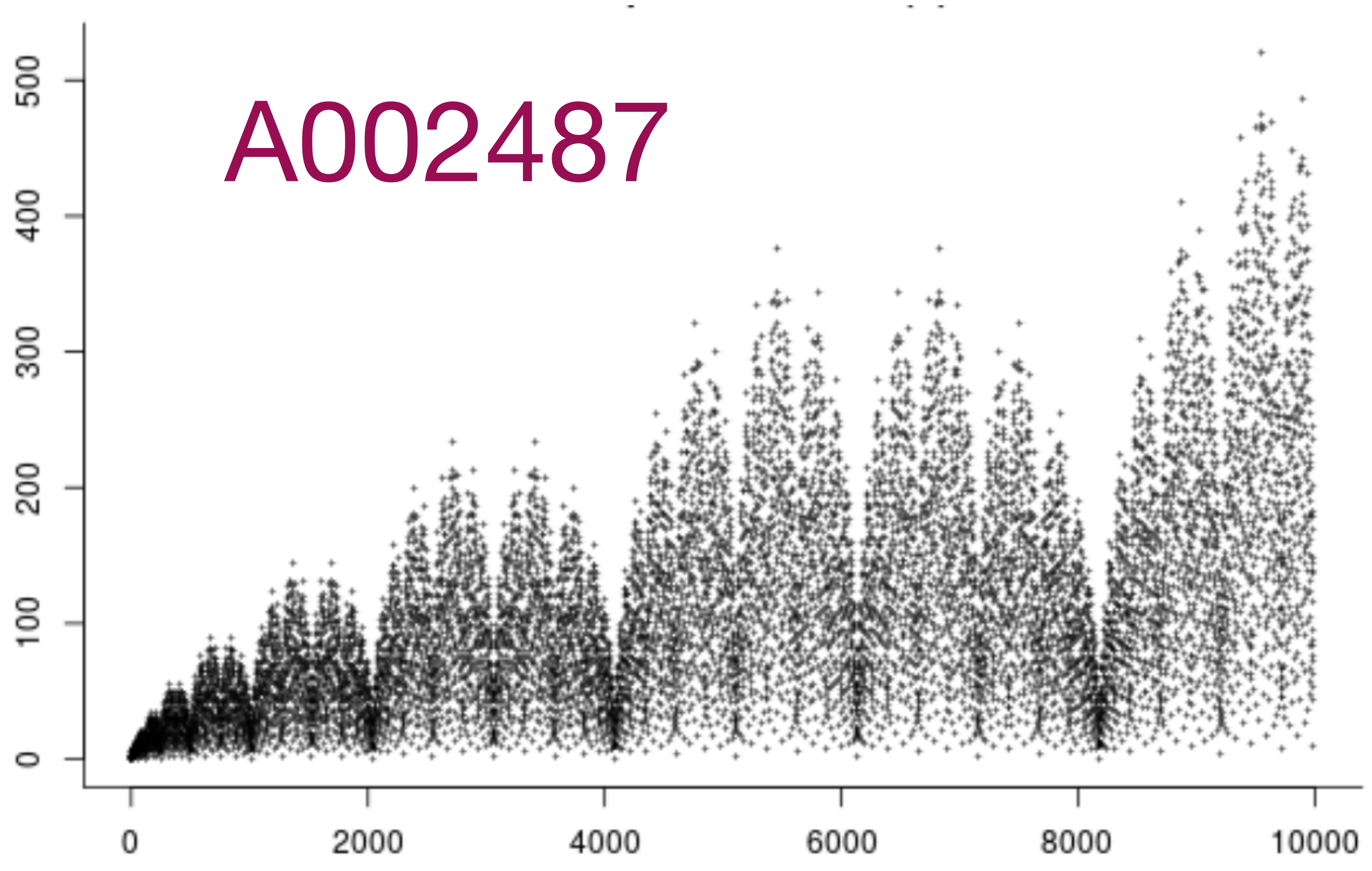
After 10^230 terms, 852655 is still missing (Ben Chaffin, 2018)

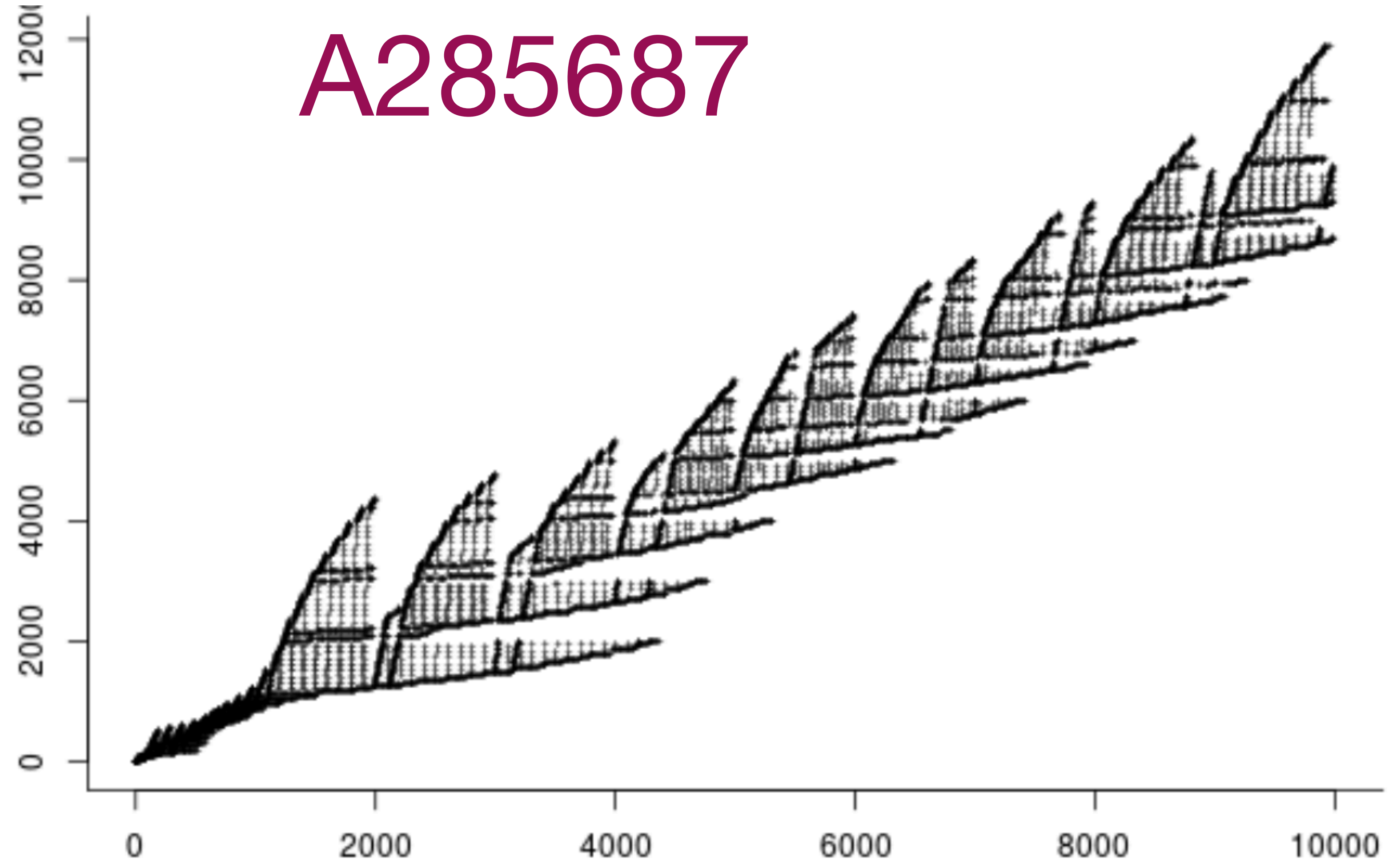
**30 years ago I believed that every number would eventually appear. Today I think that there are infinitely many missing terms, and 852655 just got lucky and is the first of many.**

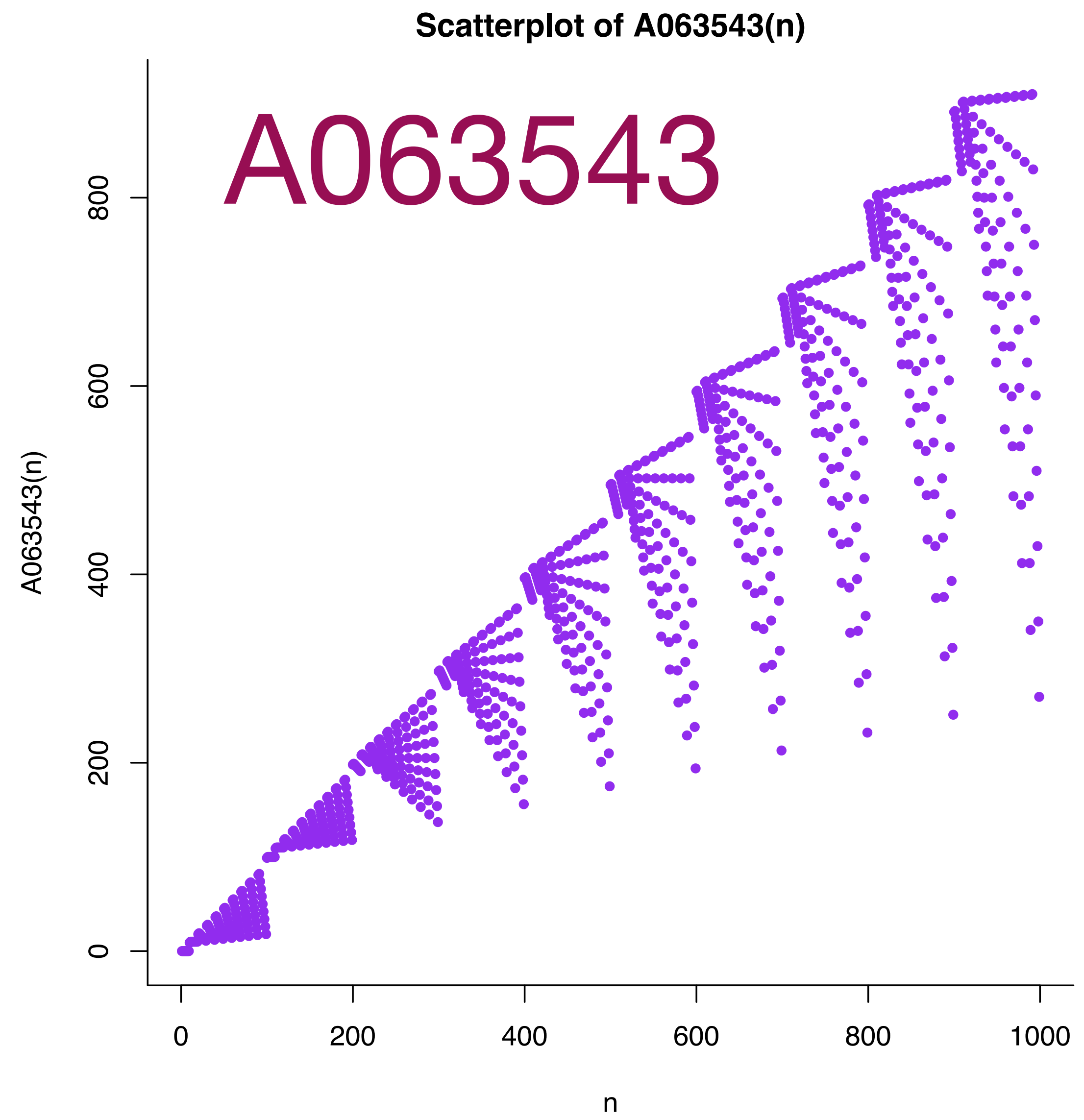
# Graphs (continued)

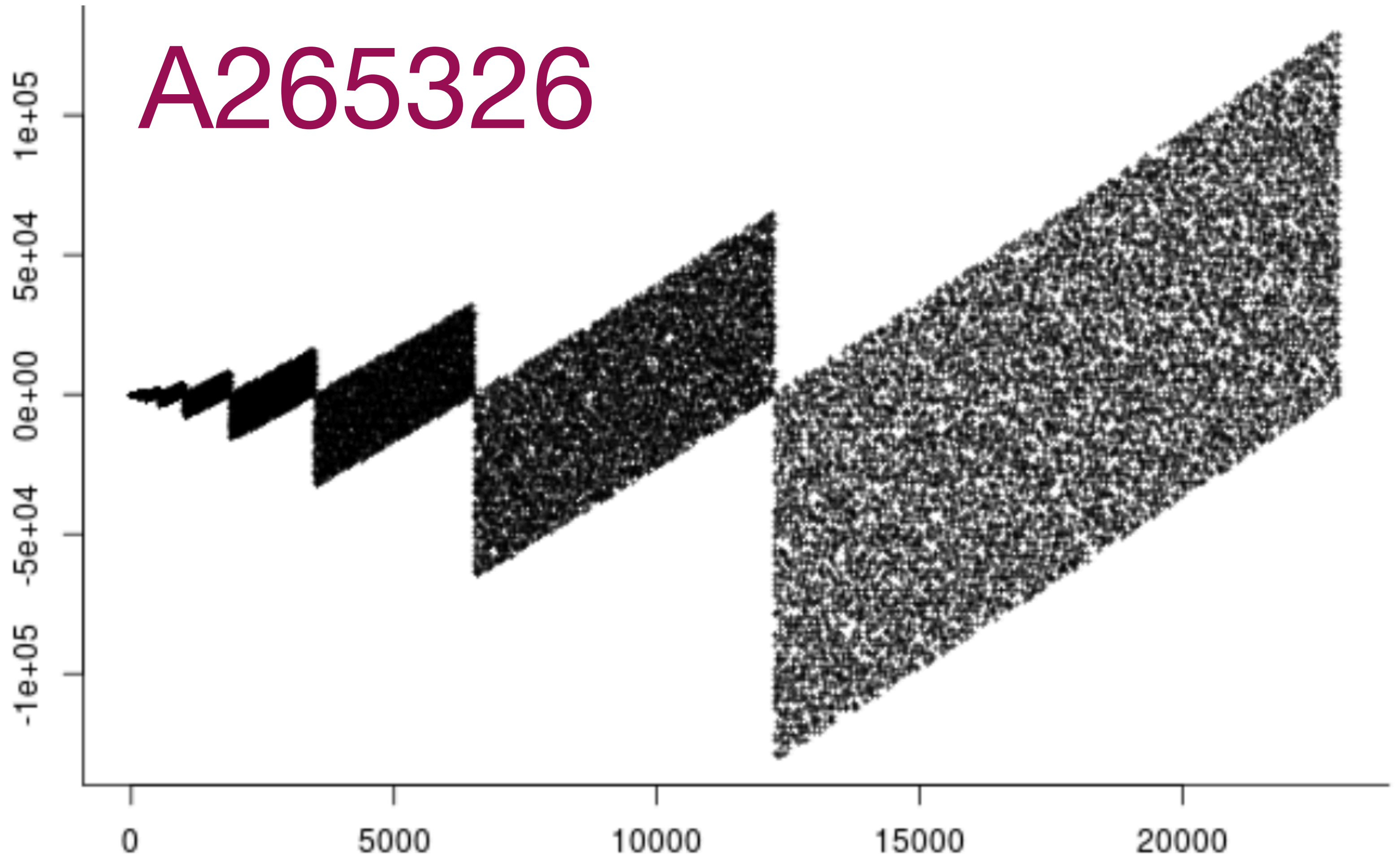
## Exotic Graphs

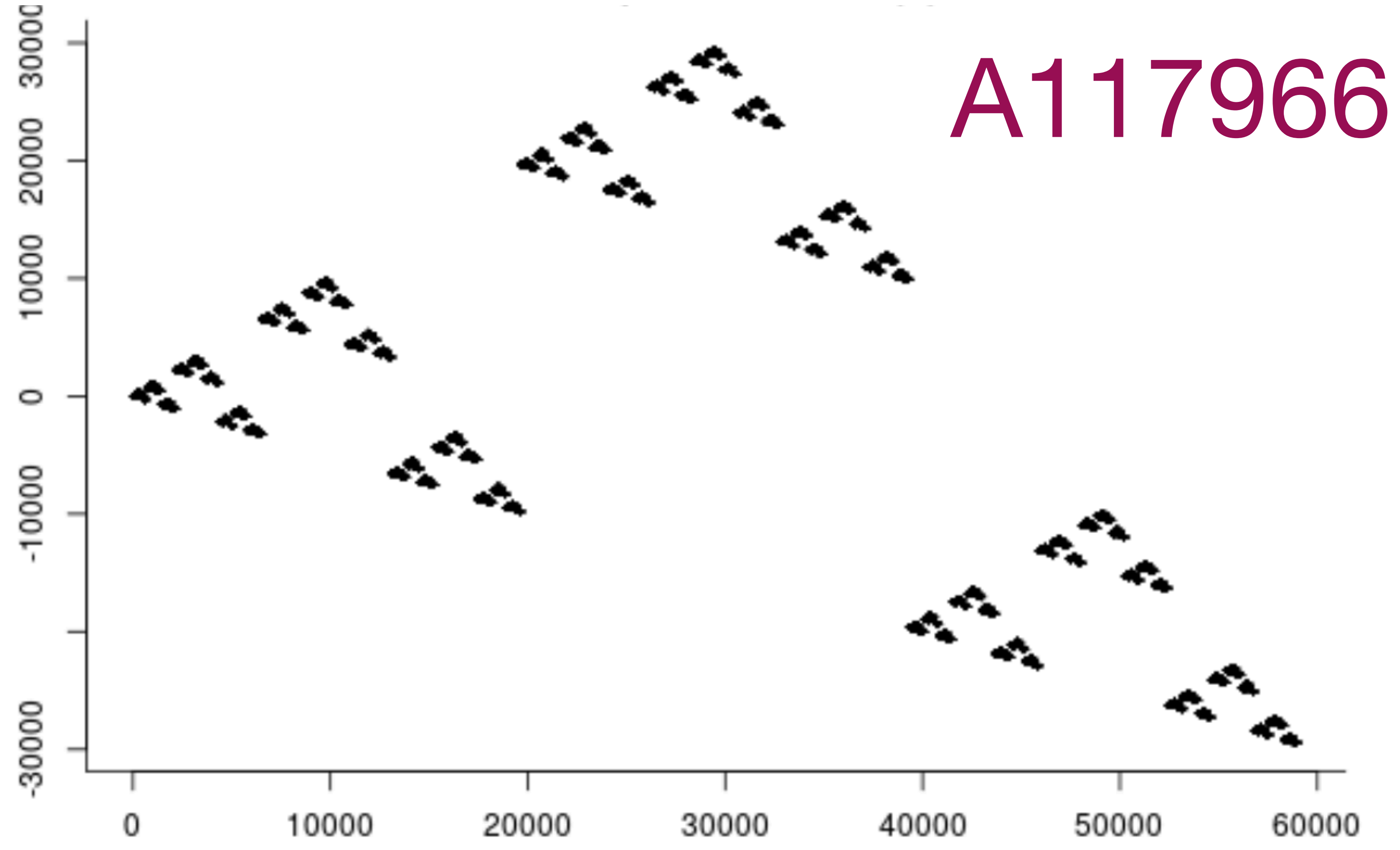
Keyword “look” means an interesting graph:

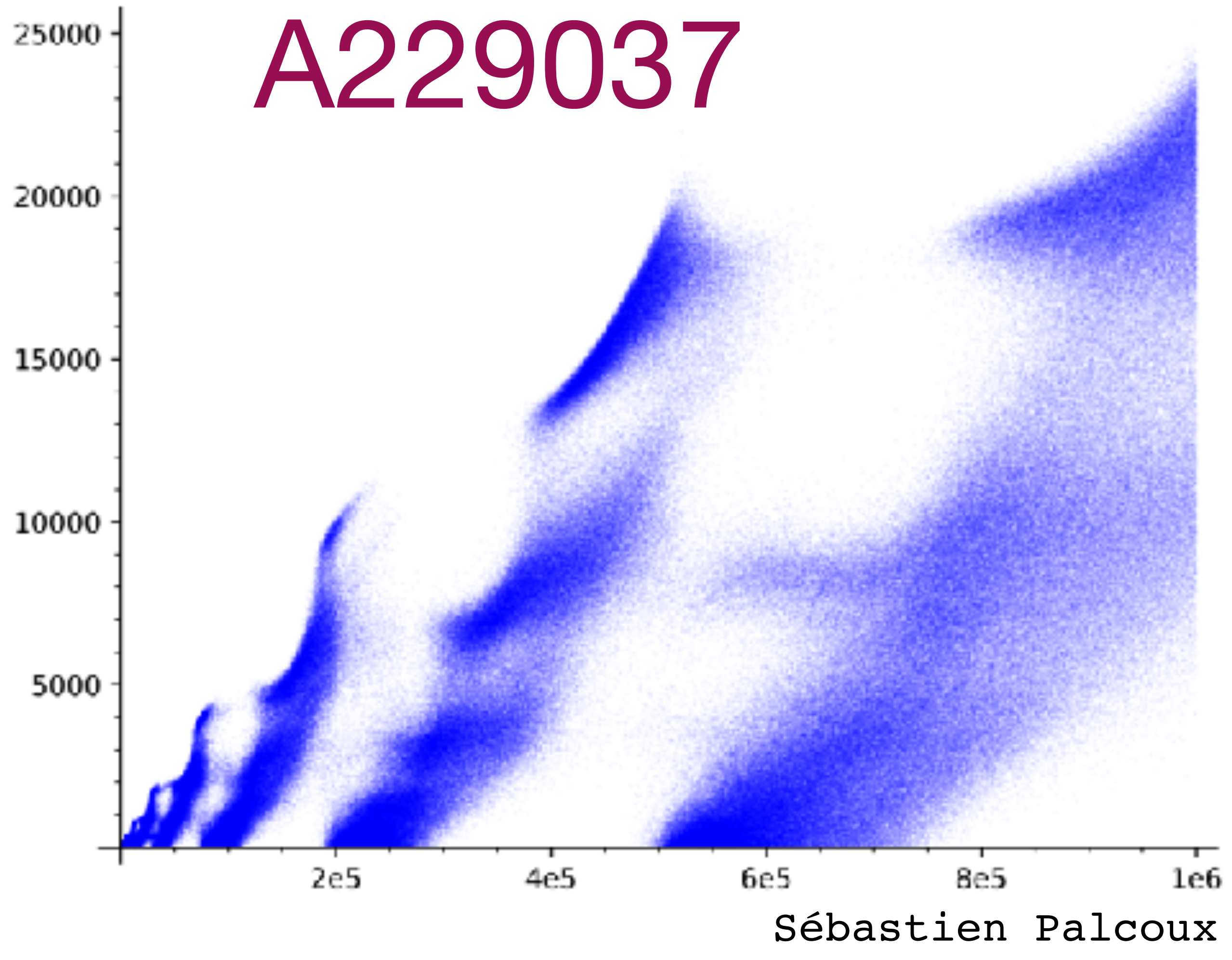


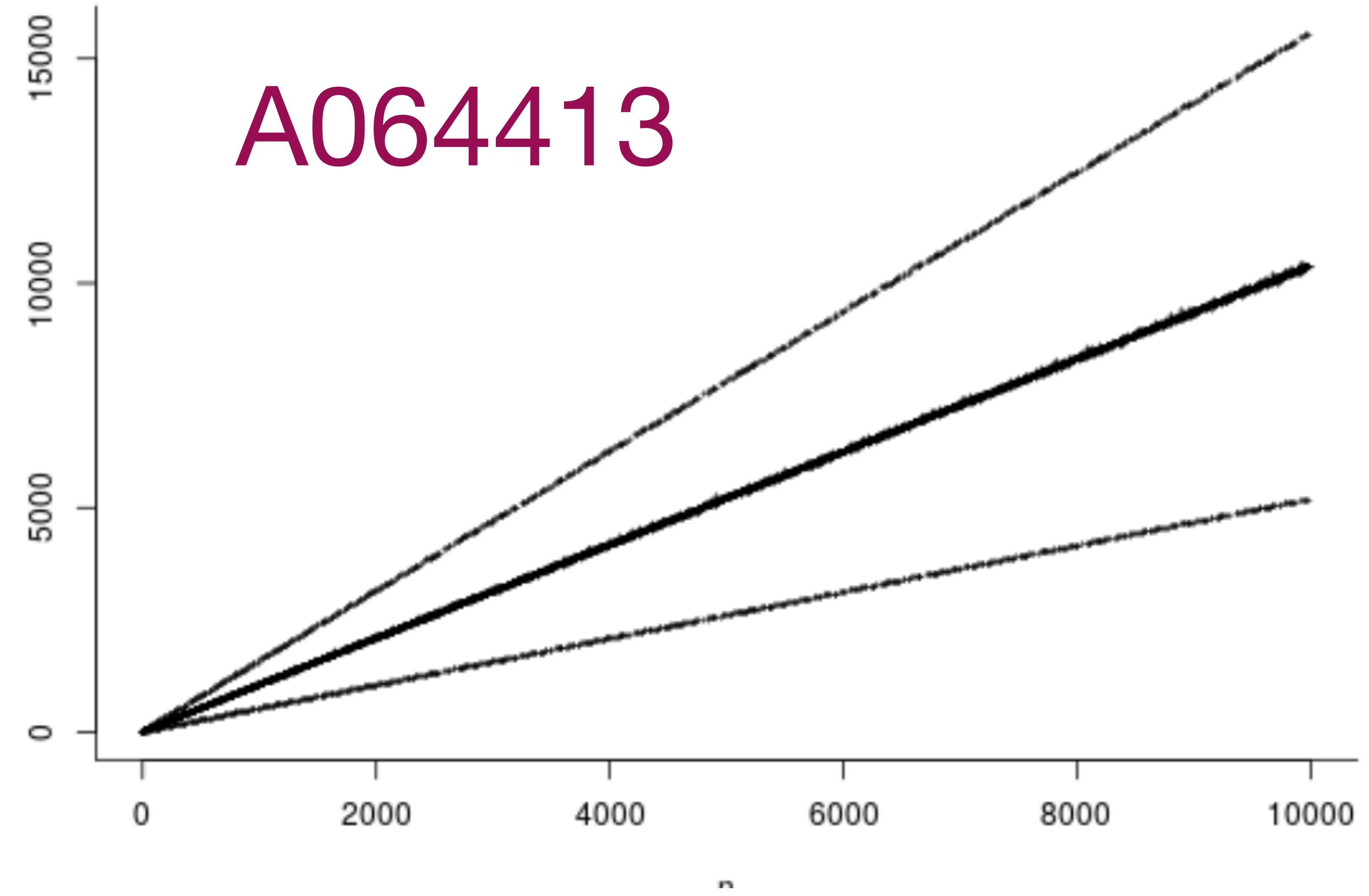



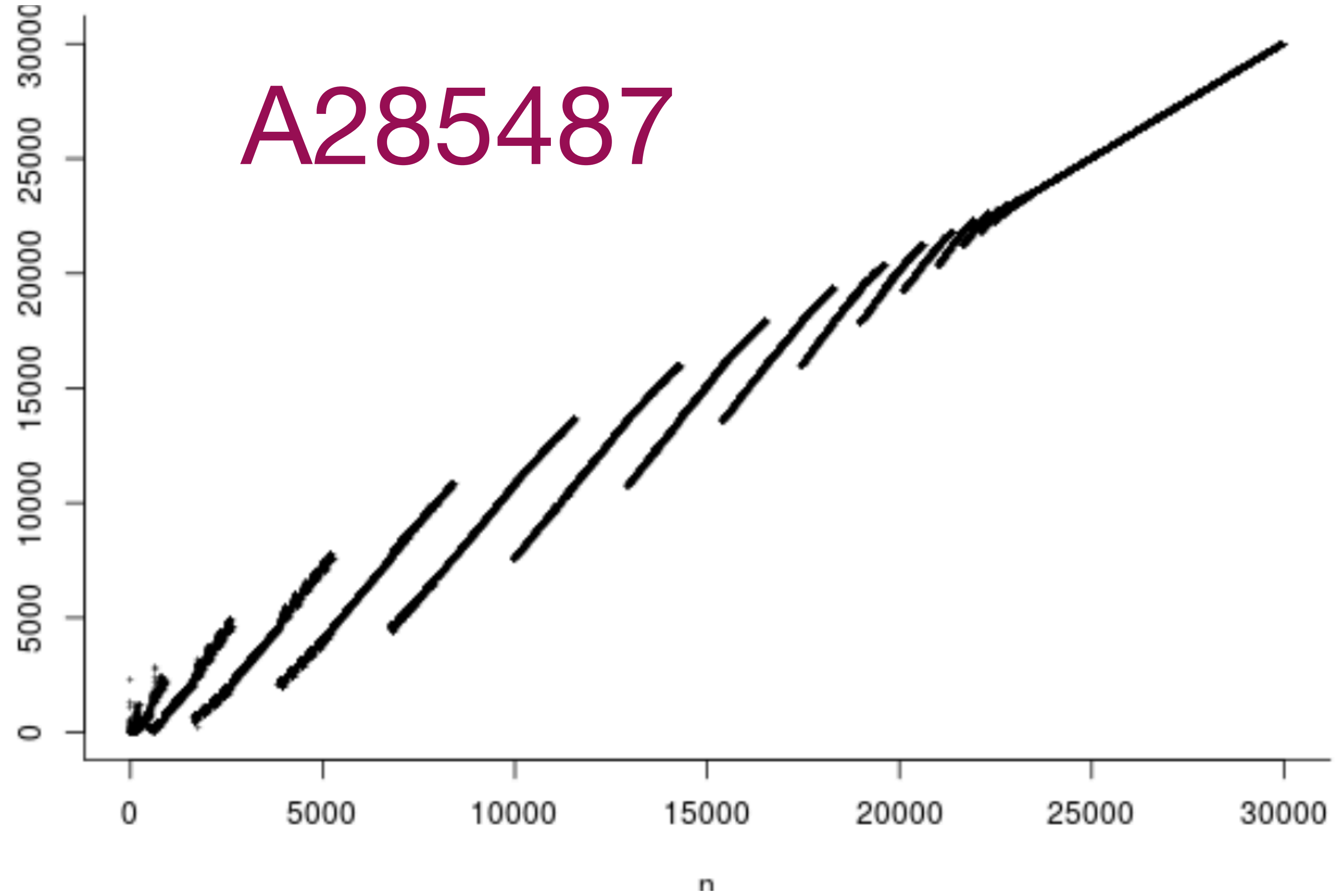



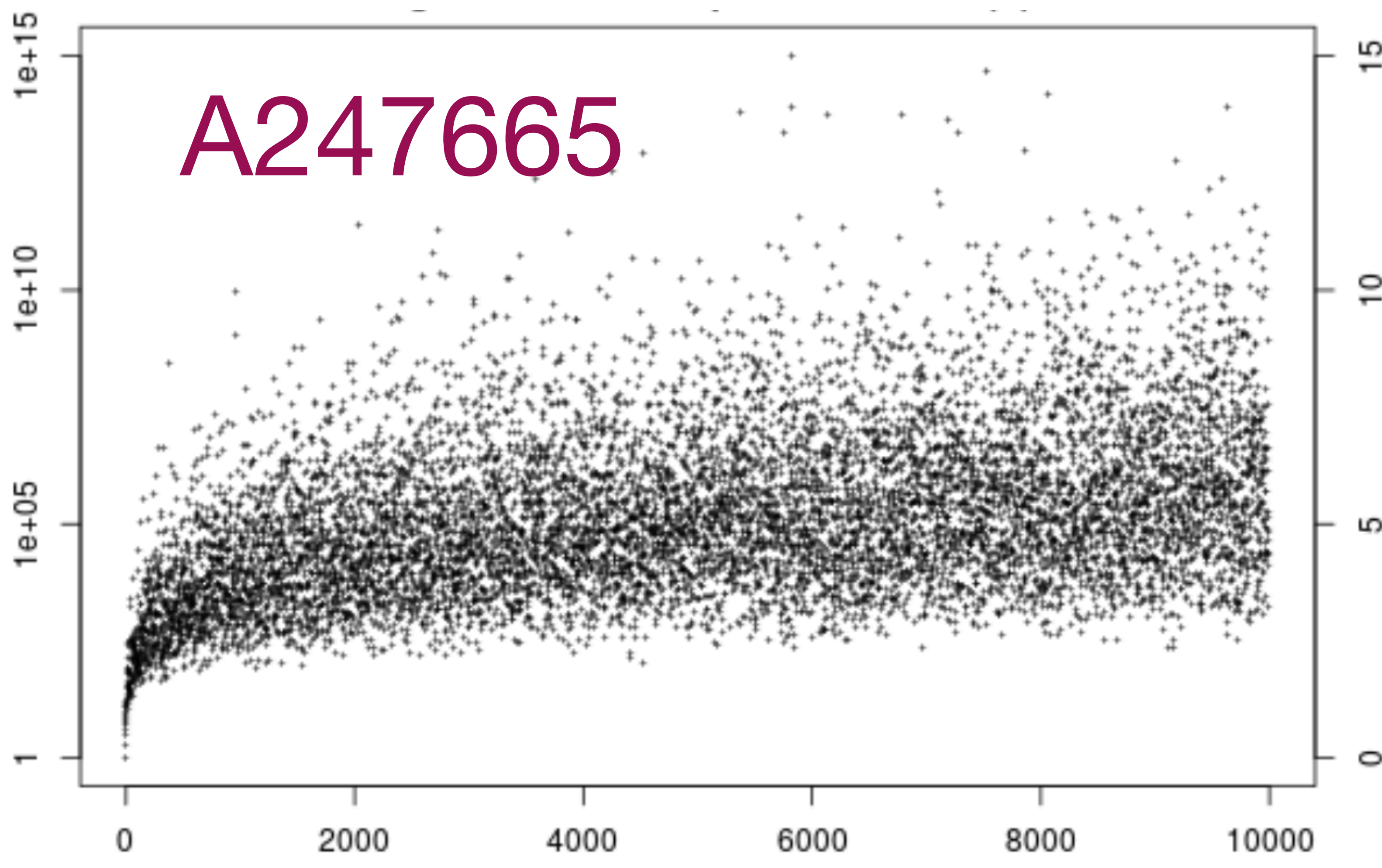



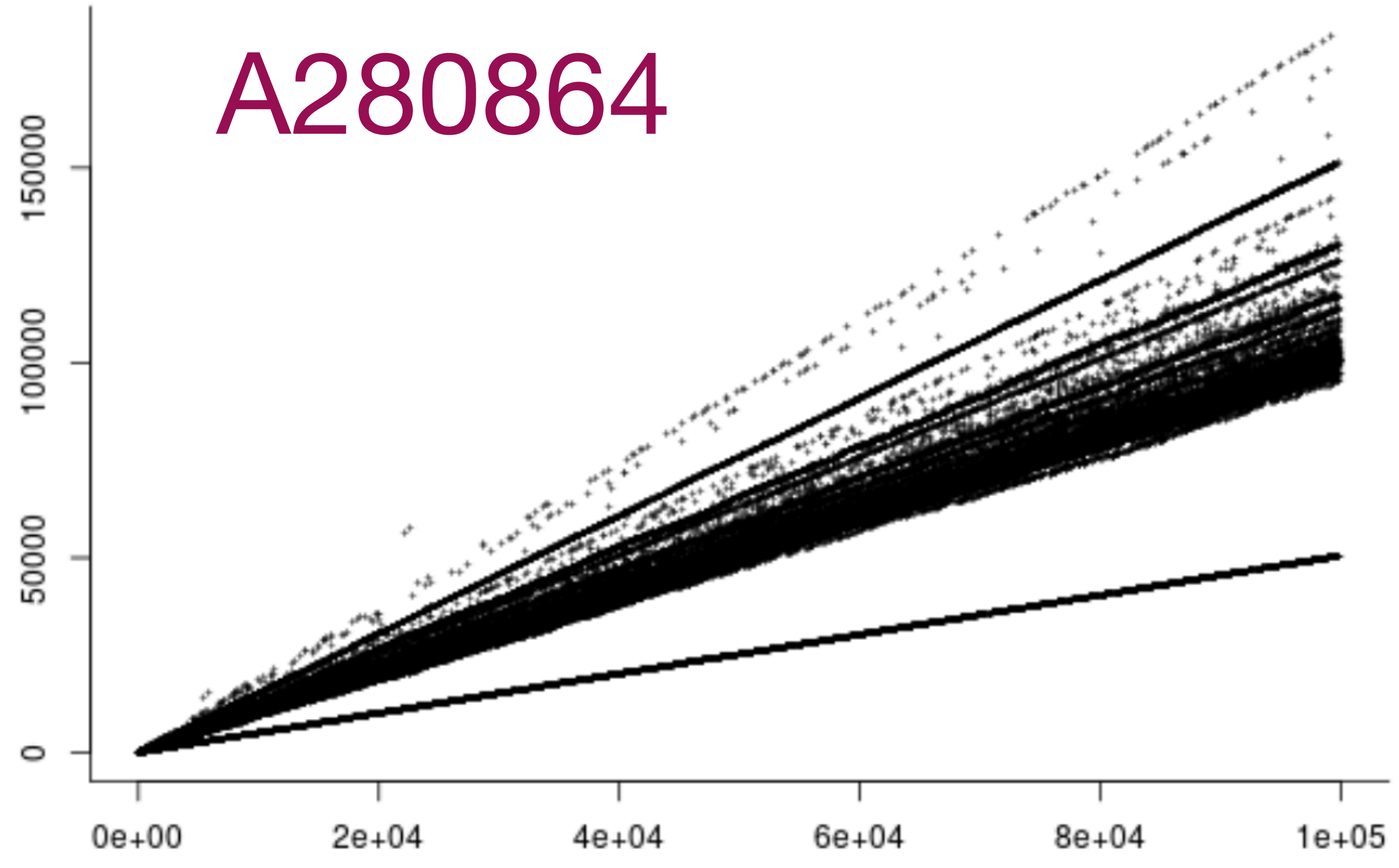
# Graphs (continued)

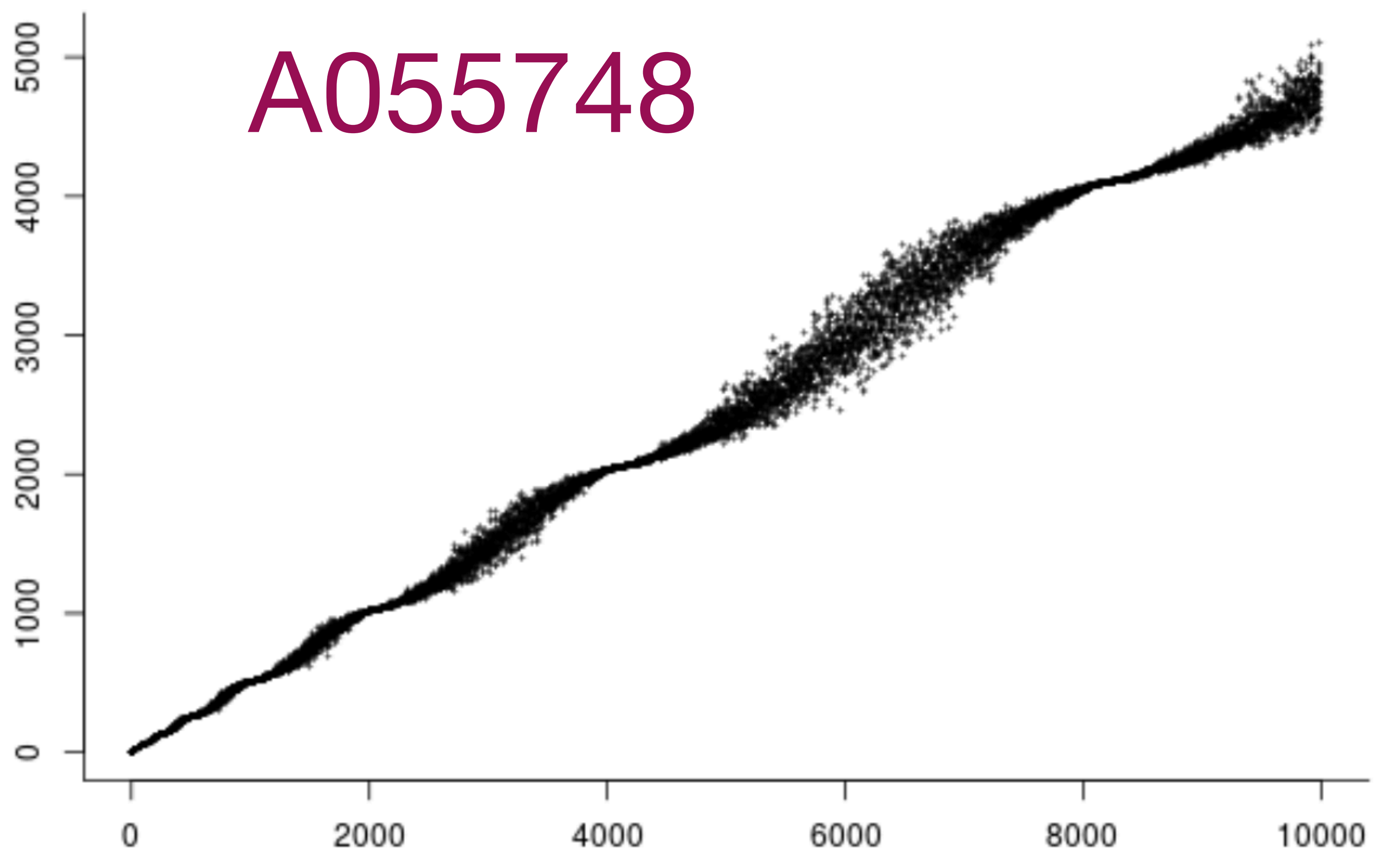
## Typical Graphs

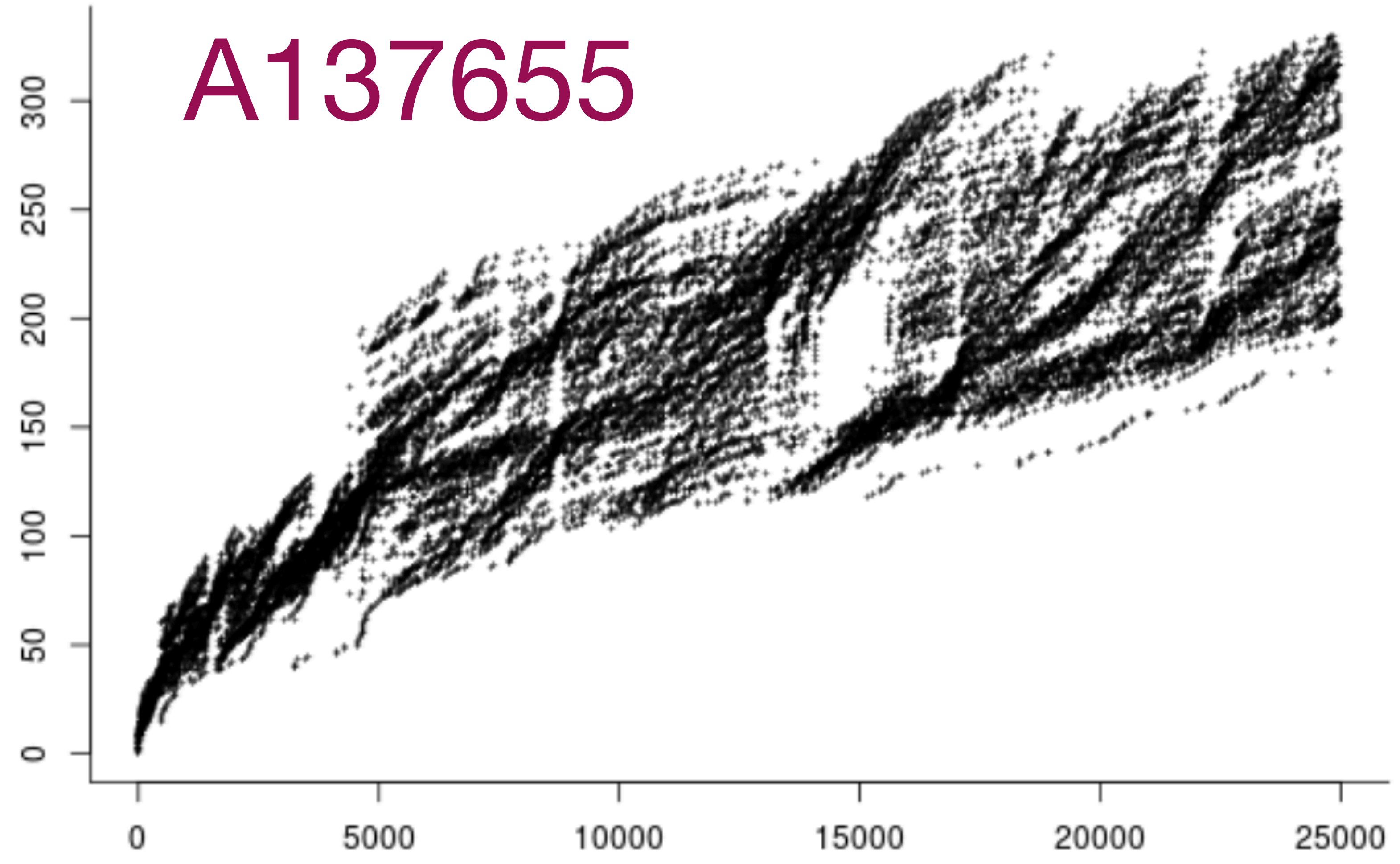


## Graphs (continued) PROJECT

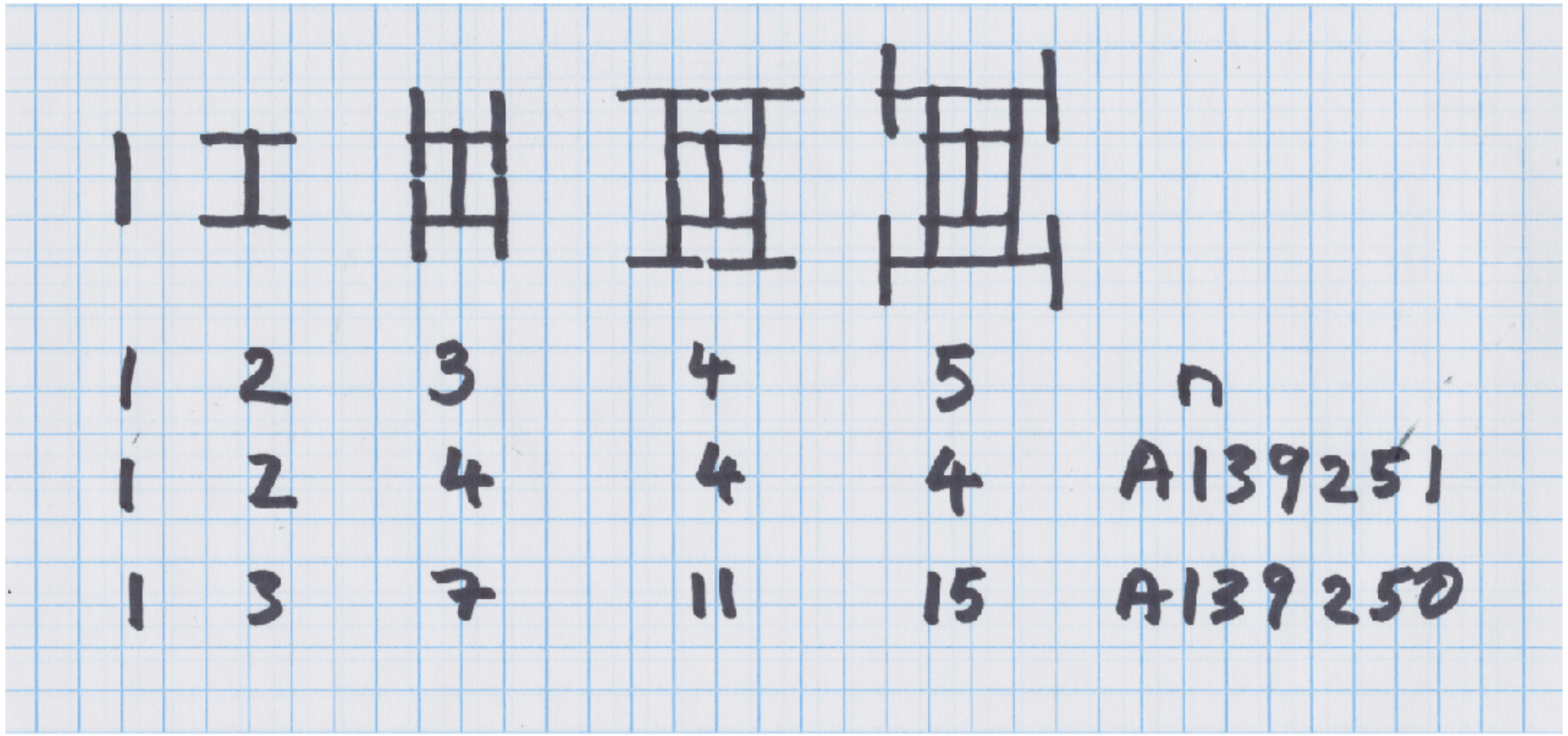
The OEIS contains over 365000 sequences, and each one has a graph (click the "graph" button).

Use facial recognition software to find closest matches to the graph of a sequence you are studying.

Would be extremely useful.

# Sequences from Cellular Automata

# The Toothpick Problem




## (Omar Pol, David Applegate, NJAS)

David L. Applegate, Omar E. Pol, and NJAS, The toothpick sequence and … Congress. Numerant., 206 (2010), 157-191.

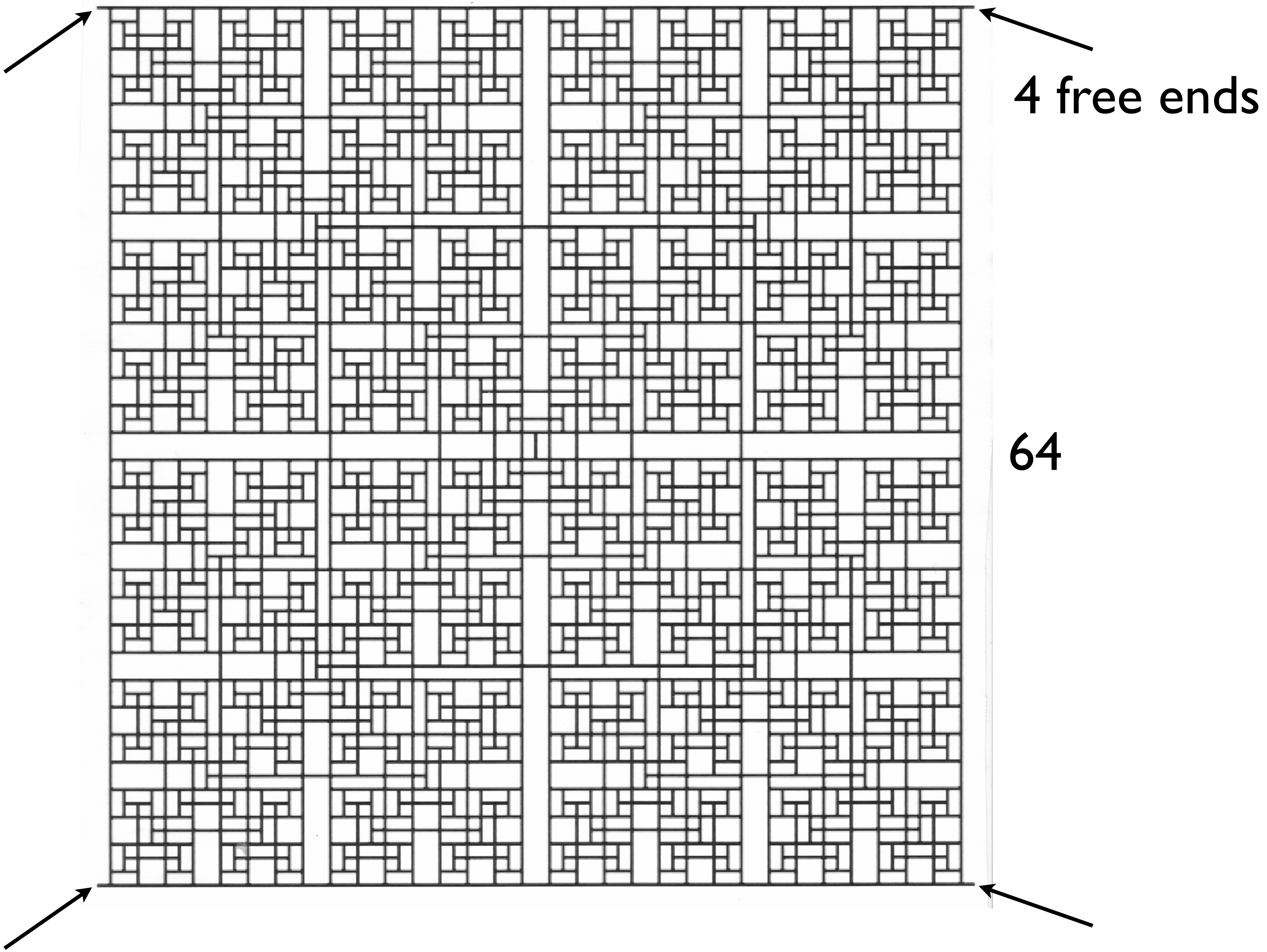
# The Toothpick Sequence (2)

## A139250

Animation created
by David Applegate

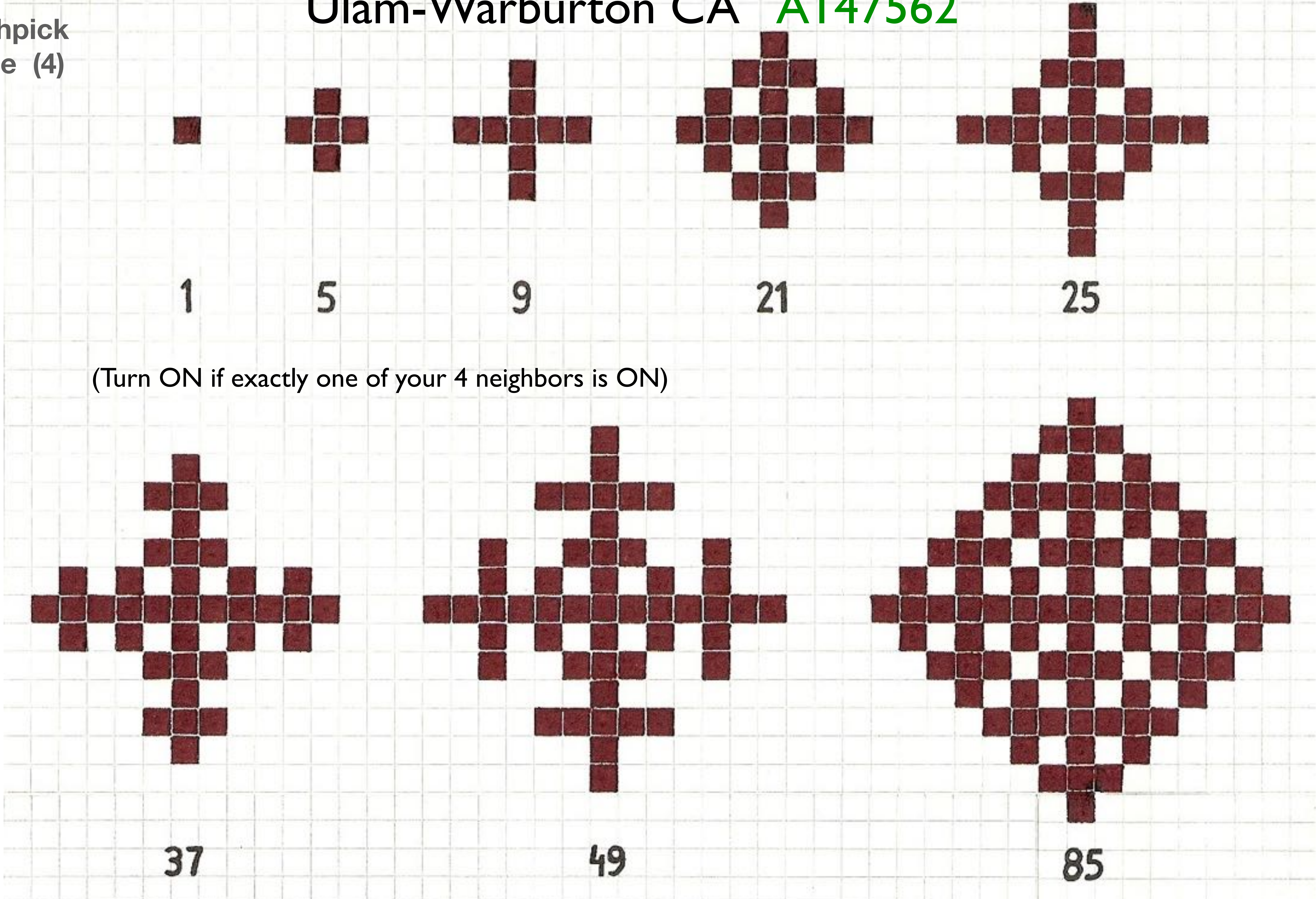
The first 32 generations of the toothpick structure.
Notice that after every power of 2 generations,
the number of new toothpicks added drops to 4.
This si the key to finding a formula for the n-th term.

The Toothpick Sequence (3)



The Toothpick Sequence (4)

# Ulam-Warburton CA A147562



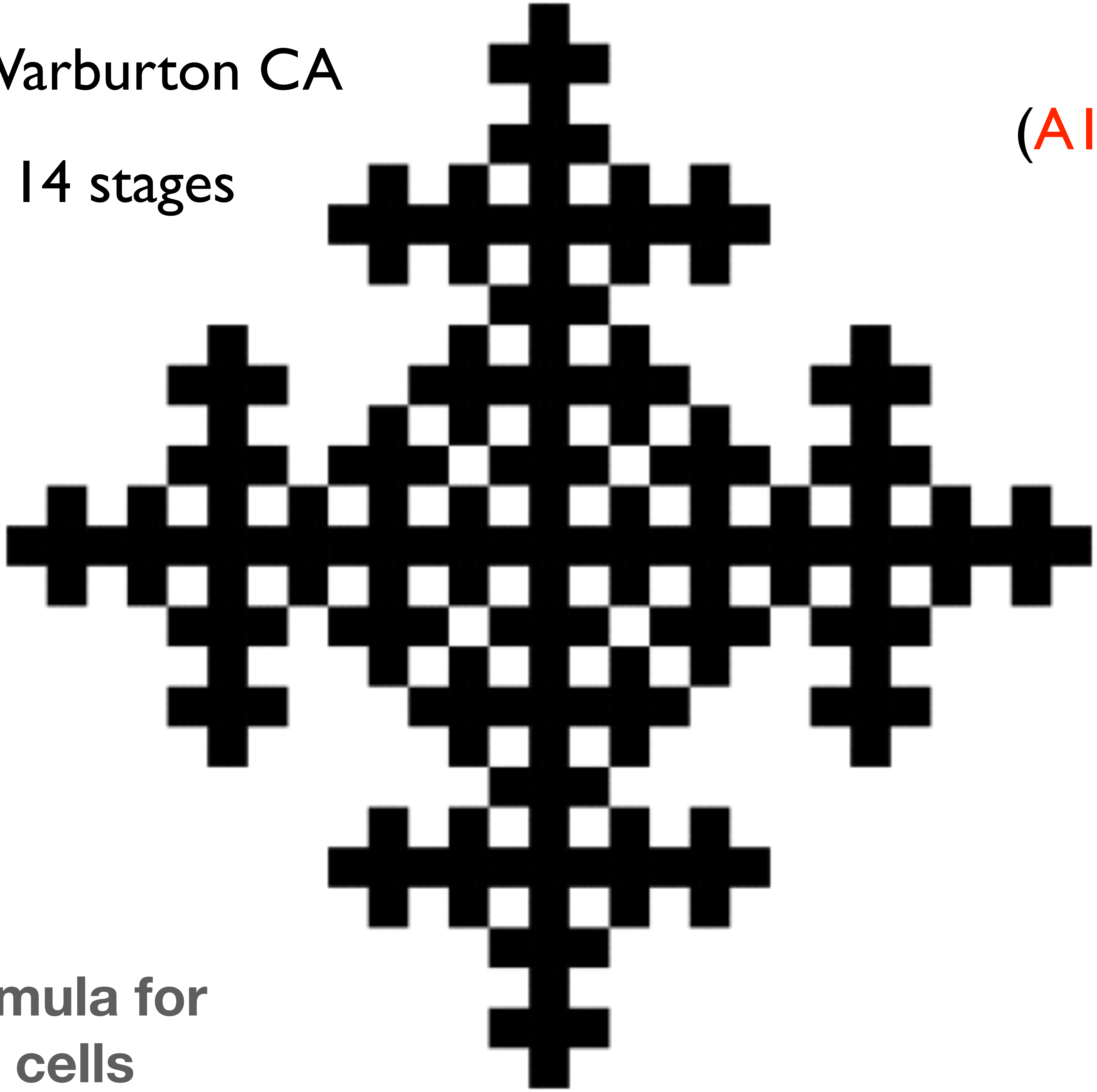

(Turn ON if exactly one of your 4 neighbors is ON)

The Toothpick Sequence (5)

Ulam-Warburton CA

after 14 stages

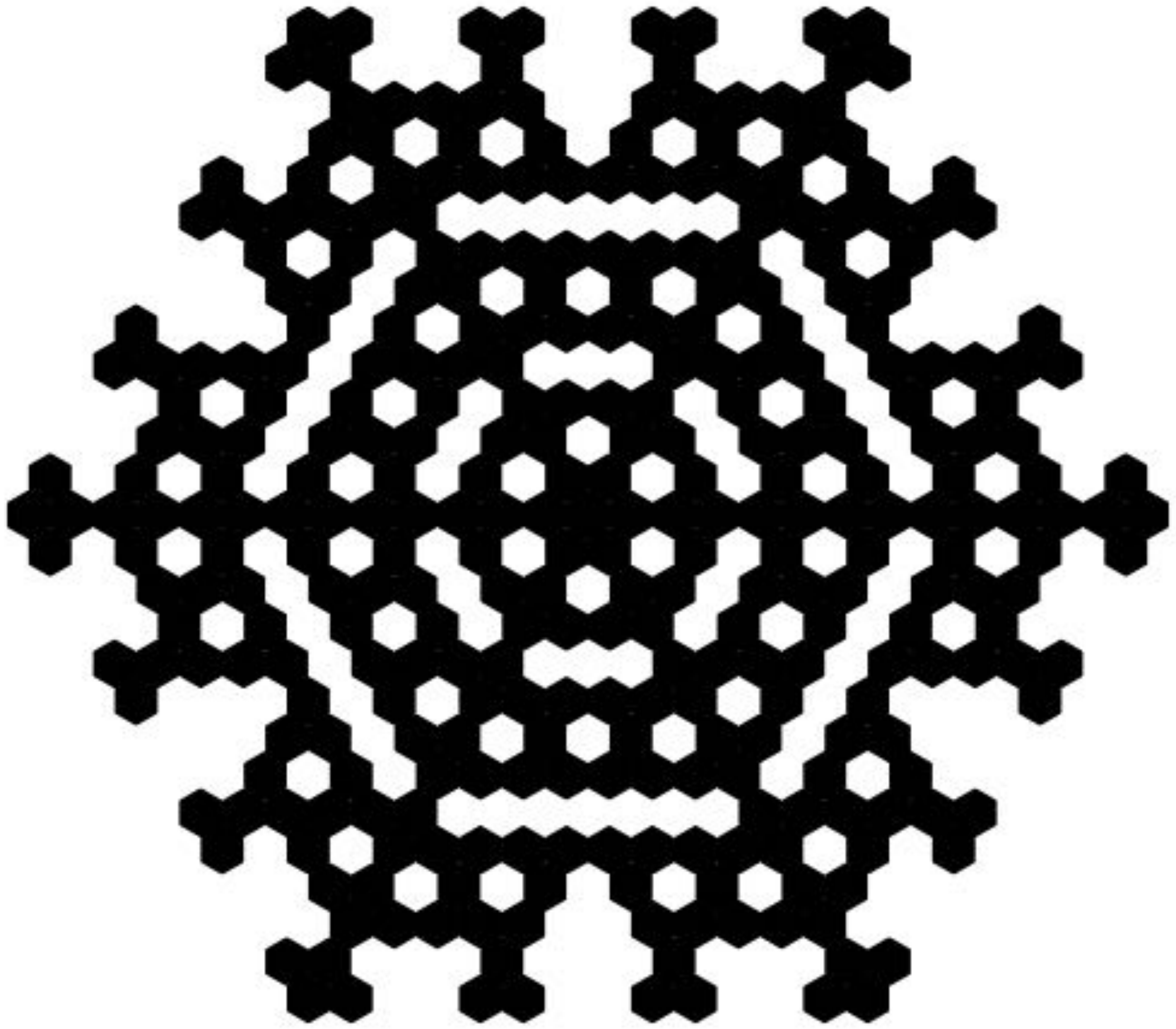
(A147582)



There is a simple formula for the number of ON cells

Another Ulam CA, on hexagonal grid:

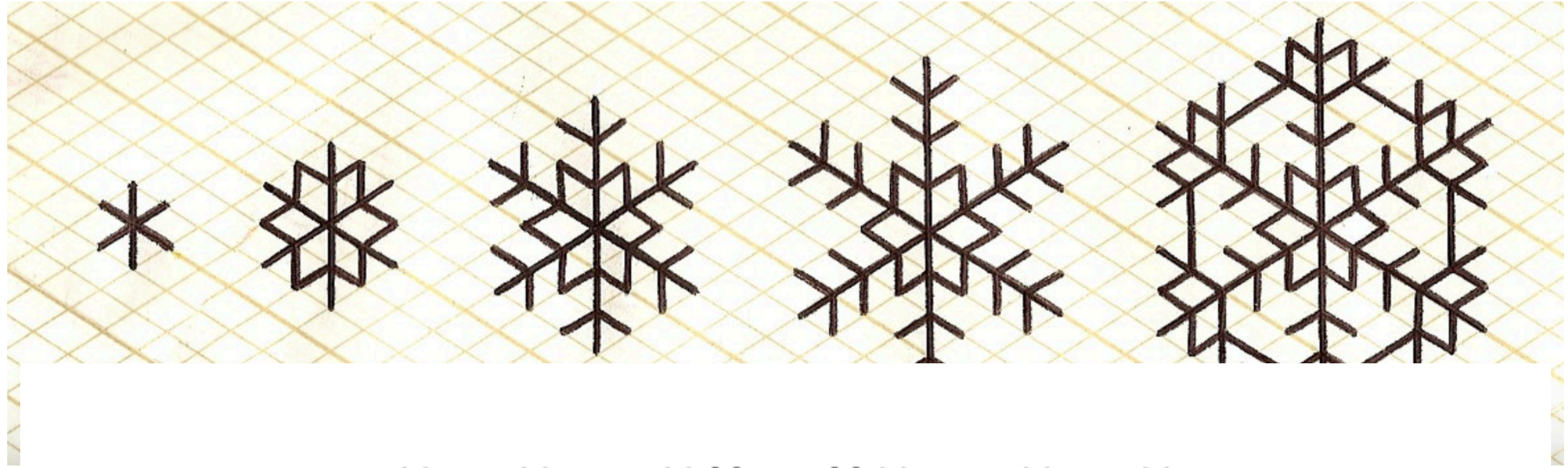
A cell turns ON iff exactly one of its 6 neighbors in ON

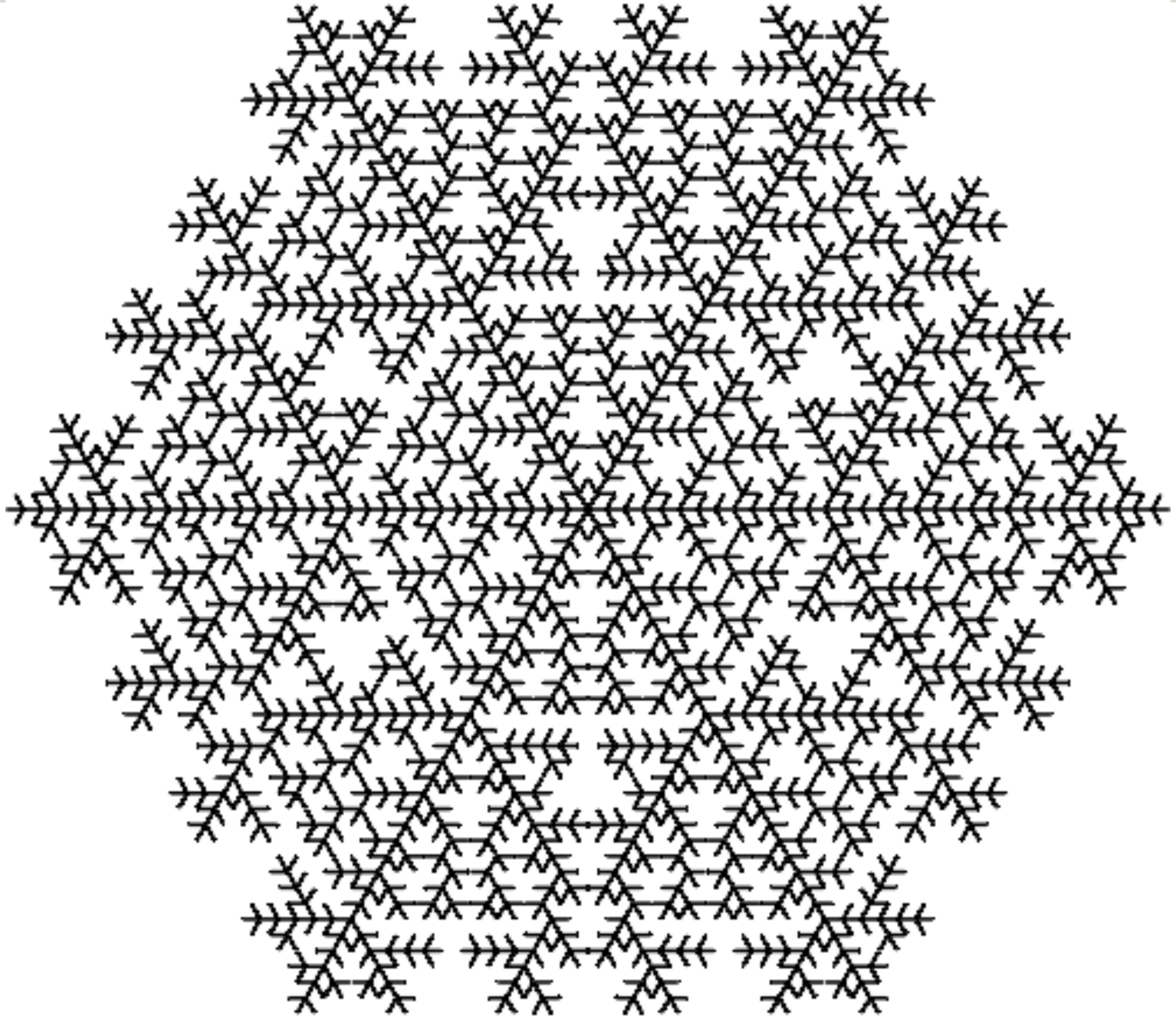


(A151724)

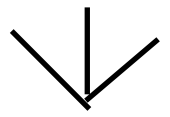
No recurrence known!

# The Toothpick Sequence (7)





The snow-flake automaton



(trident)

Generations 1 (2), 2 (8), 3 (14), 4 (20), 5 (38), … 32 (1124), …
Sequence A161330.

David L. Applegate, Omar E. Pol, and NJAS, The toothpick sequence and … Congress. Numerant., 206 (2010), 157-191.

# Coordination Sequences

**Joint work with Chaim Goodman-Strauss**

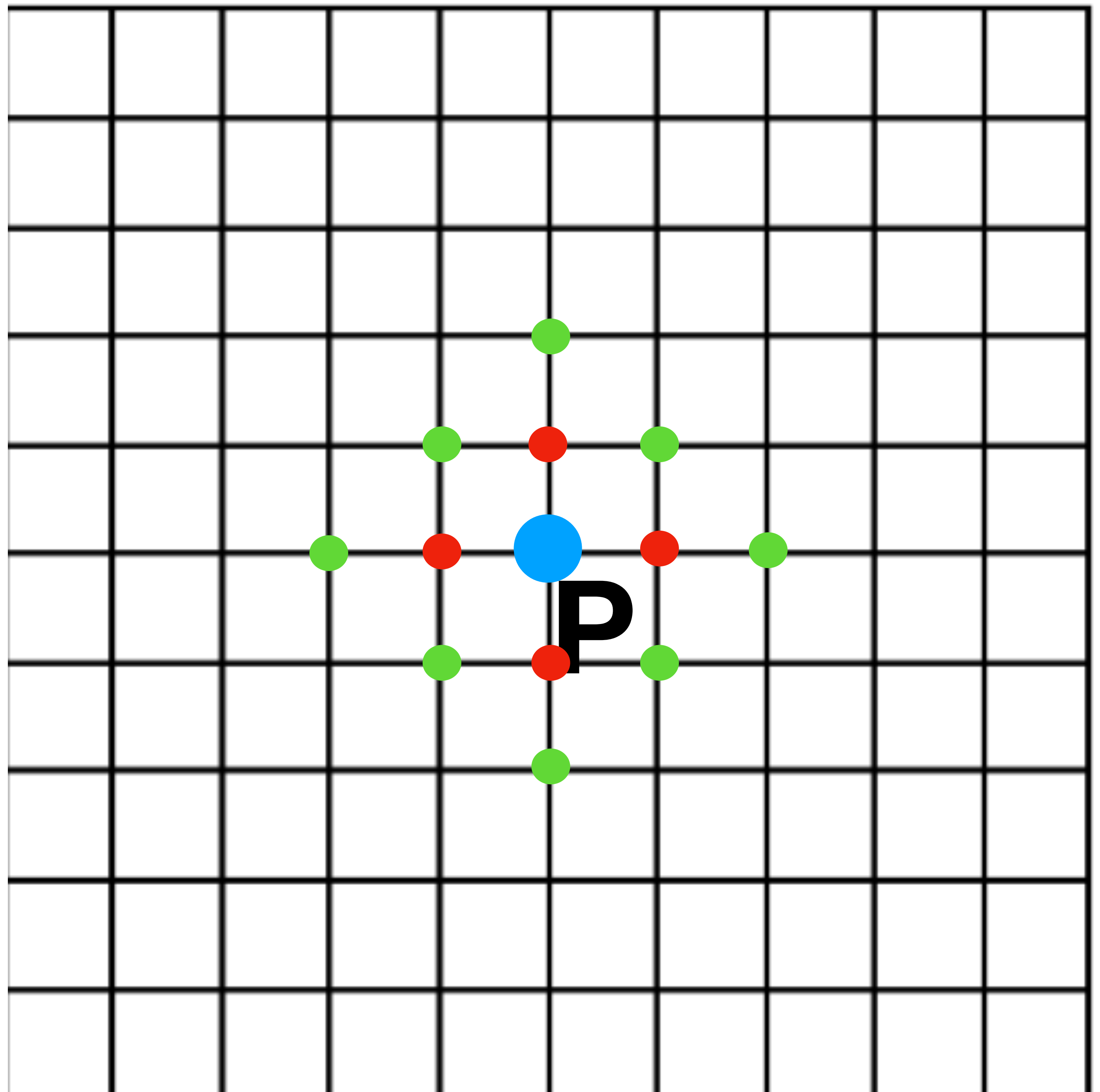
# Coordination Sequences

## Joint work with Chaim Goodman-Strauss

With thanks to Jean-Guillaume Eon, Brian Galebach, Joseph Myers, Davide Proserpio, Rémy Sigrist, Allan Wechsler, and others

**Definition.** G = graph, P = node,
the coordination sequence w.r.t P:
a(n) = number of nodes at edge-distance n from P

G

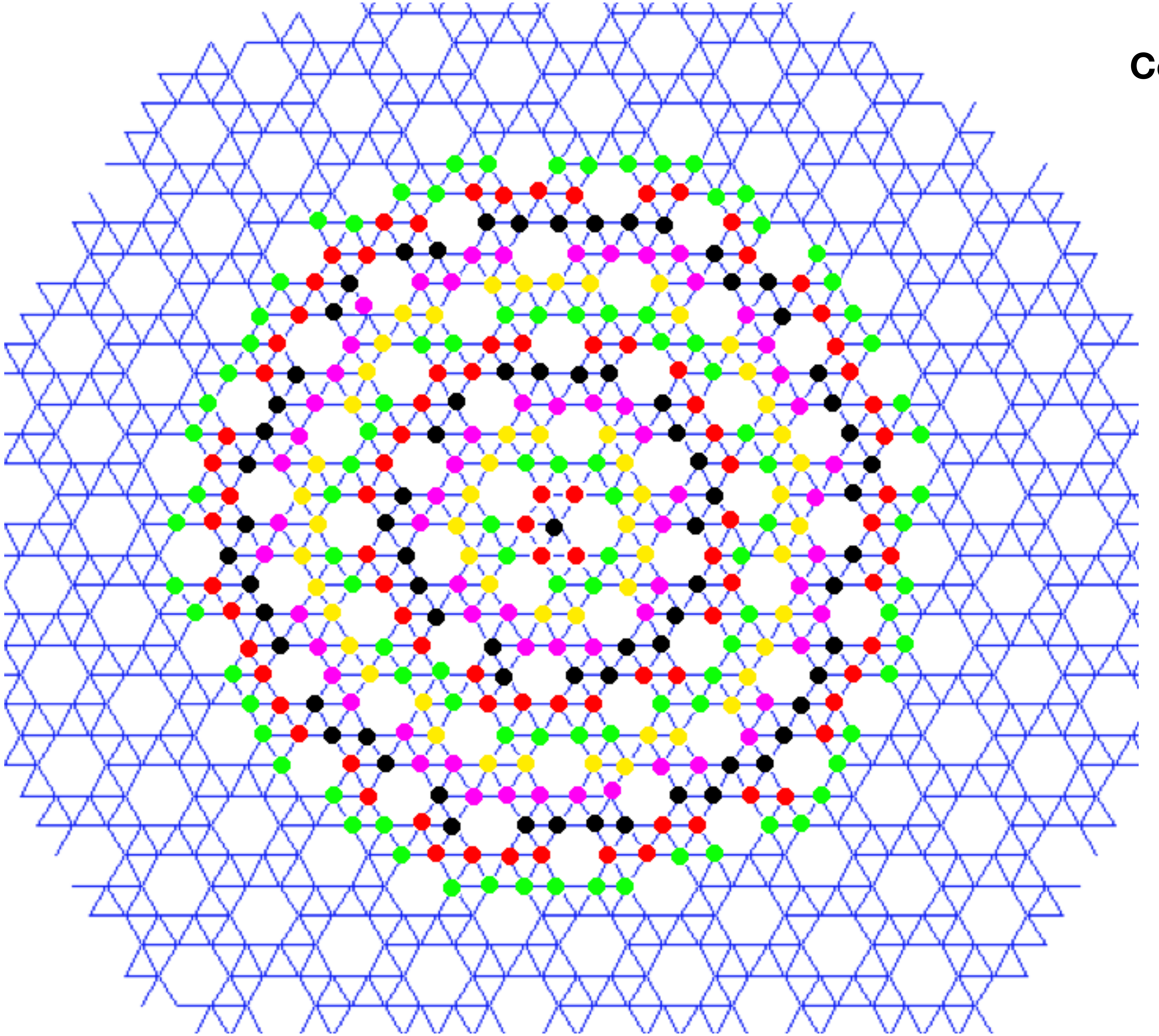



A8574

CS is 1, 4, 8, 12, 16, 20, 24, 28, …

G.f. = (1+2x+2x^2+2x^3+…)^2

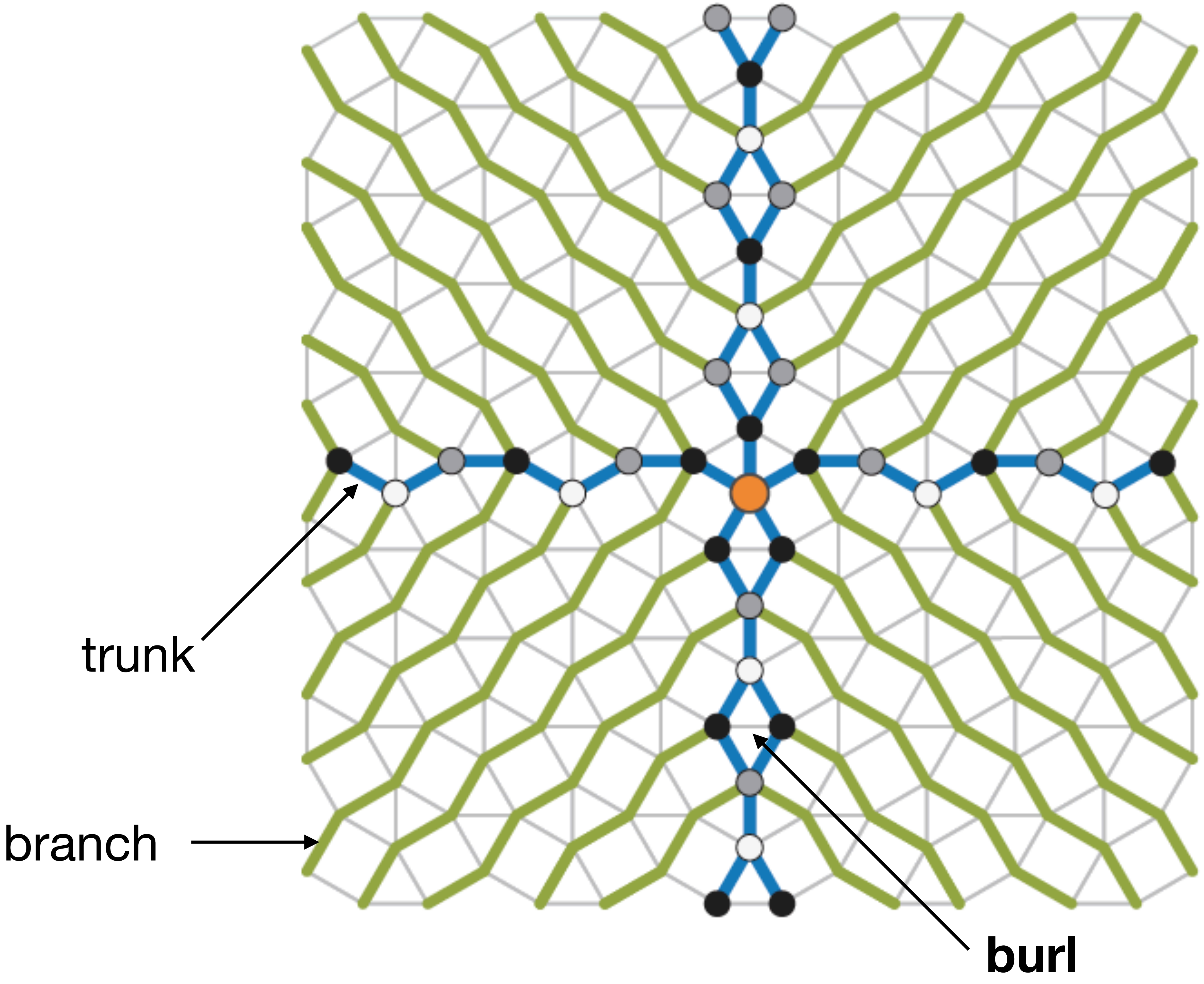
# The 3.3.3.3.6 uniform tiling (A250120)



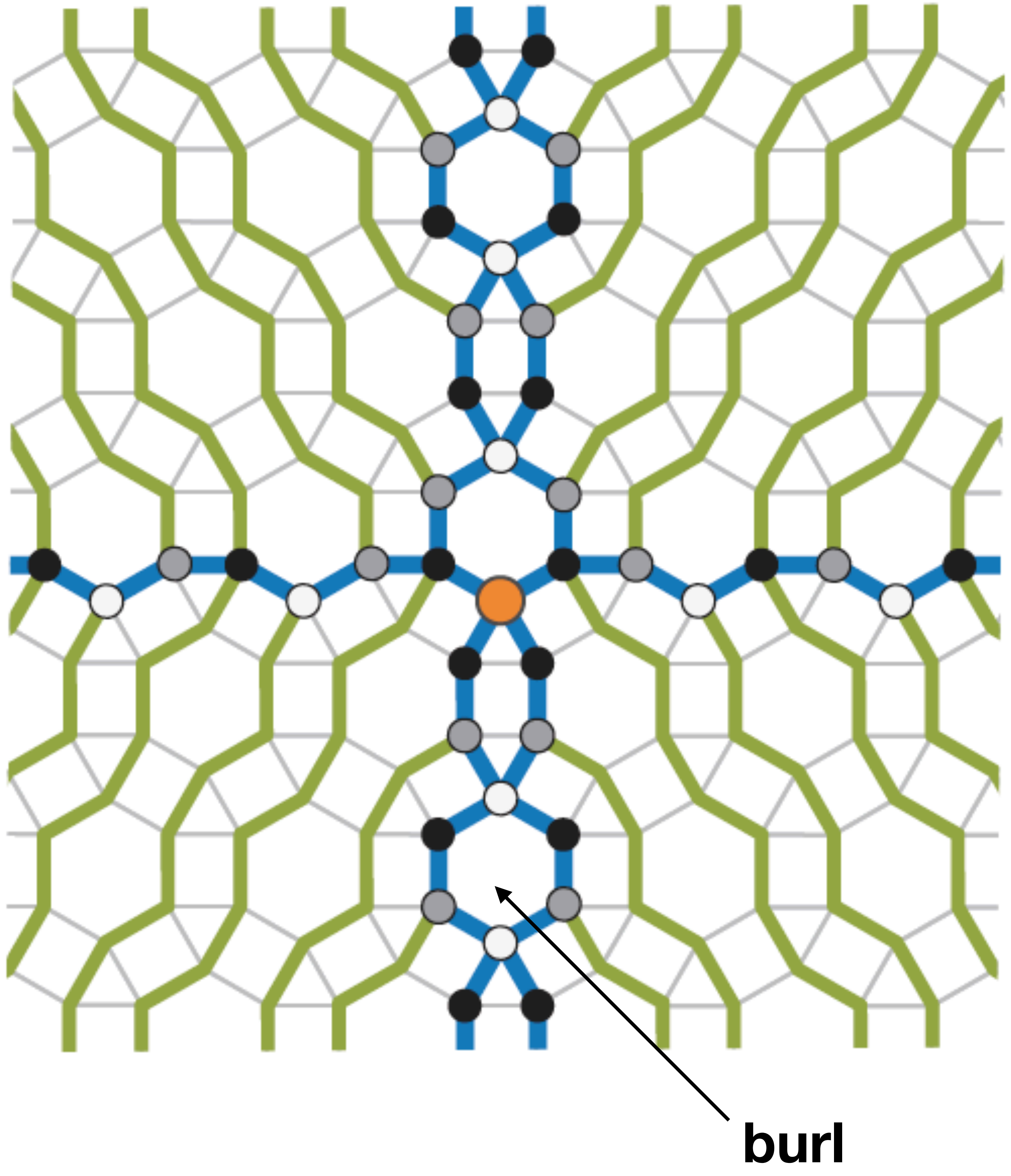
Coordination sequence
1,5,9,15,19,...

Conjecture
a(n+5)=a(n)+24
for n > 2

# Trunks and Branches for 2 of the 11 Uniform Tilings




**3.3.4.3.4 (dual to Cairo), A219529**



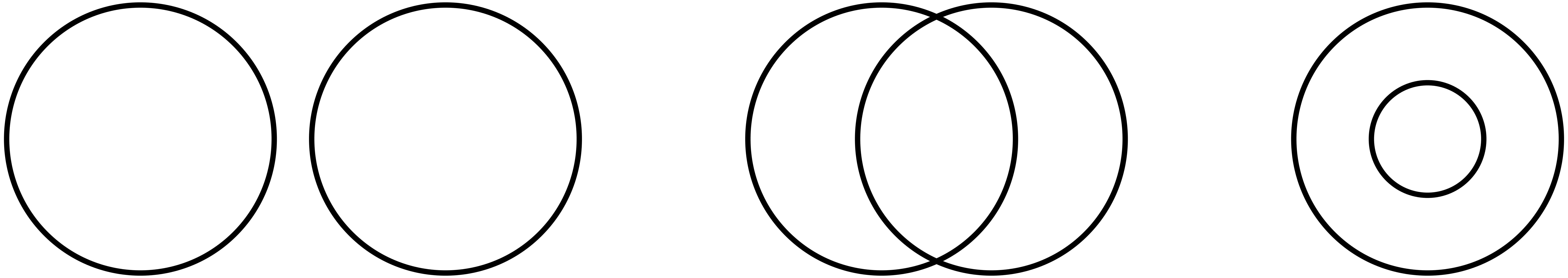

**3.4.6.4, A8574 again!**

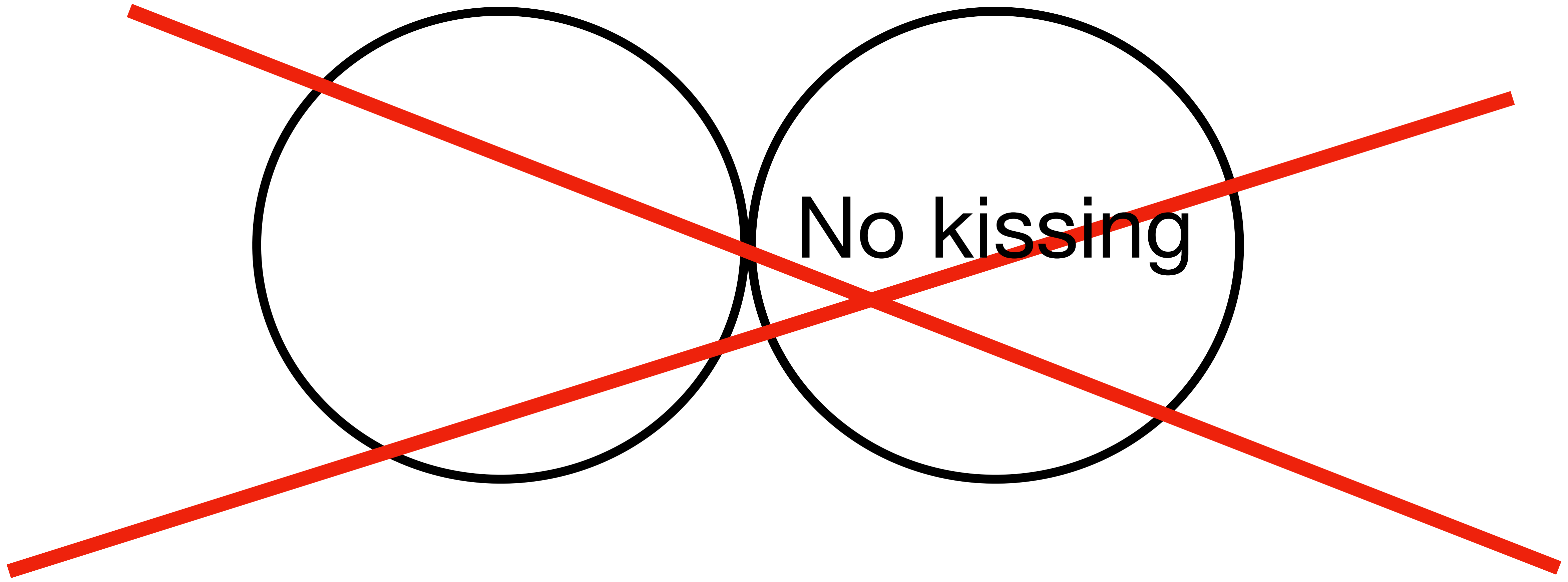
# Number of ways to draw n circles

Jonathan Wild
Music Department, McGill

A250001

a(2)=3





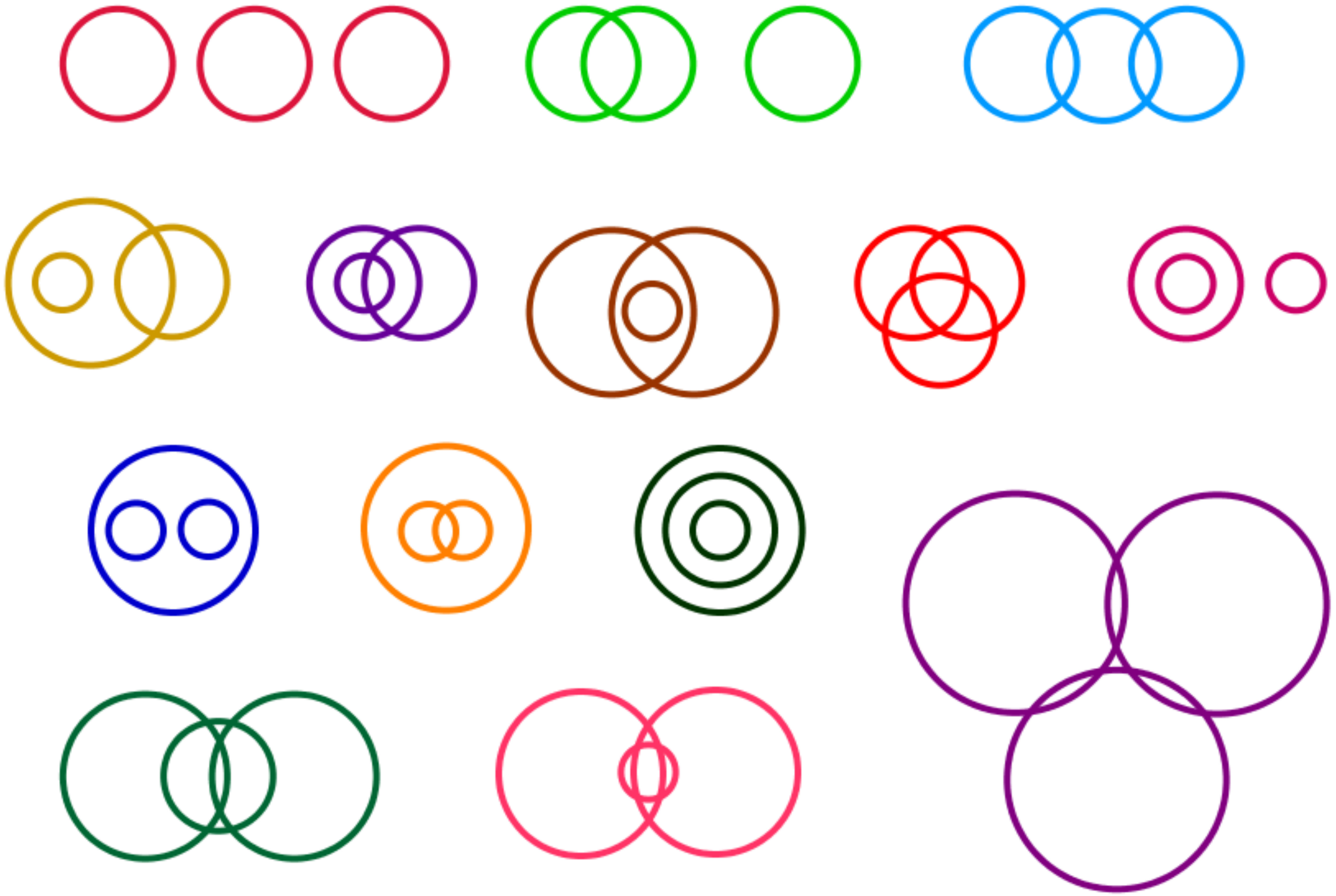
No. of arrangements of n circles in the plane

A250001

1, 3, 14, 173, 16951

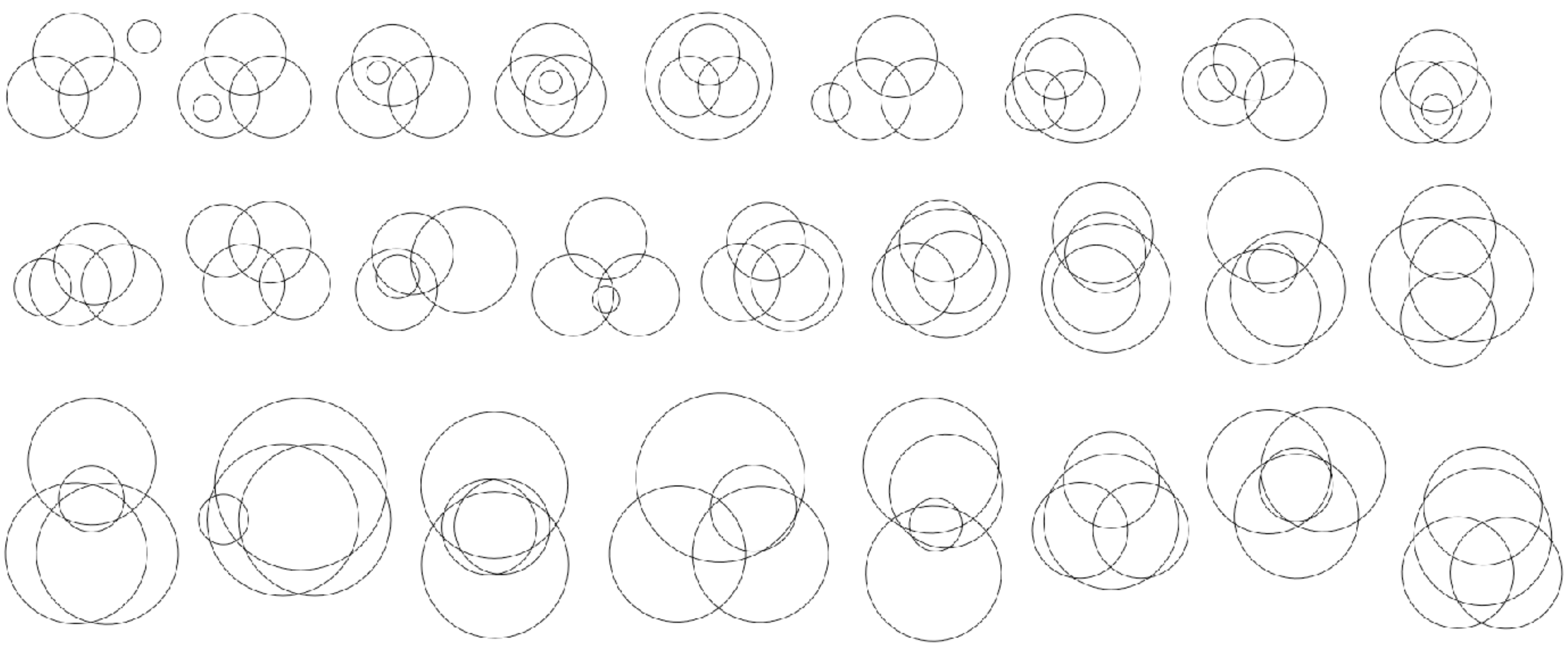
Jonathan Wild

a(3) = 14:



A250001

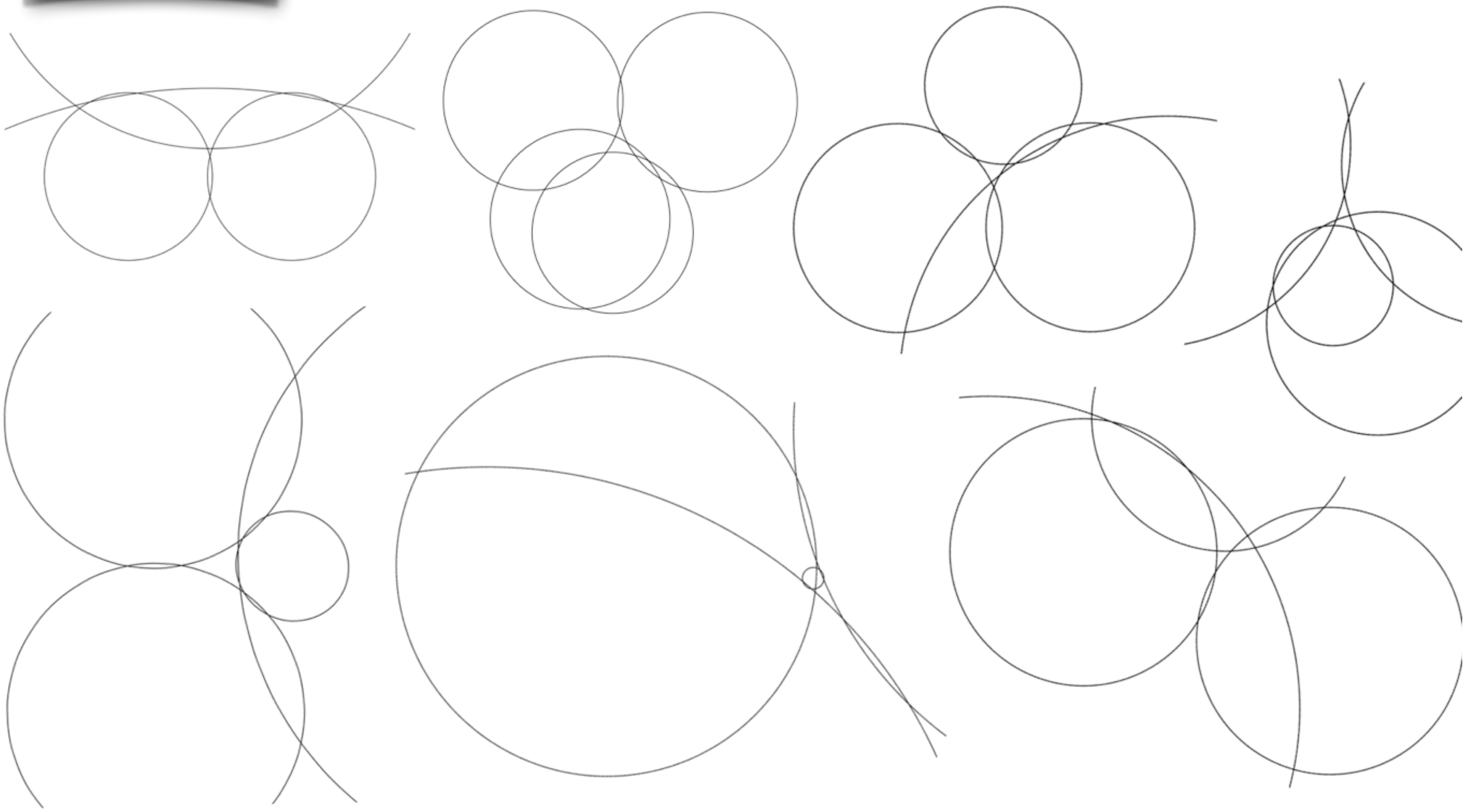
# Some of the 173 arrangements of 4 circles



Counted (and drawn) by Jon Wild

A250001

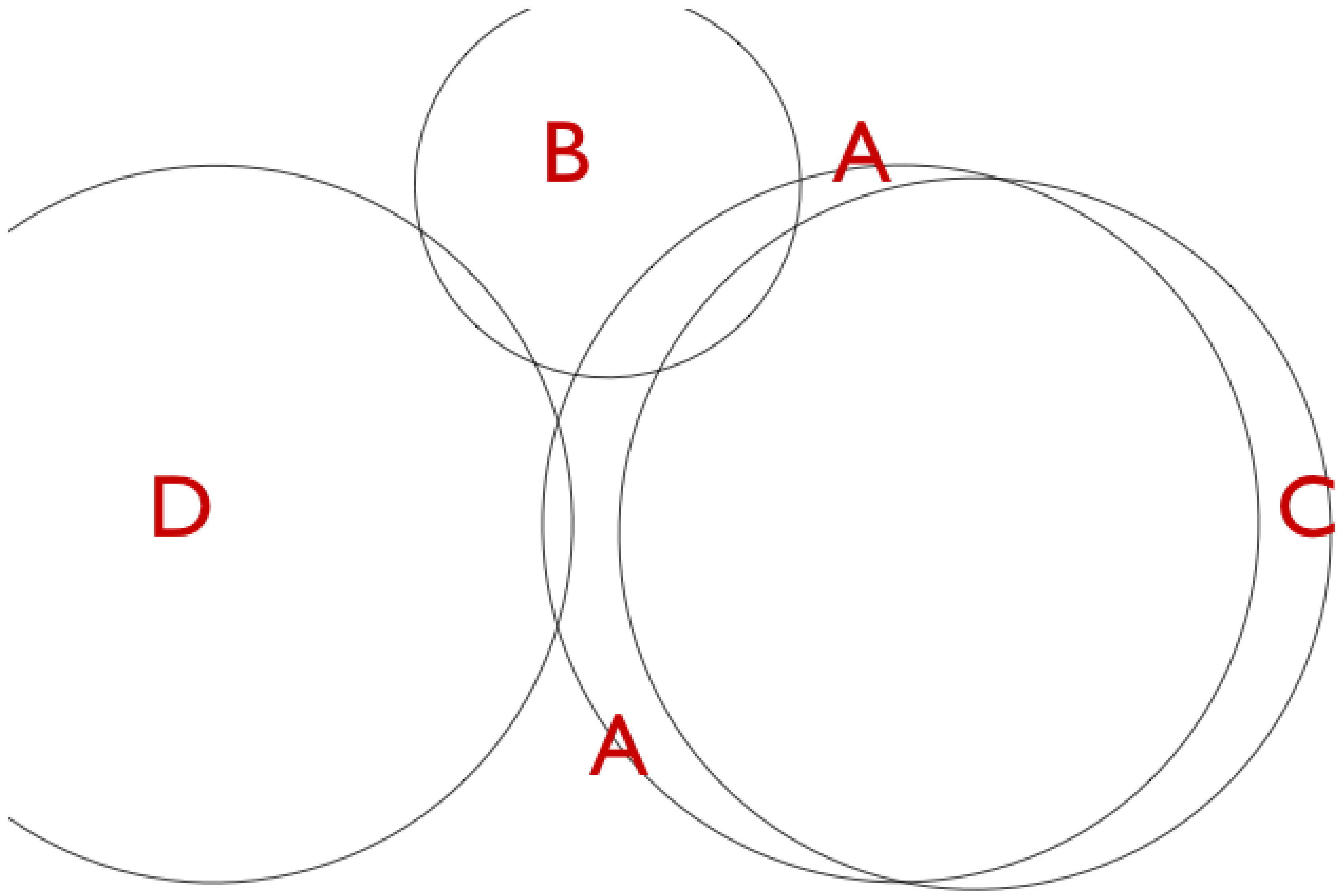
# More of the 173 arrangements of 4 circles

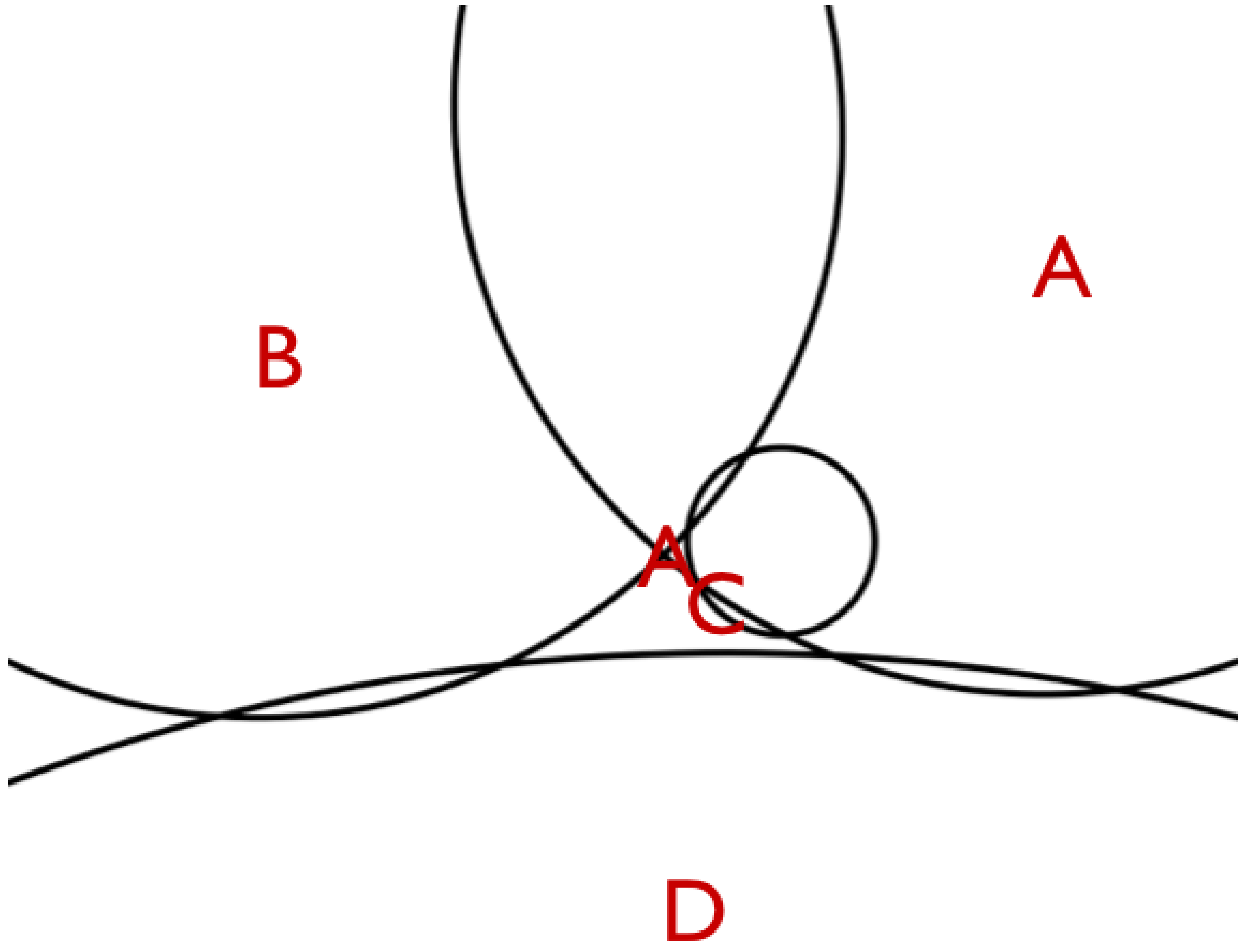


Counted (and drawn) by Jon Wild

Two distinct arrangements with same “truth table” of intersections

Jonathan Wild



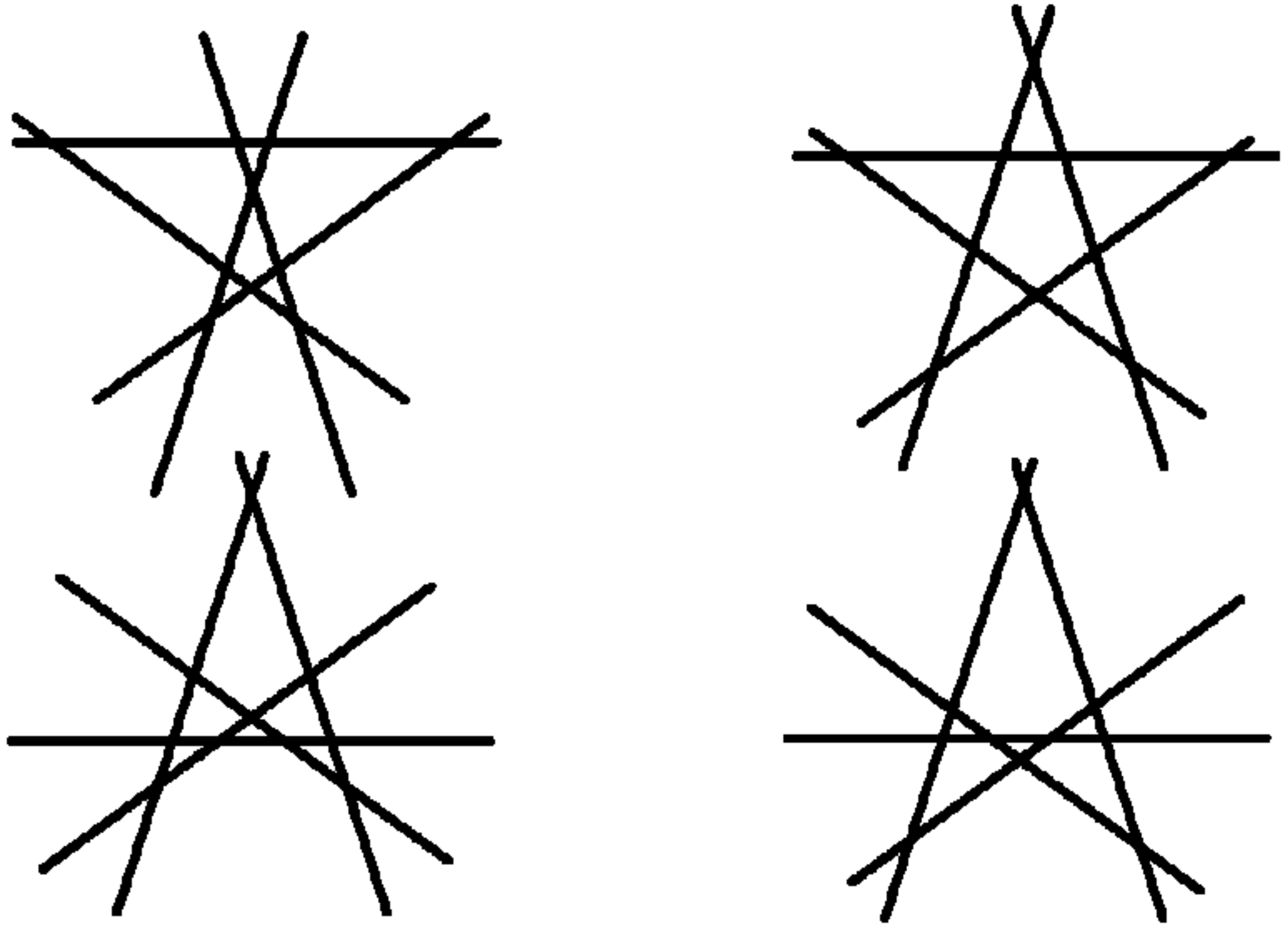

Open Problem: How many ways to draw six circles?

A90338

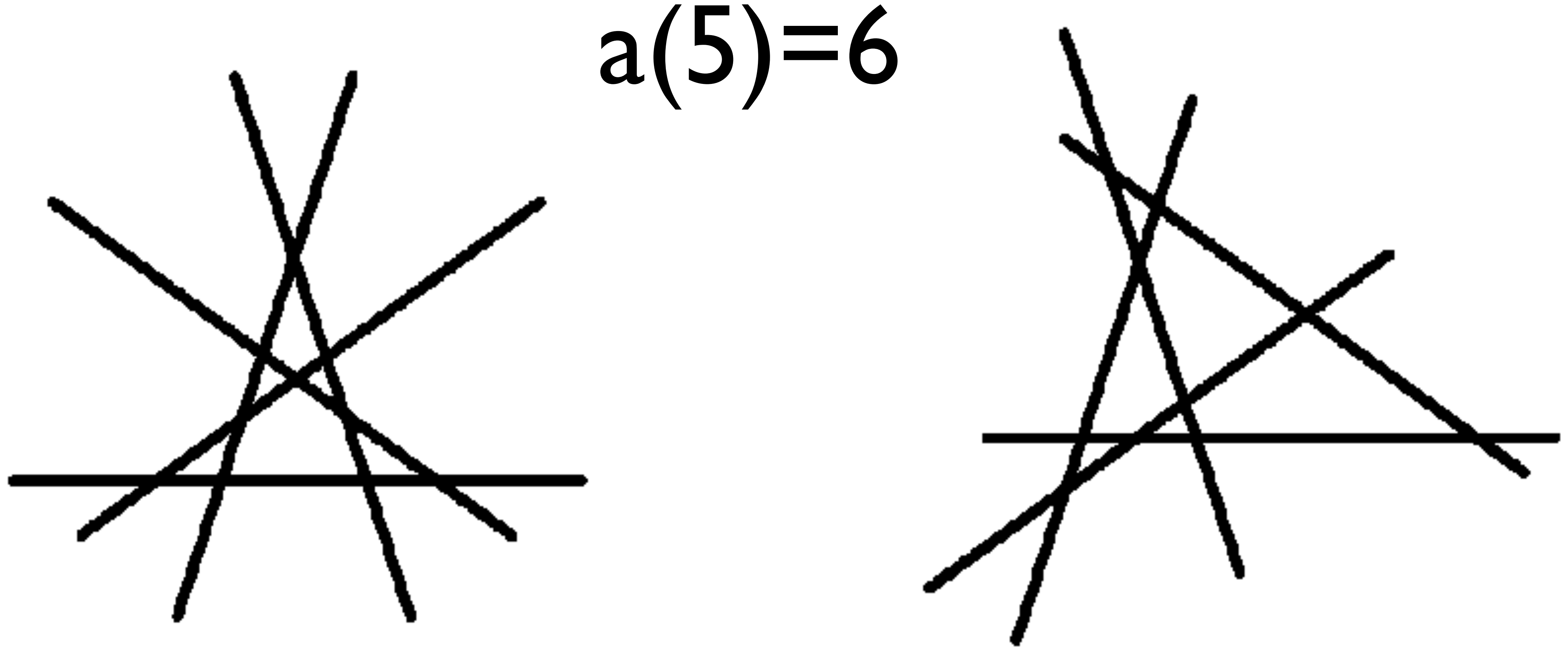
A subset: n lines in general position

1,1,1,1, 6, 43, 922, 38609

Wild and Reeves, 2004



a(5)=6



# Dissecting Polygons into Squares, Rectangles, or Monotiles (Gavin Theobald)

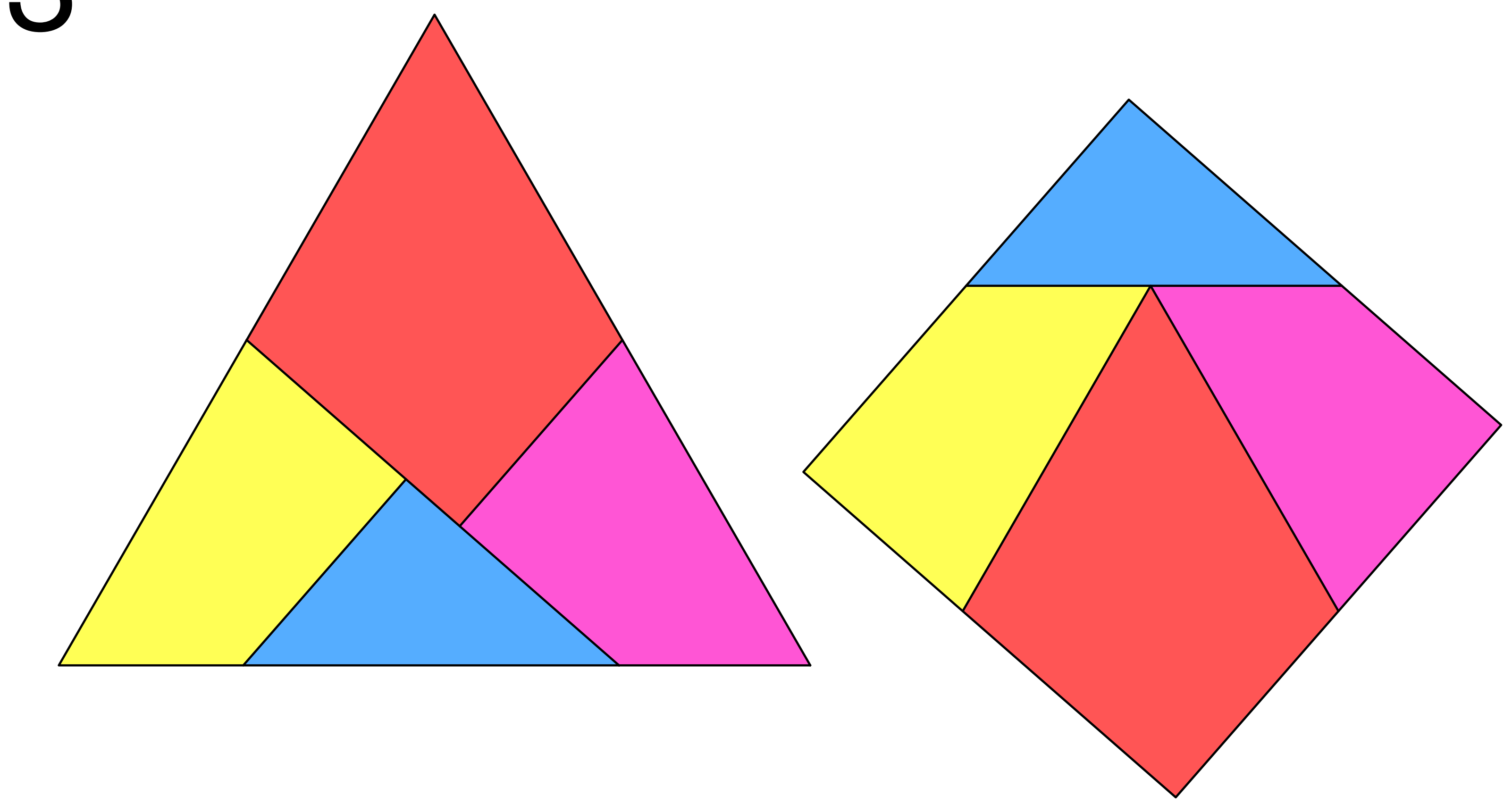
# Three fundamental sequences from geometry

**s(n) = min number of pieces needed to dissect regular n-gon to a square**

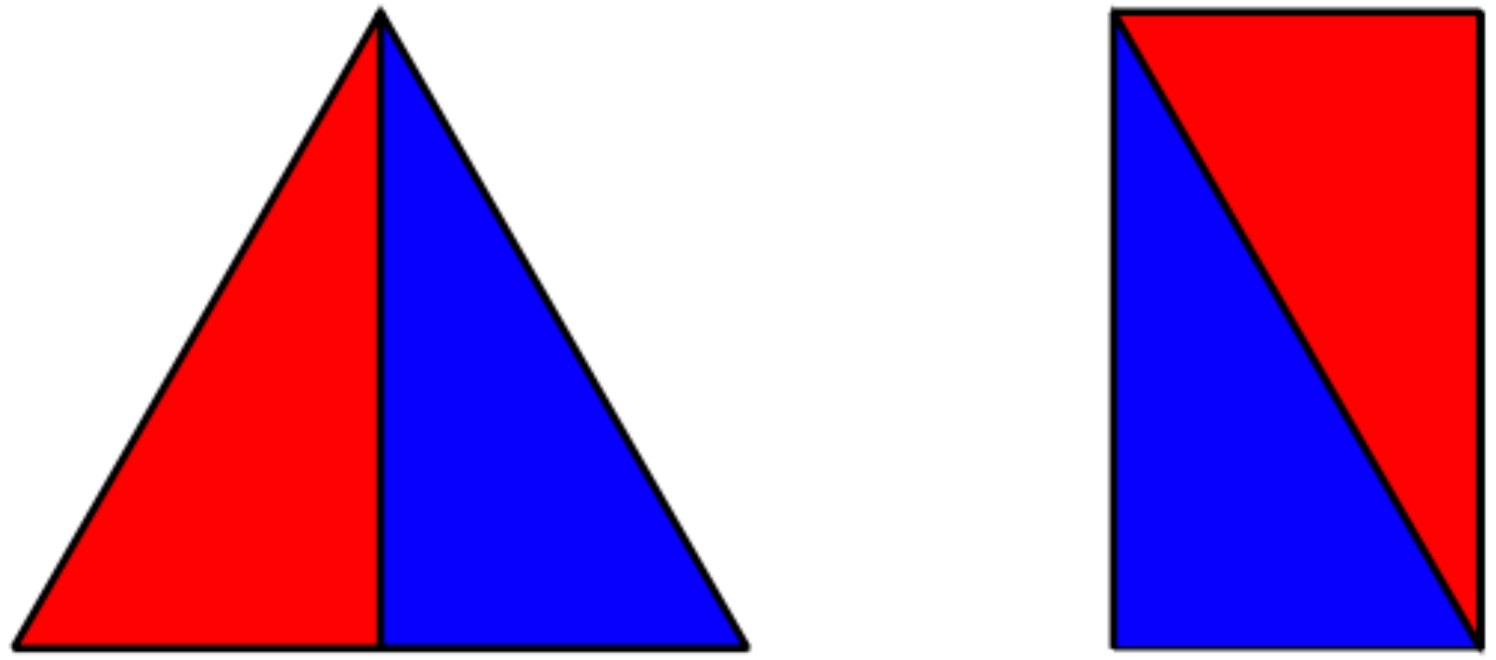
**r(n) = ........................................................................ a rectangle**

**q(n) = ........................................................................ a monotile**

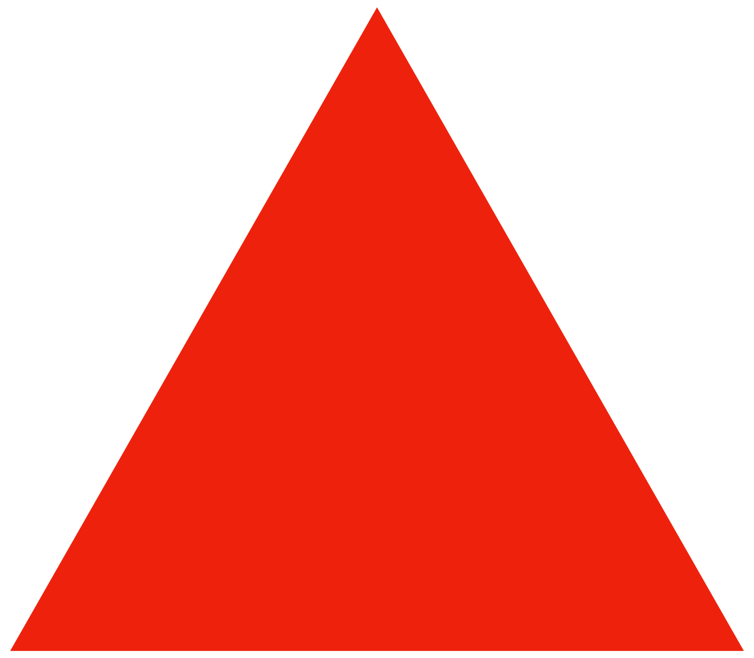
n=3



s(3) = 4 ?

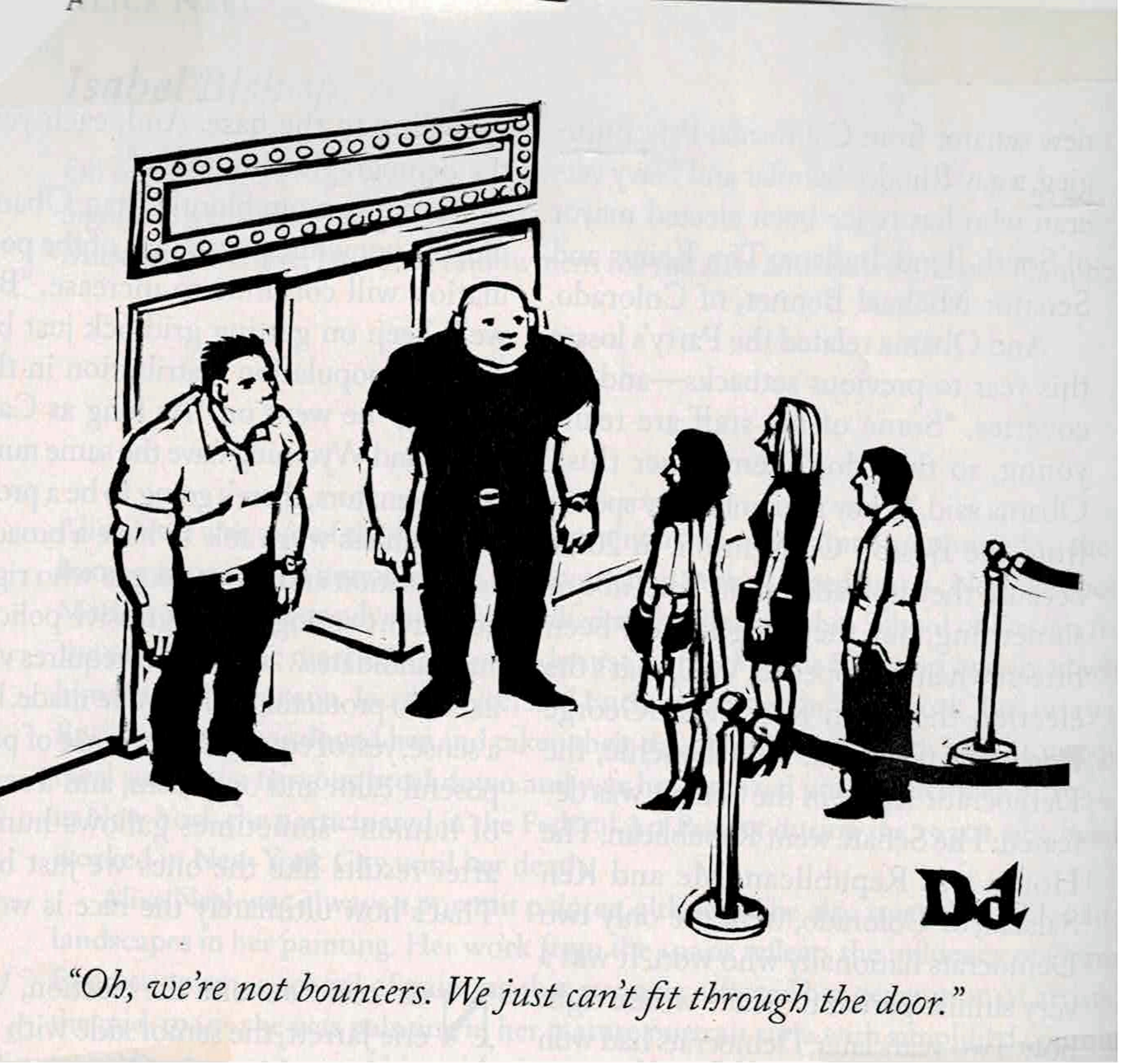


r(3) = 2



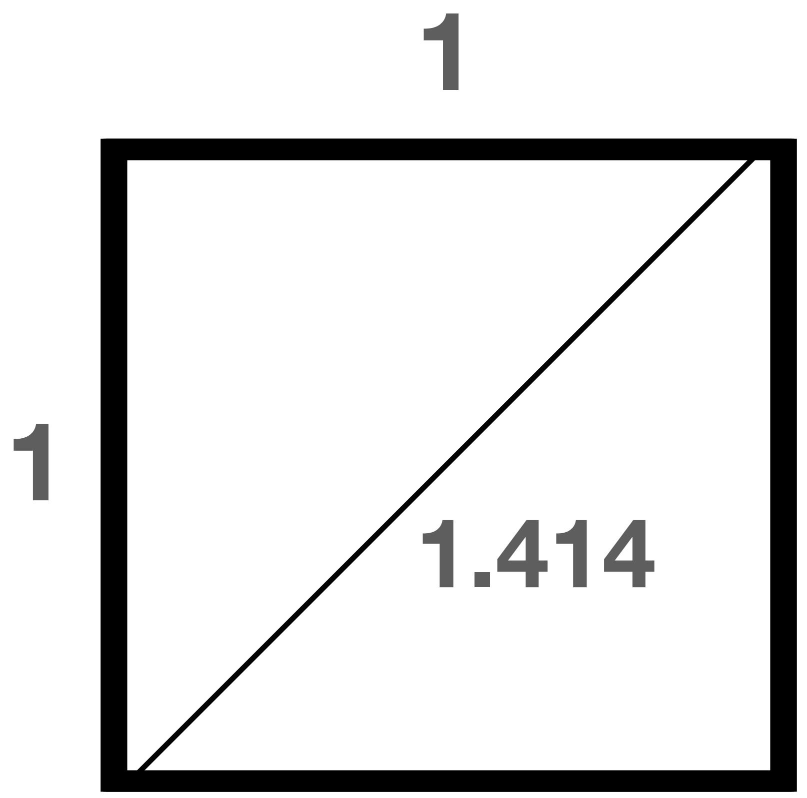
q(3) = 1

Rules: cuts are simple curves, turning over is allowed

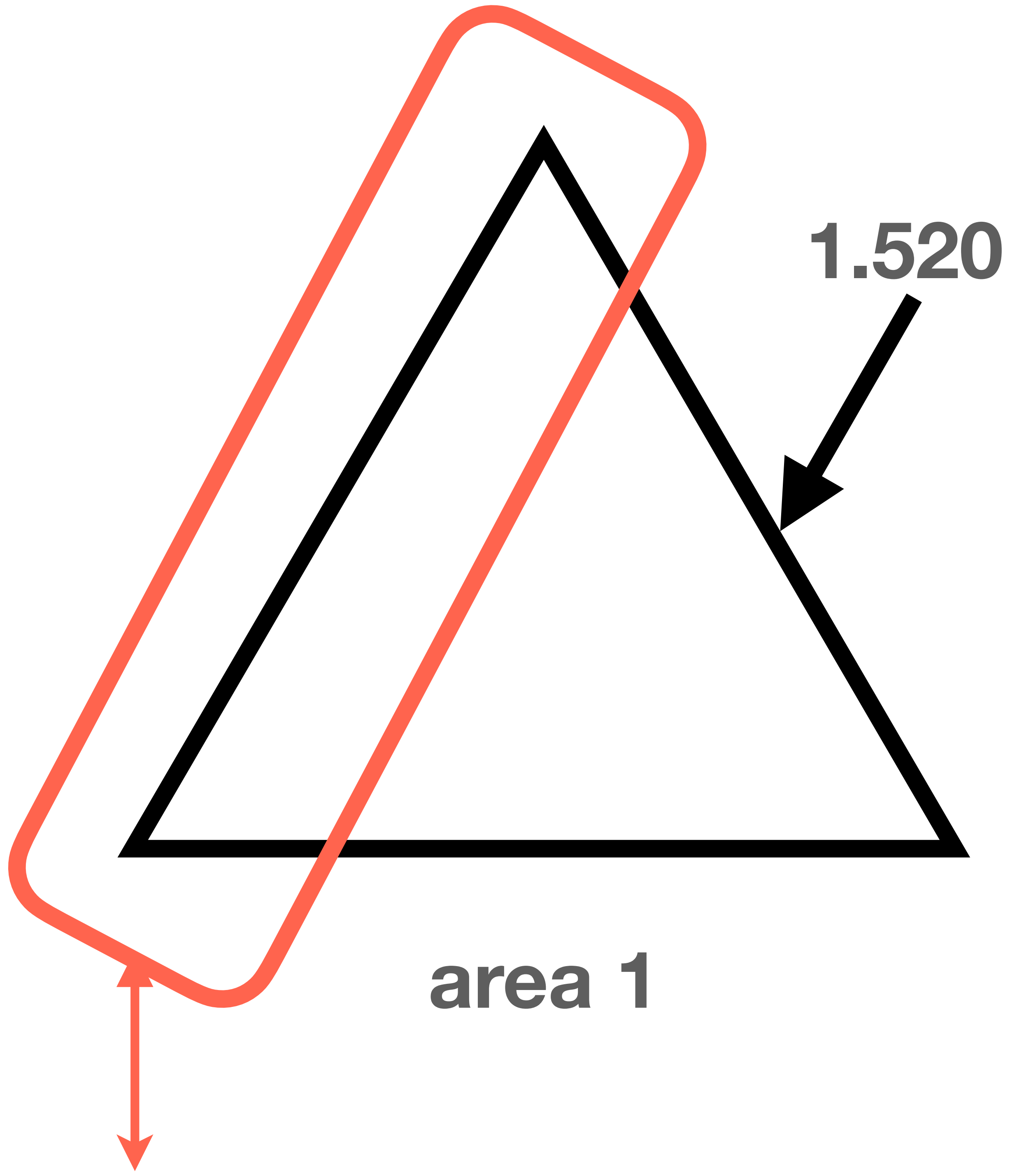



Oh, we're not bouncers. We just can't fit through the door.

# Theorem $s(3) > 2$



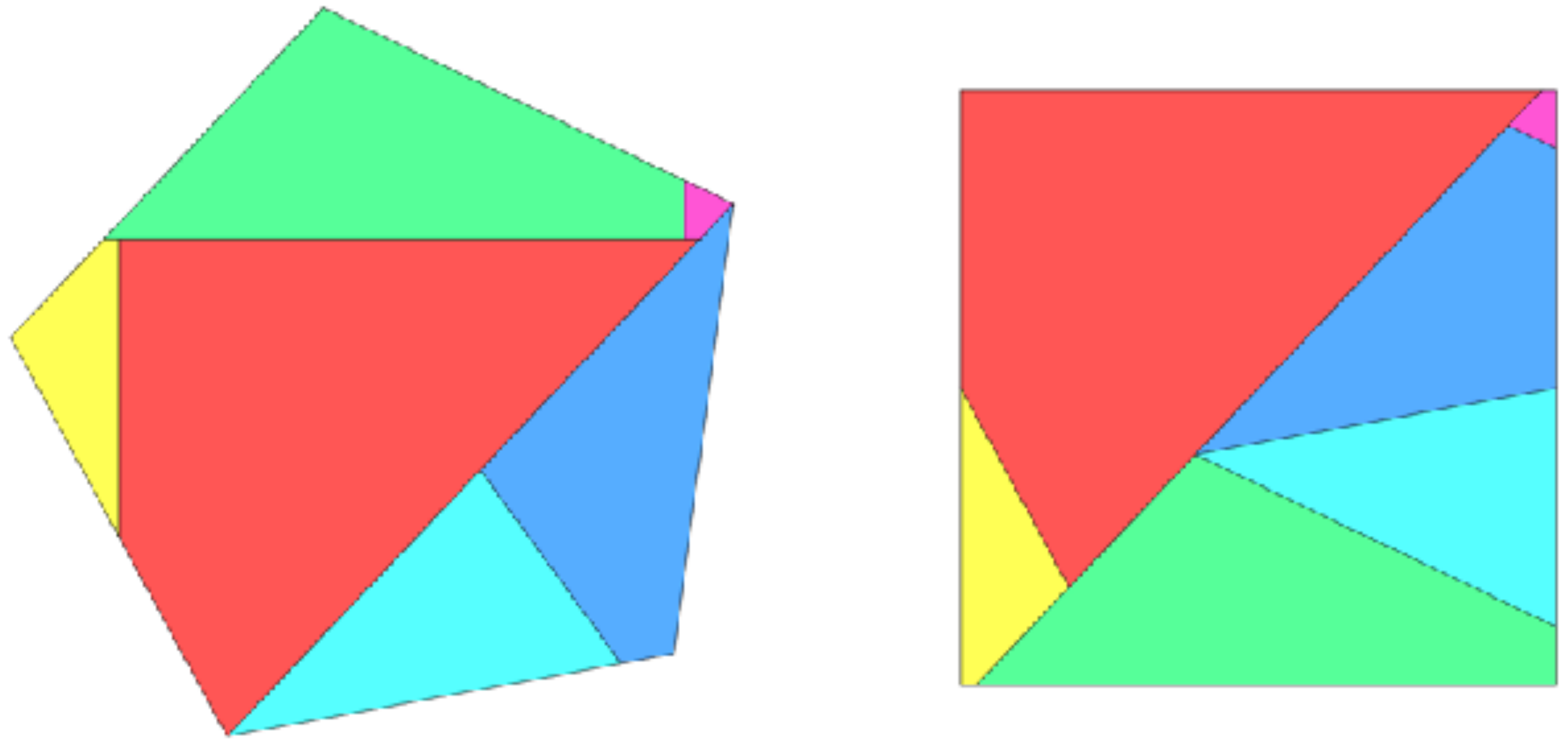

"door"



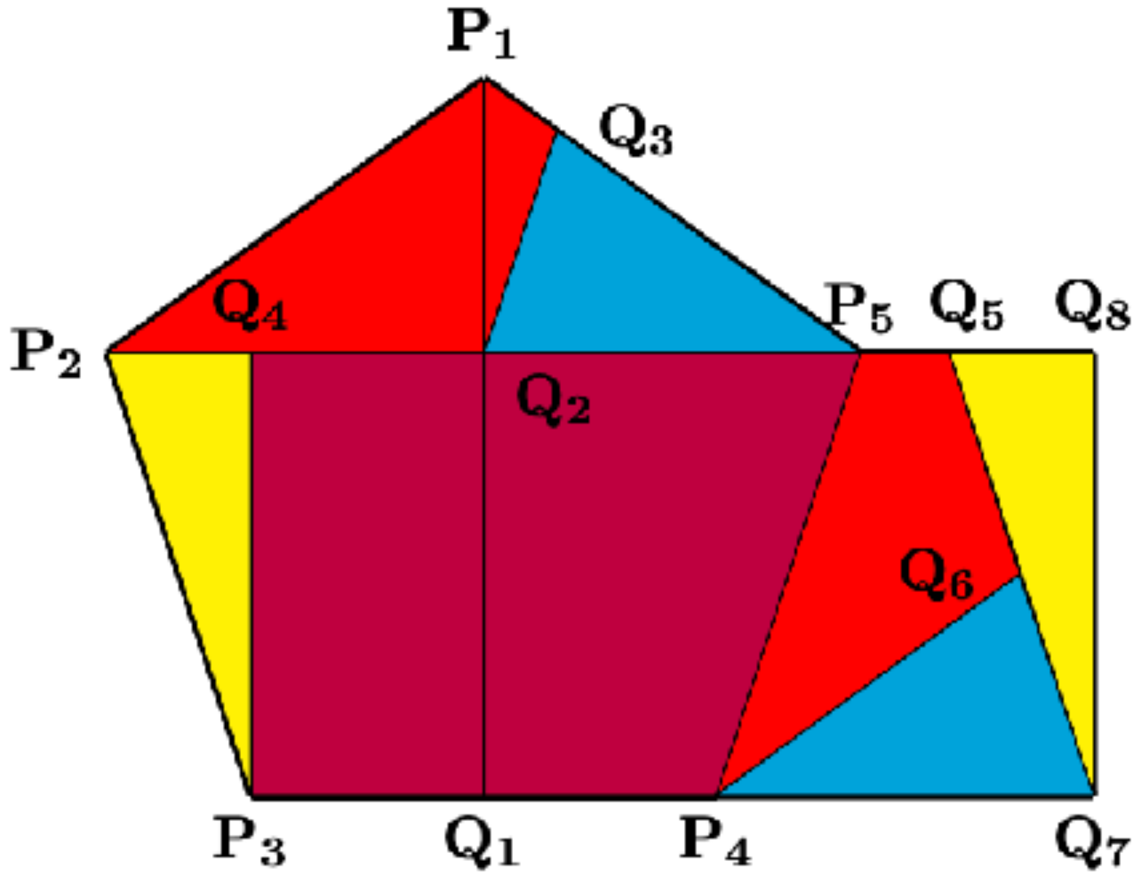

Each piece can contain at most one vertex, so at least 3 pieces!

# Three fundamental sequences from geometry (3)

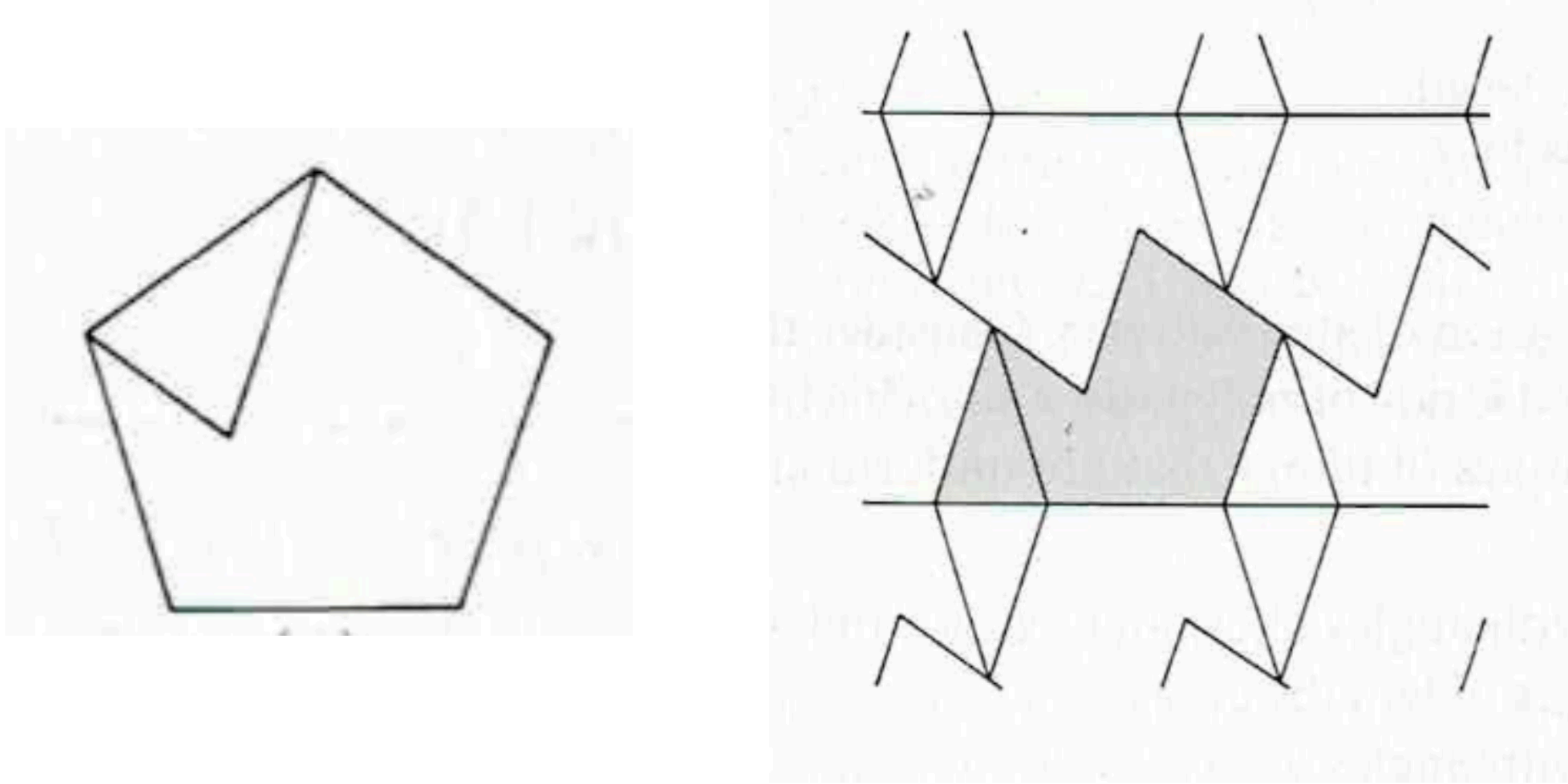
n = 5



s(5) <= 6

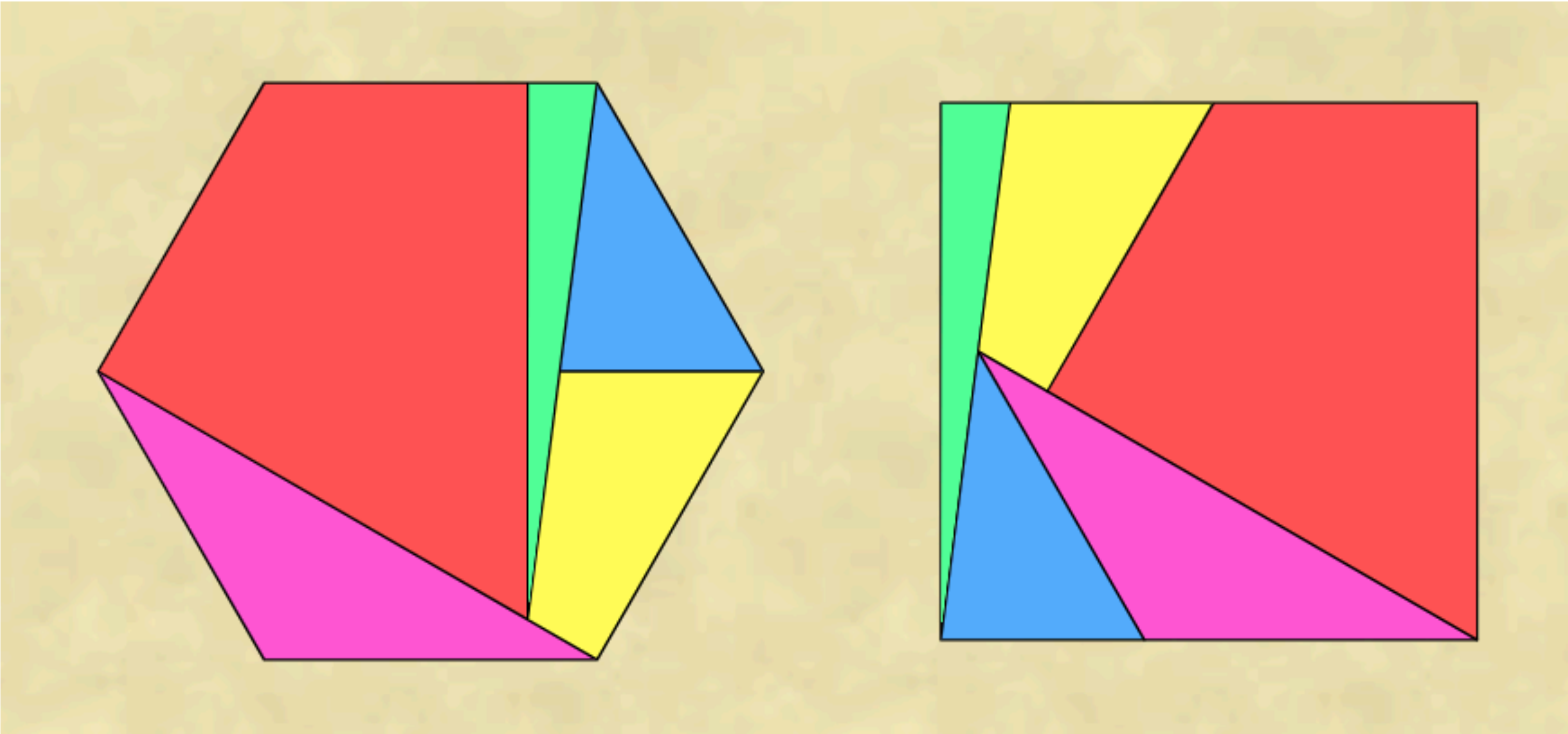



r(5) <= 4

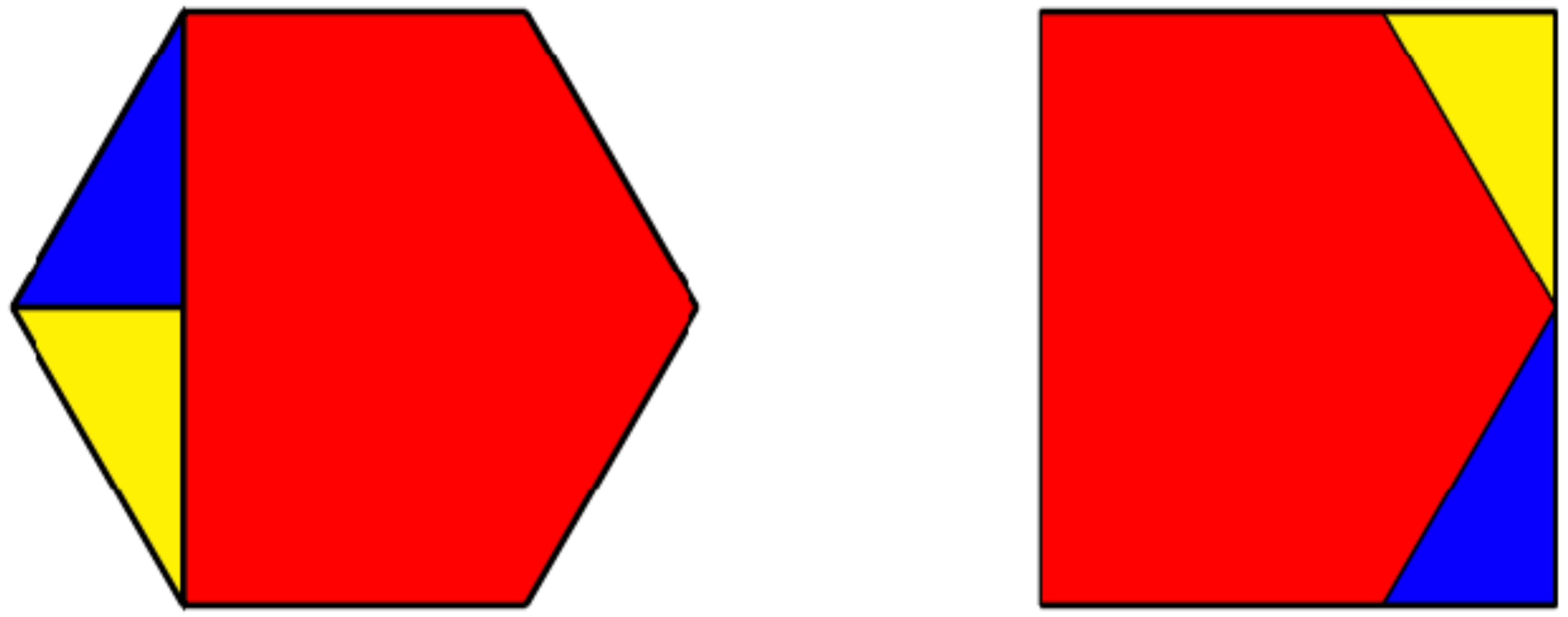


q(5) = 2

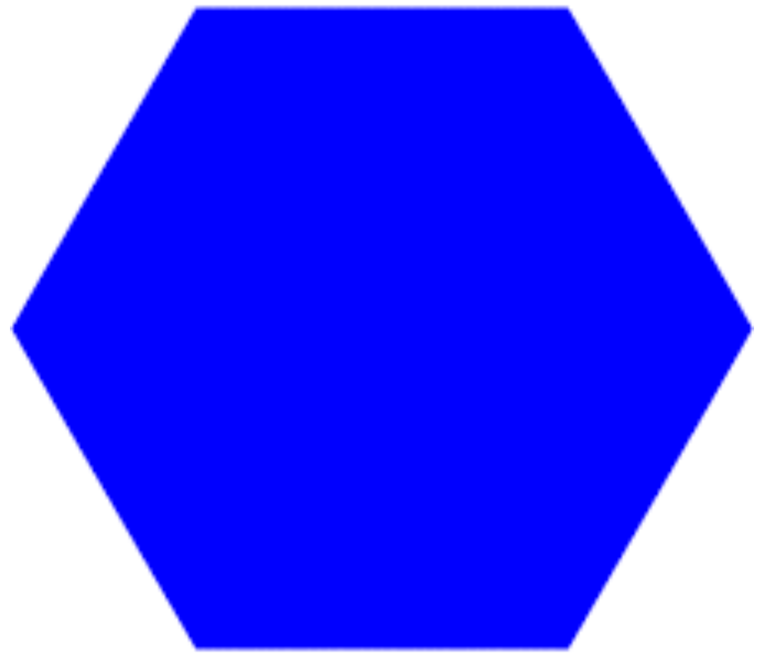
n = 6



s(6) <= 5



r(6) = 3 ?



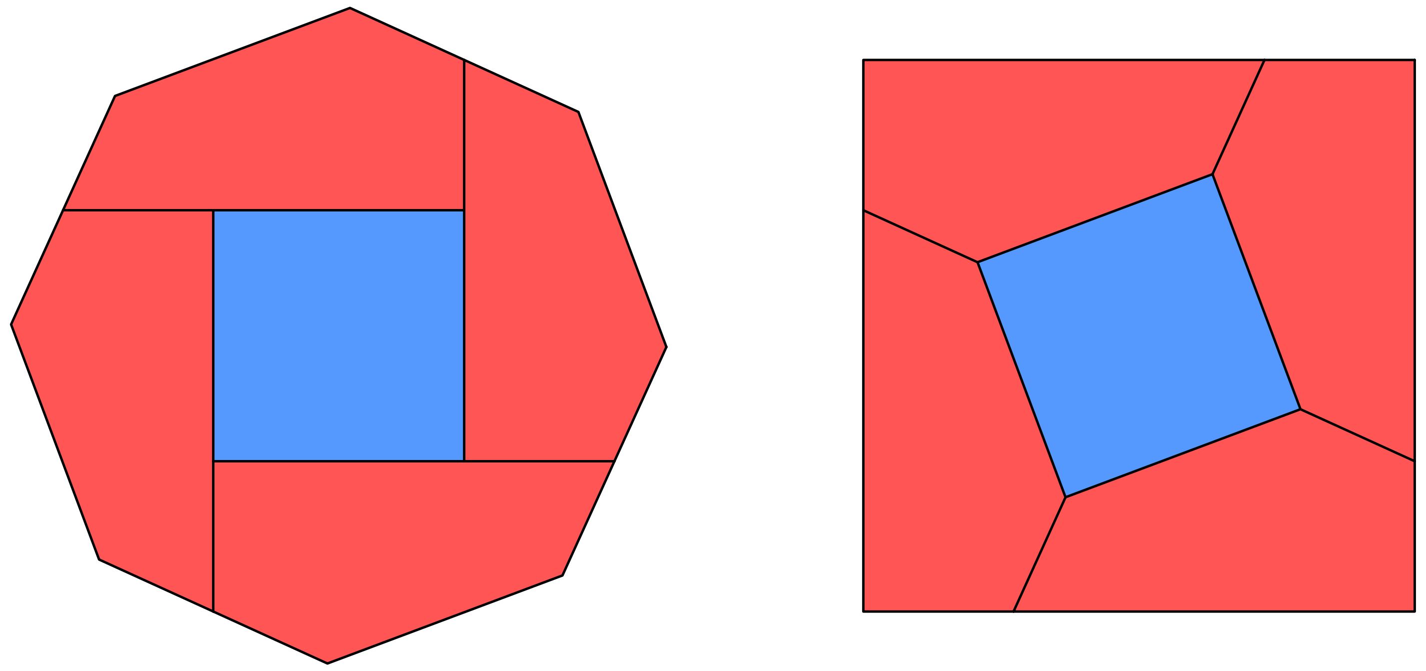
q(6) = 1

# Three fundamental sequences from geometry (4)

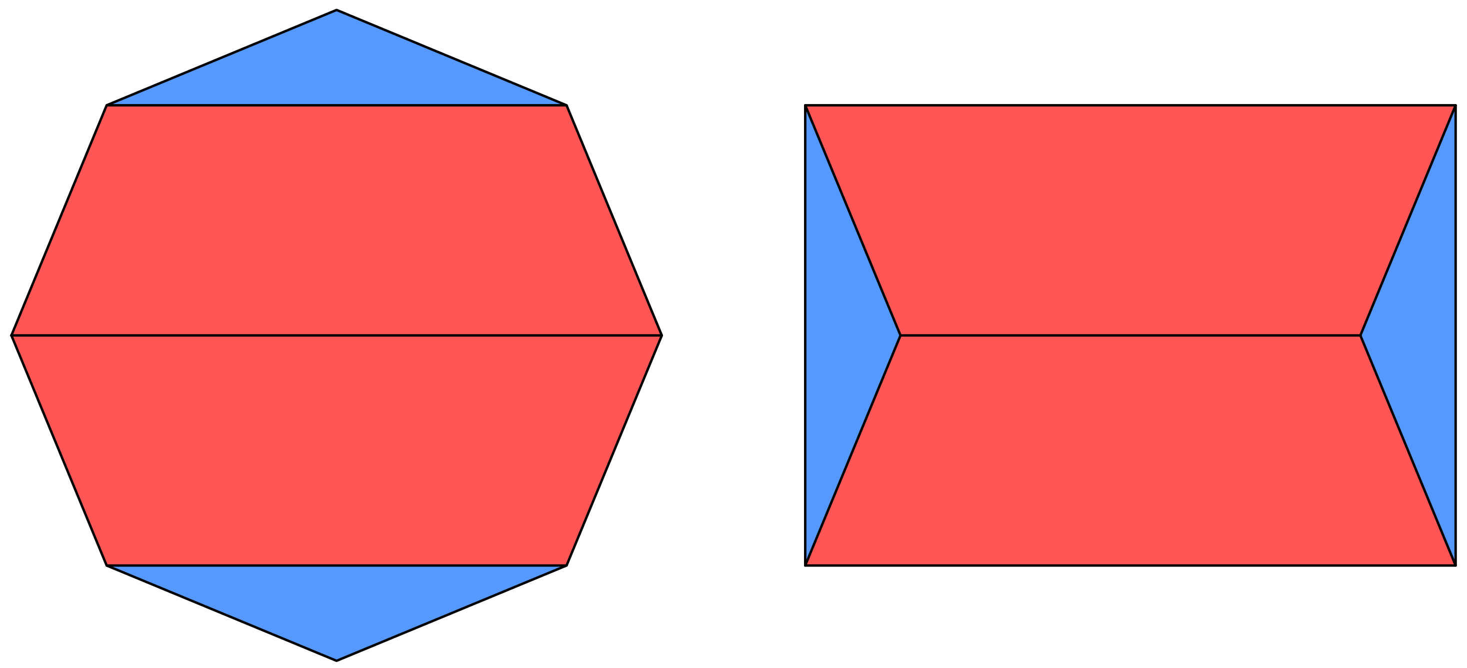
n = 8

Octogon — Monotile

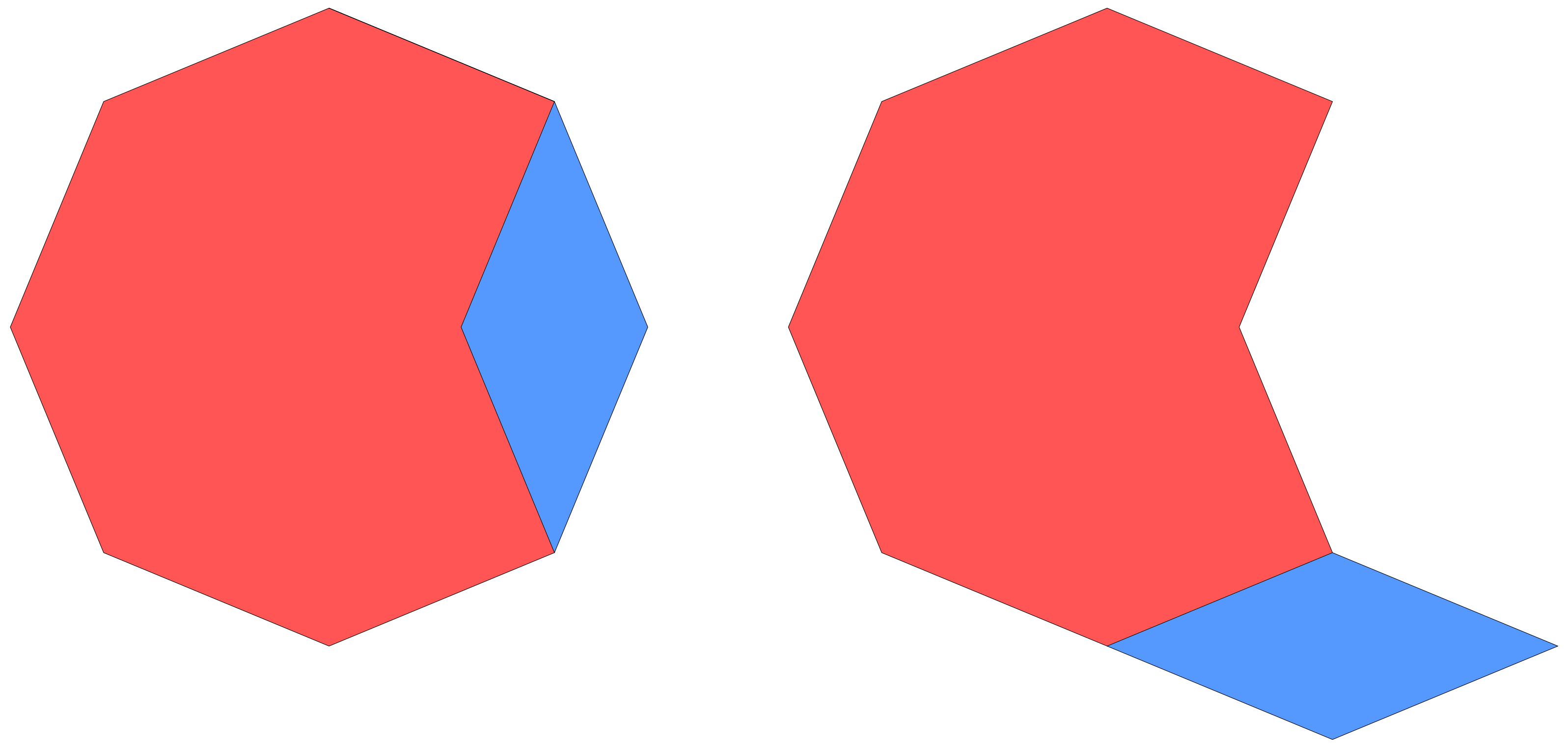
(2 pieces)



s(8) <= 5



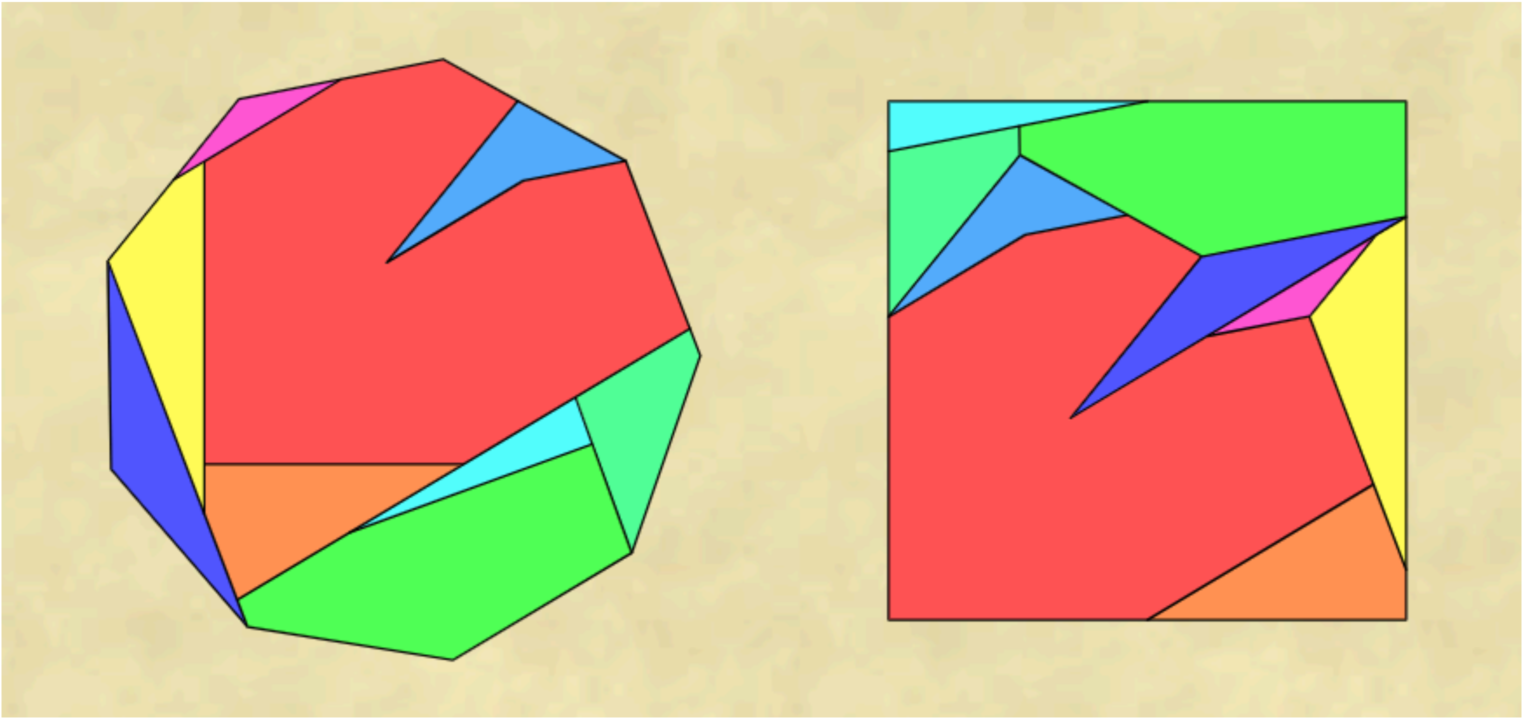
r(8) <= 4



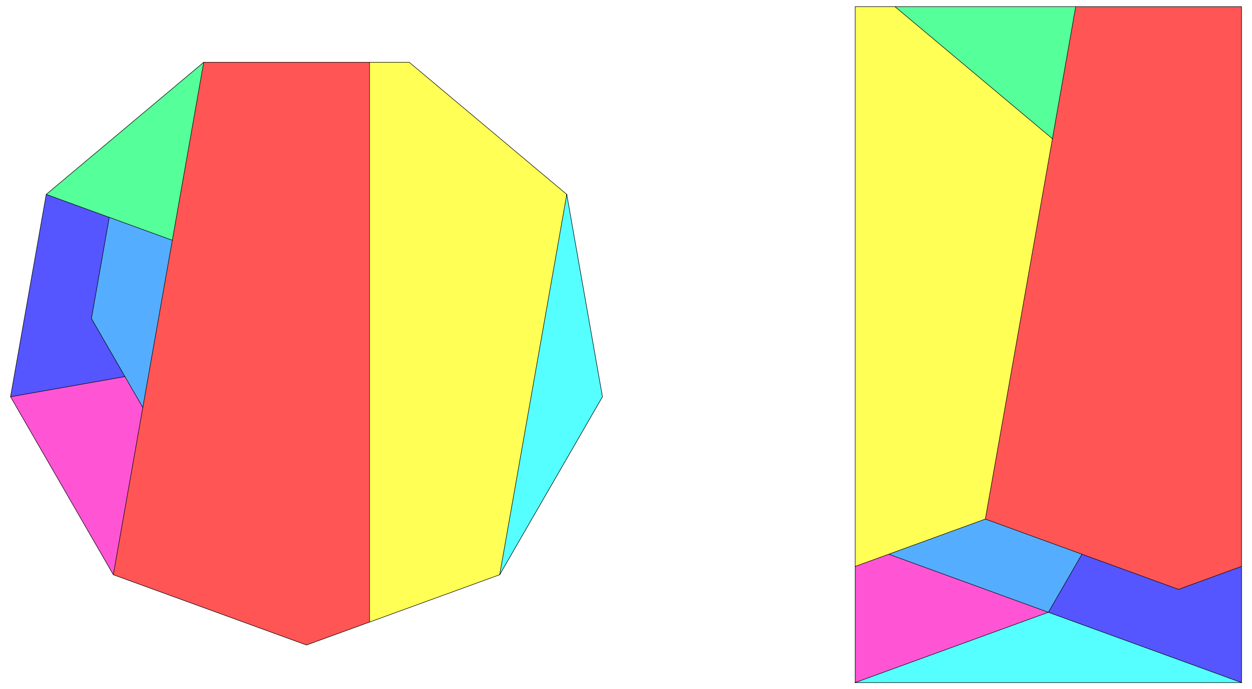
q(8) = 2

# Three fundamental sequences from geometry (5)

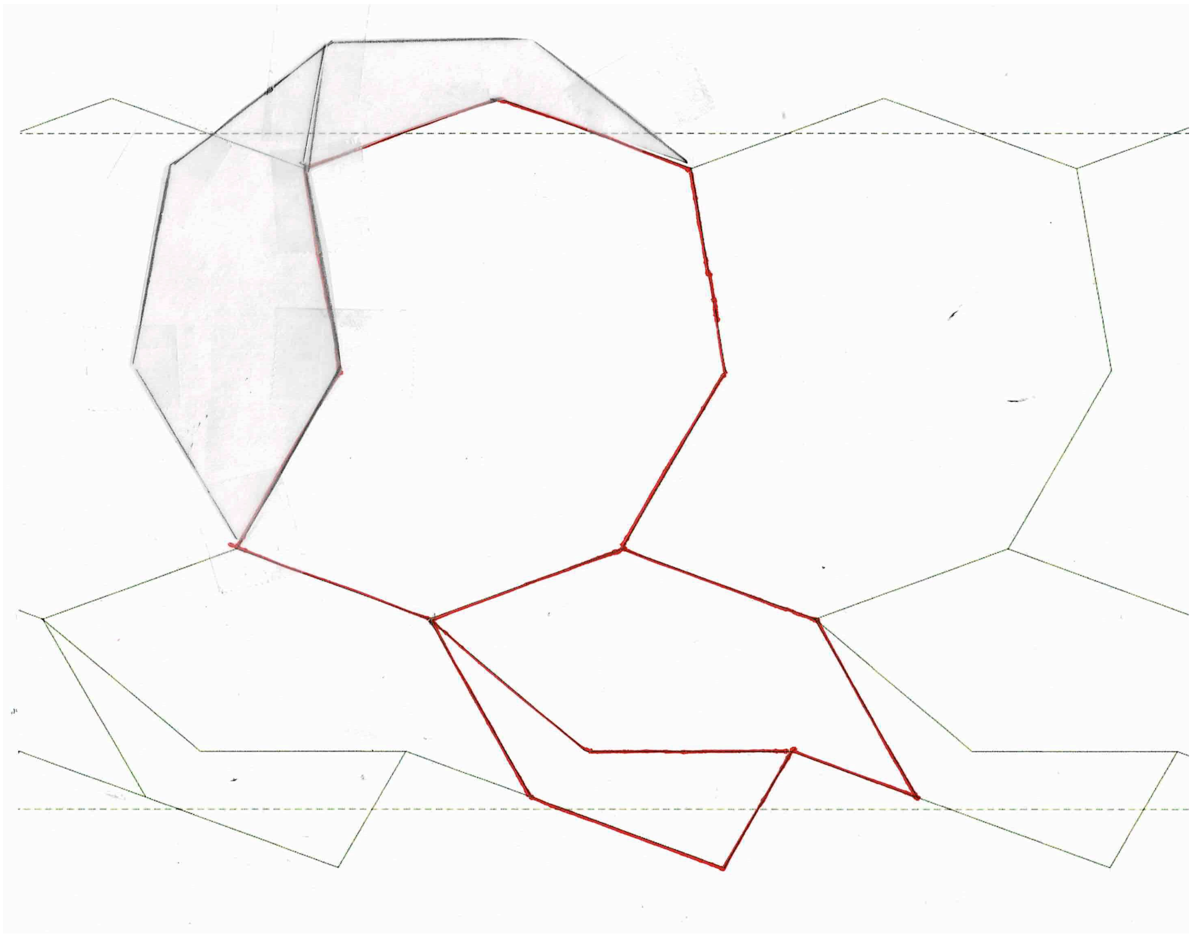
n = 9



s(9) <= 9



r(9) <= 7



q(9) <= 3

# Three fundamental sequences from geometry (6)

| n = | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 11 | 12 | OEIS |
|---|---|---|---|---|---|---|---|---|---|---|---|
| s(n) <= | 4 | 1 | 6 | 5 | 7 | 5 | 9 | 7 | 10 | 6 | A110312 |
| r(n) <= | 2 | 1 | 4 | 3 | 5 | 4 | 7 | 4 | 9 | 5 | A362939 |
| q(n) <= | 1 | 1 | 2 | 1 | 3 | 2 | 3 | 2 | 4 | 3 | A362938 |

General reference:
Gavin Theobald, Geometric Dissections database, http://gavin-theobald.uk

For r(n), see N. J. A. Sloane and Gavin A. Theobald, On Dissecting Polygons into Rectangles, arXiv:2309.14866 [math.CO], 2023.

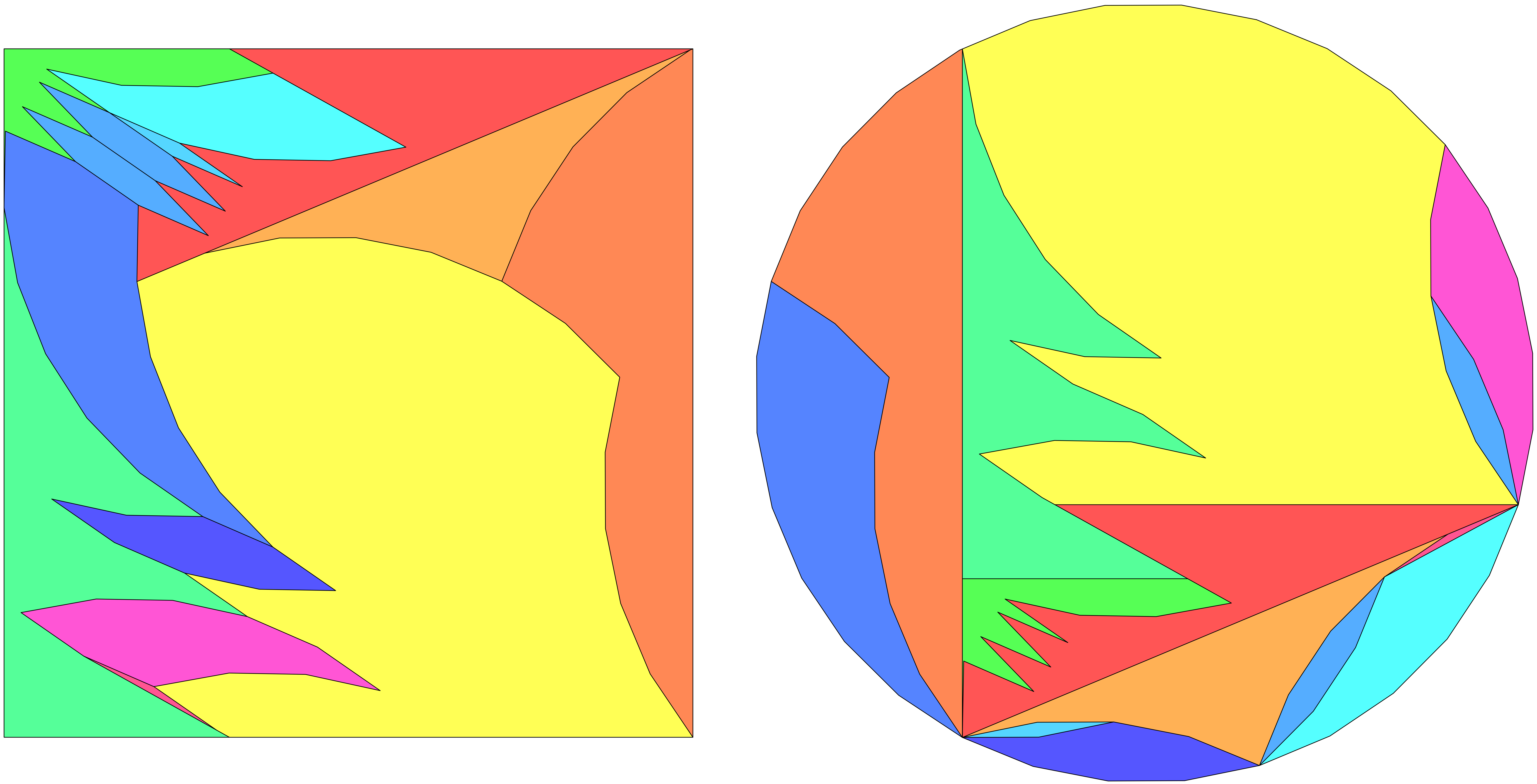
The three OEIS entries also have further illustrations

# Square — 32-gon

(14 pieces)

s(32) <= 14 from Gavin Theobald's

Geometric Dissections Database

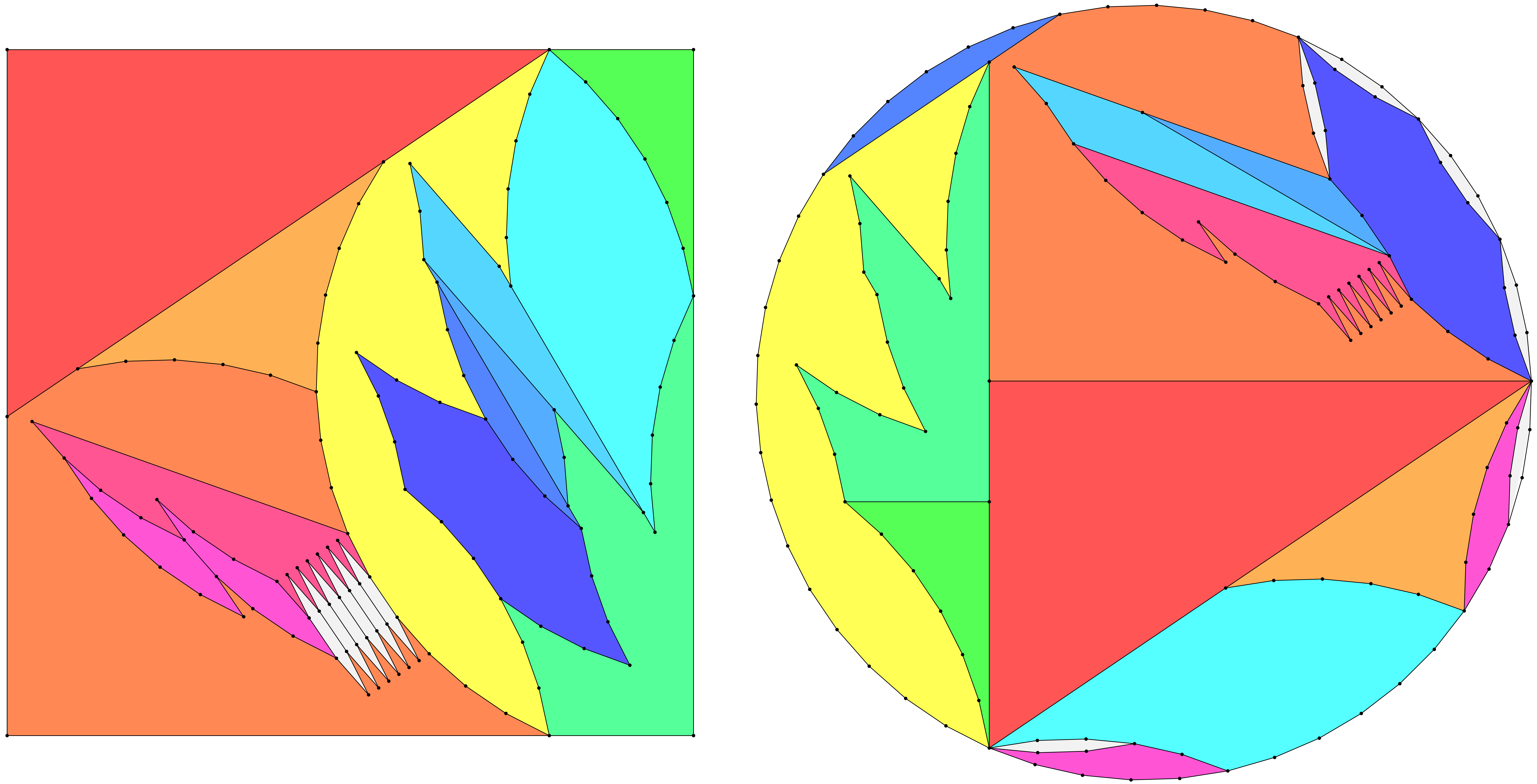


Square — 50-gon

(20 pieces)

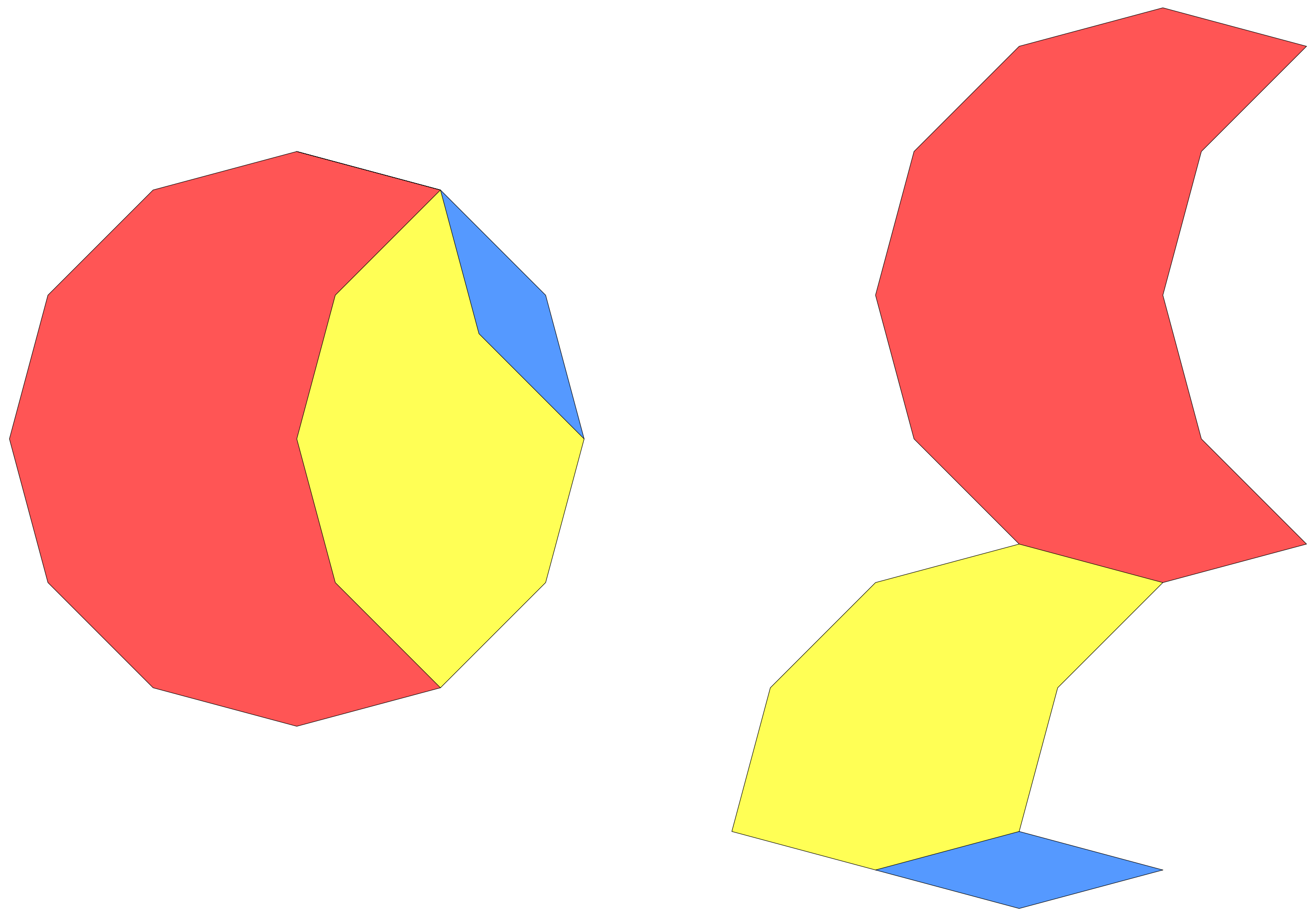
s(50) <= 20 from Gavin Theobald's

Geometric Dissections Database

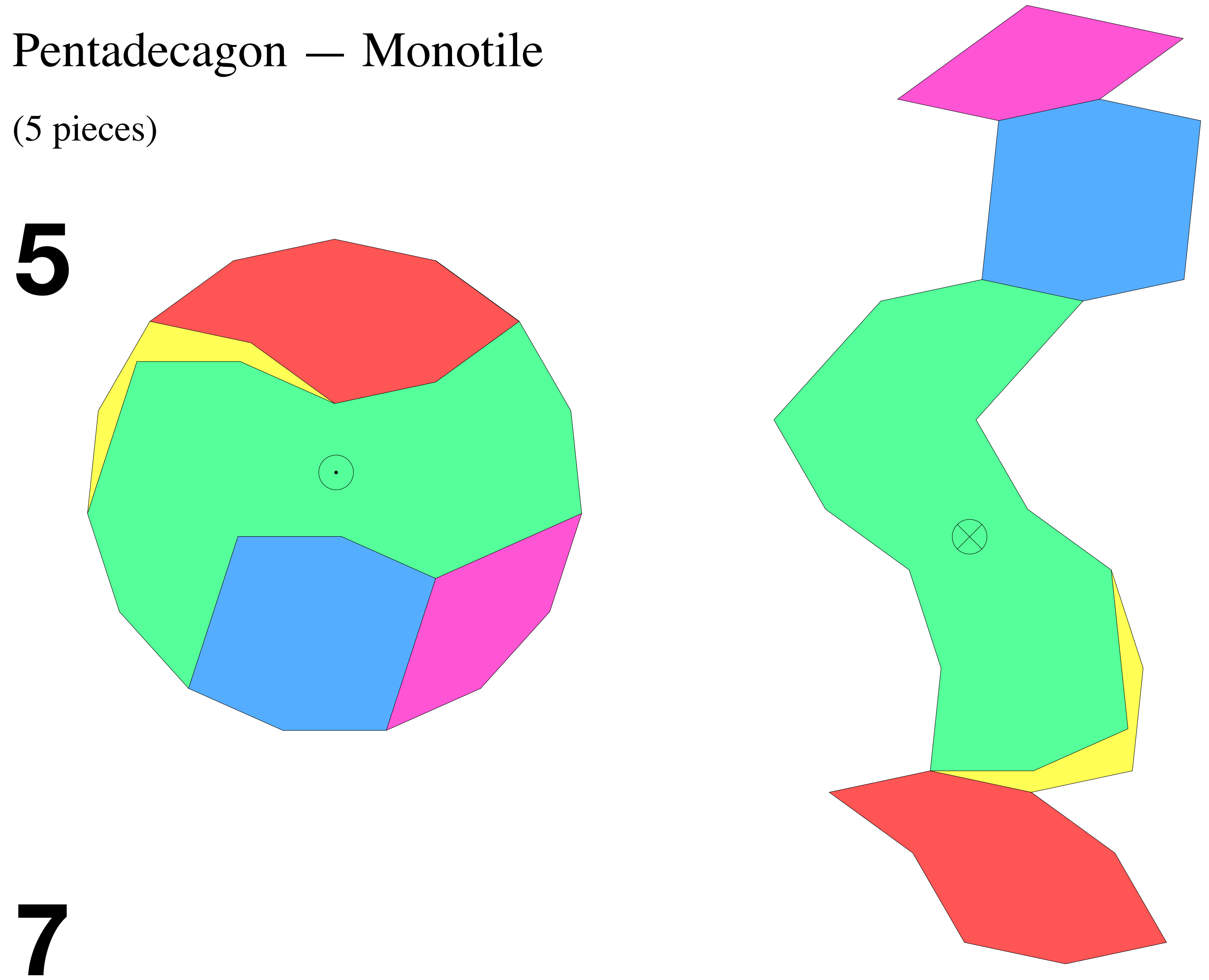


# Monotile dissections from Gavin Theobald

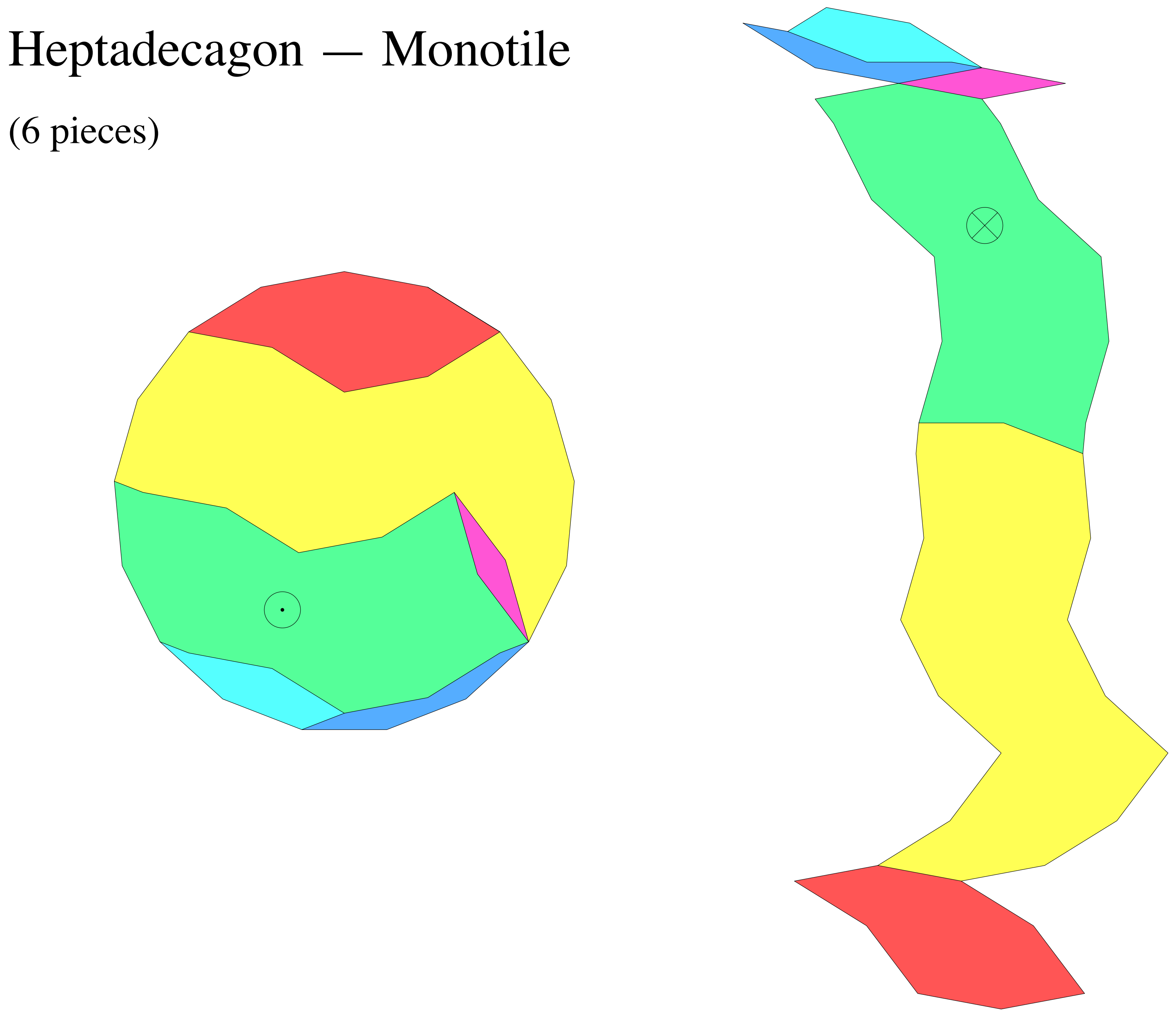
## q(12) <= 3



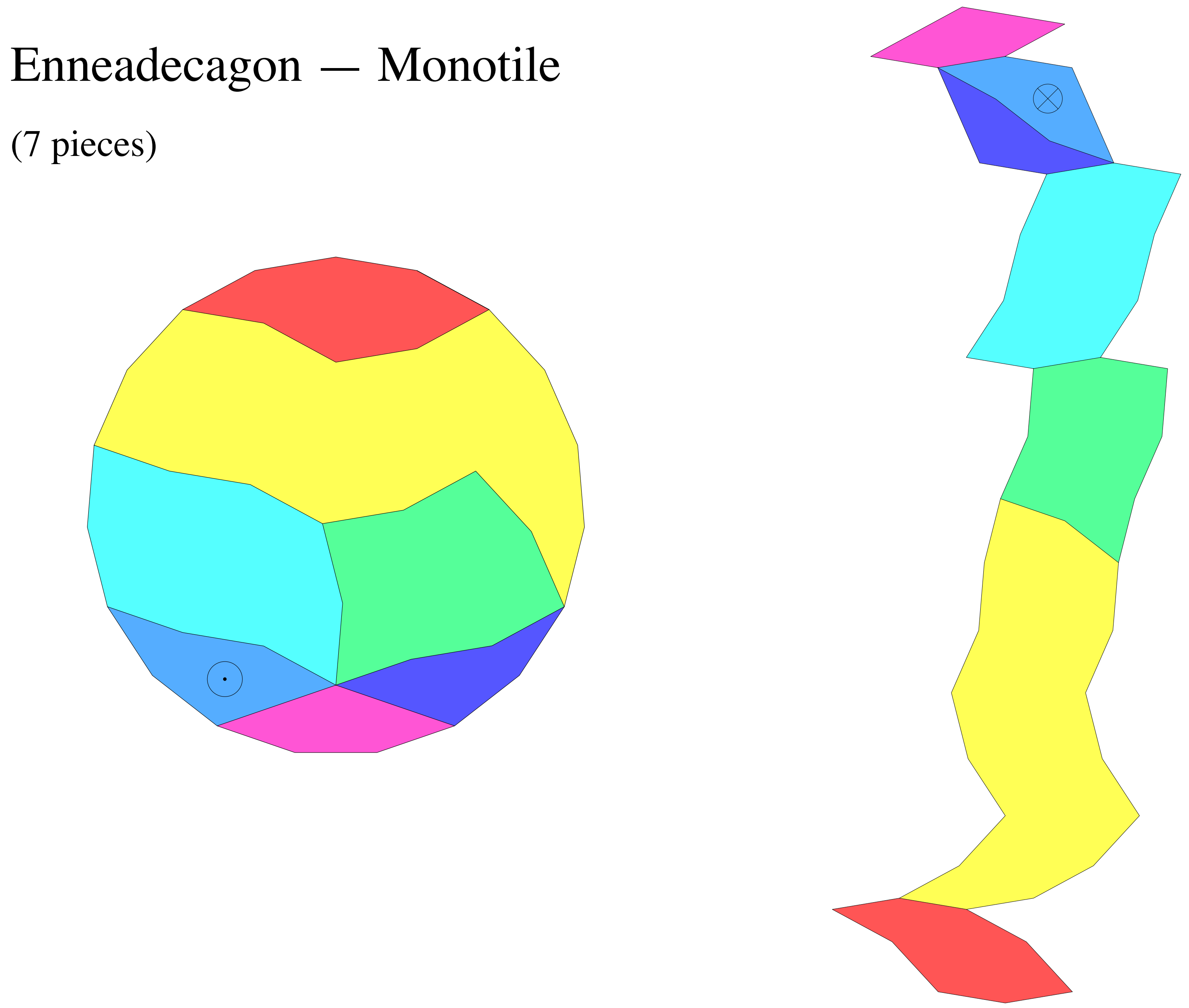
## q(15) <= 5




## q(17) <= 6




## q(19) <= 7




See A362938

# Graphical Enumeration and Stained Glass Windows

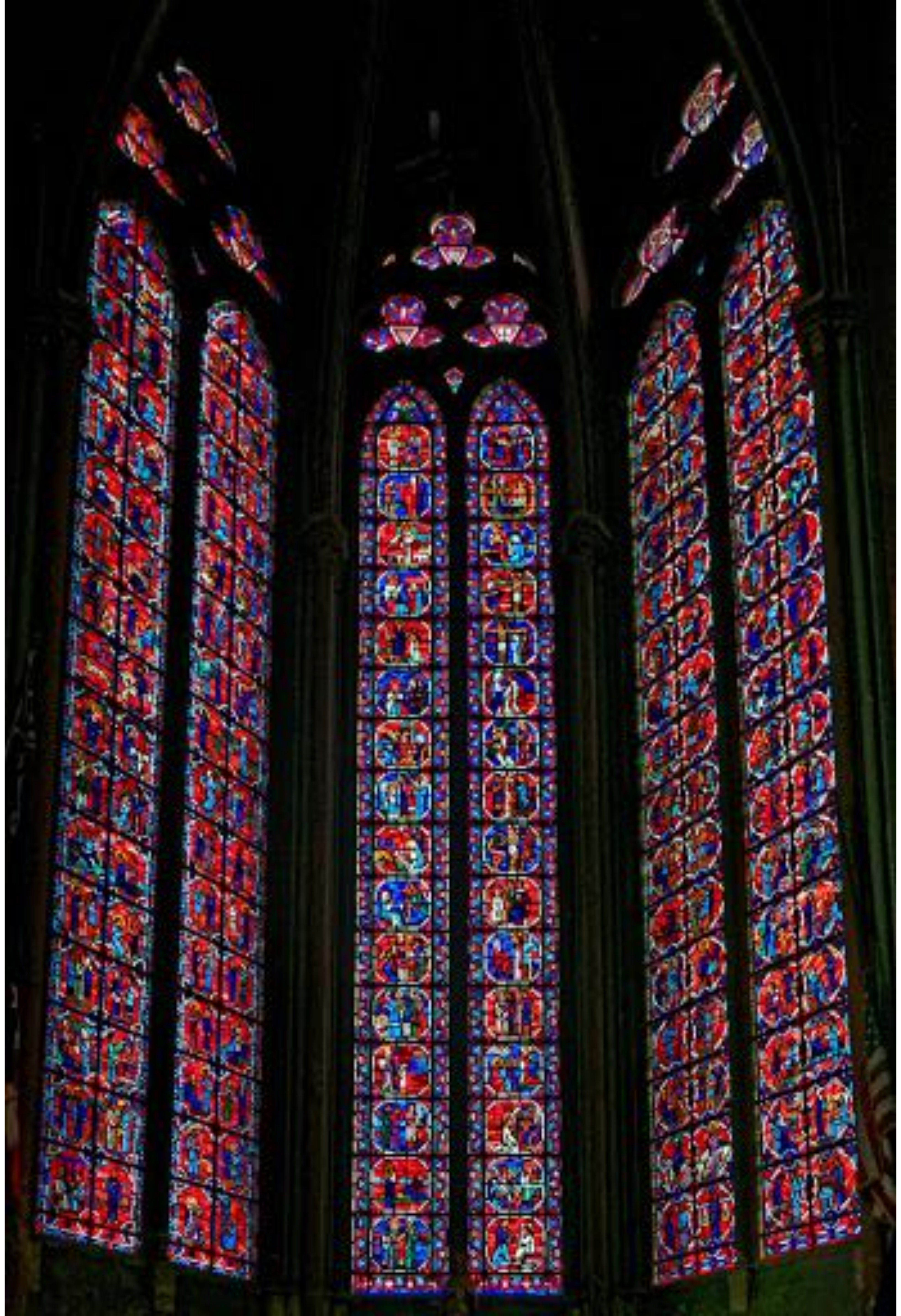
**Lars Blomberg, Scott Shannon, and NJAS**

**Part 1 is on the arXiv and has been published in INTEGERS**

Rose window

Amiens, France

**Sainte-Chapelle, Paris**



# Motivation

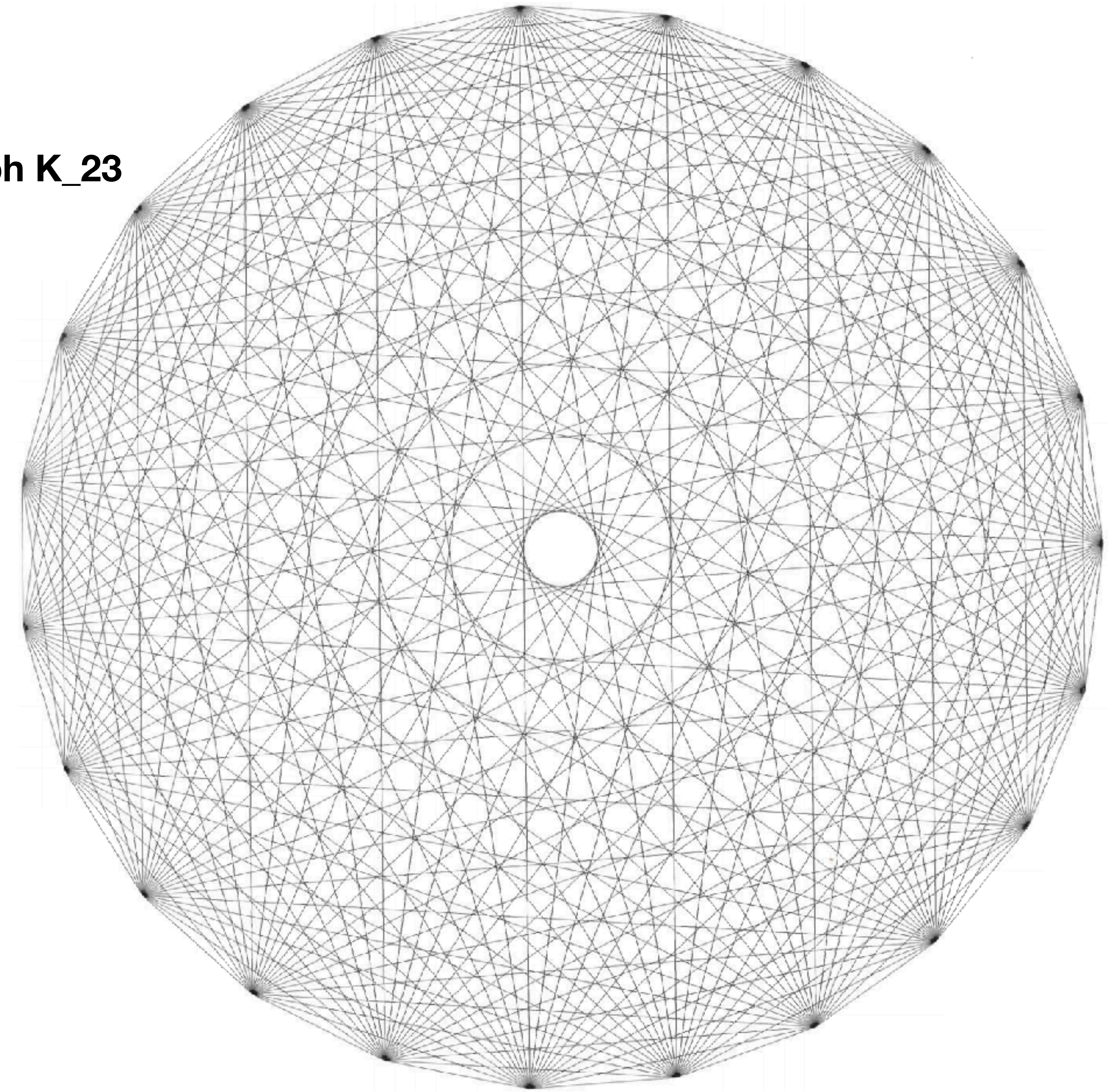
1. Extend work of Poonen-Rubinstein on K_n, and Legendre-Griffiths on K_{n,n} to other families of graphs

2. Desire to create our own stained glass windows, in homage to Amiens, Sainte-Chapelle, Chartres, Strasbourg.

**Our motto: "If you can't solve it, make art"**

# Planar Graphs and Stained Glass Windows (2)

Complete graph K_23



9086 cells (R)
8878 nodes (V)
17963 edges (E)

Solved by Poonen and Rubinstein 1998

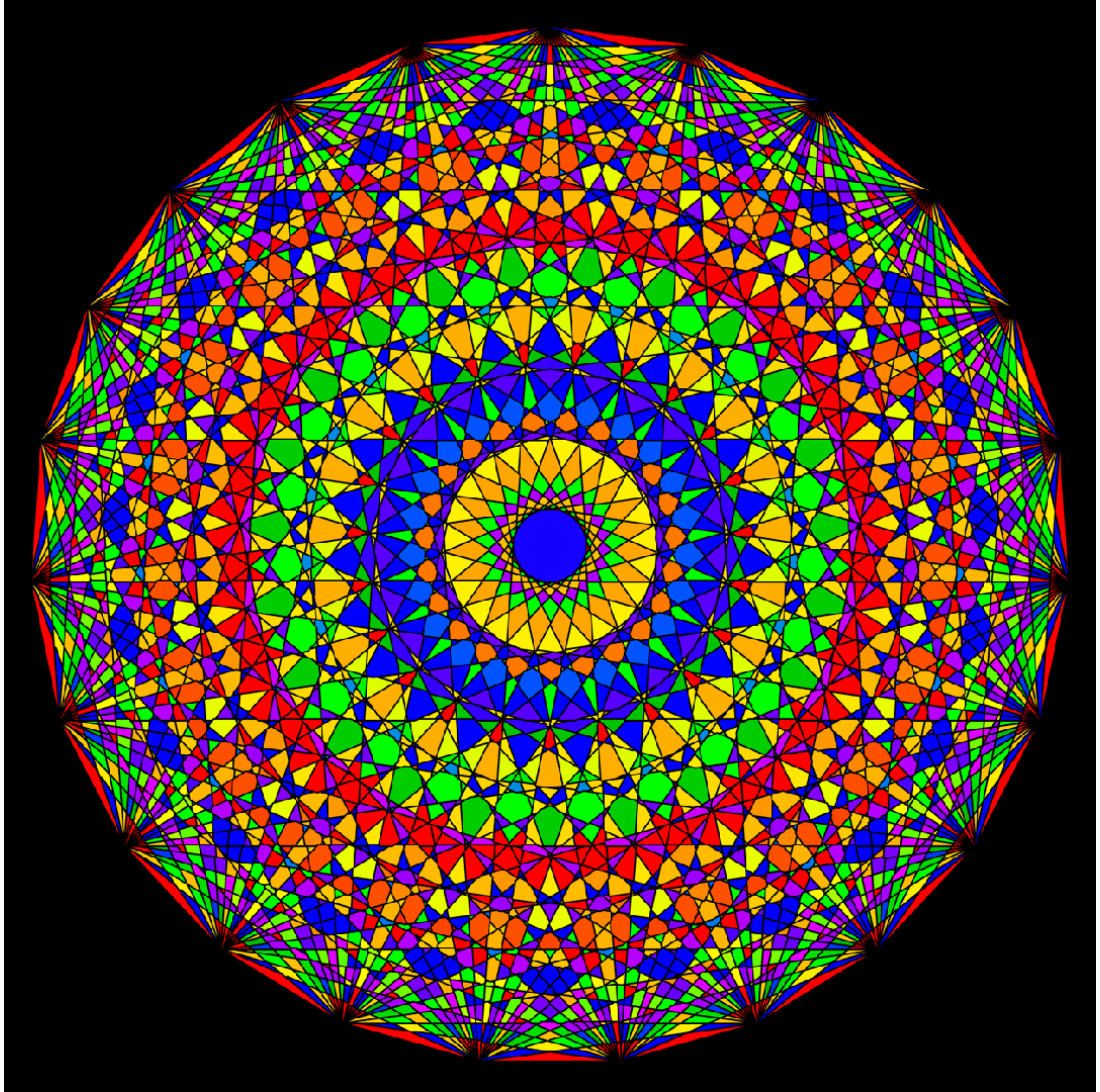
Euler says
E = R+V-1.

R and V about equal tells us most crossings are simple.

Here n is odd, so all crossings are simple.

## Planar Graphs and Stained Glass Windows (3)

**Complete graph**
**K_23**
**with 9086 cells.**
**Colored by our**
**special algorithm.**

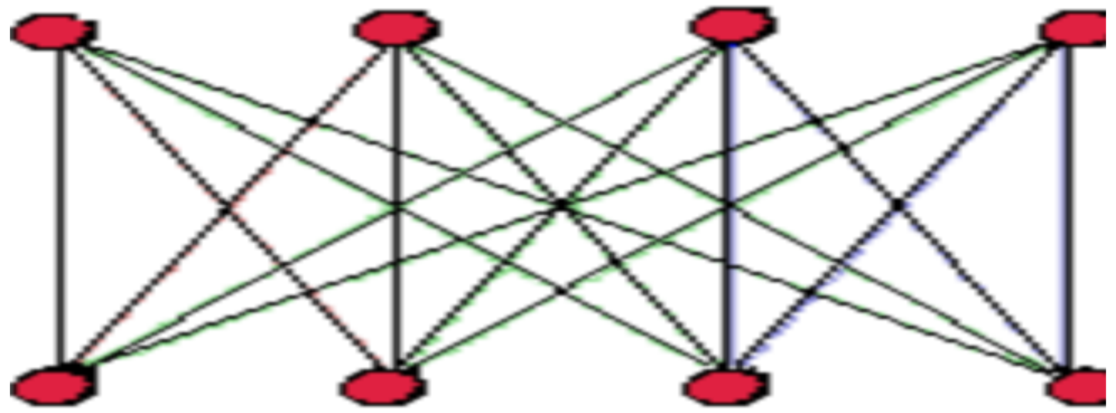


# The Two Known Results

1. **Poonen and Rubinstein, 1998: Number of nodes and cells in K_n :**

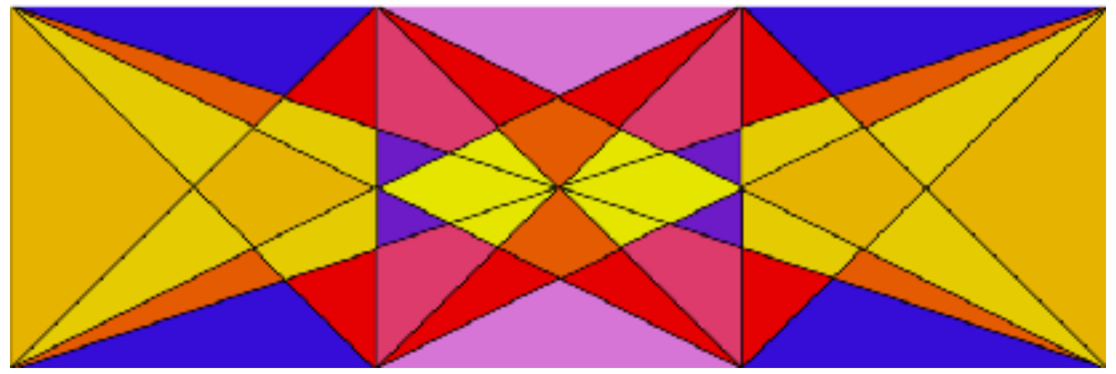
**Basically** $\binom{n}{4}$ **minus complicated correction terms.**

2. **Legendre (2009), Griffiths (2010), ditto for K_{n,n}.**



**K_{4,4}**

**or equivalently**



**= BC(1,3)**

## Typical problem:

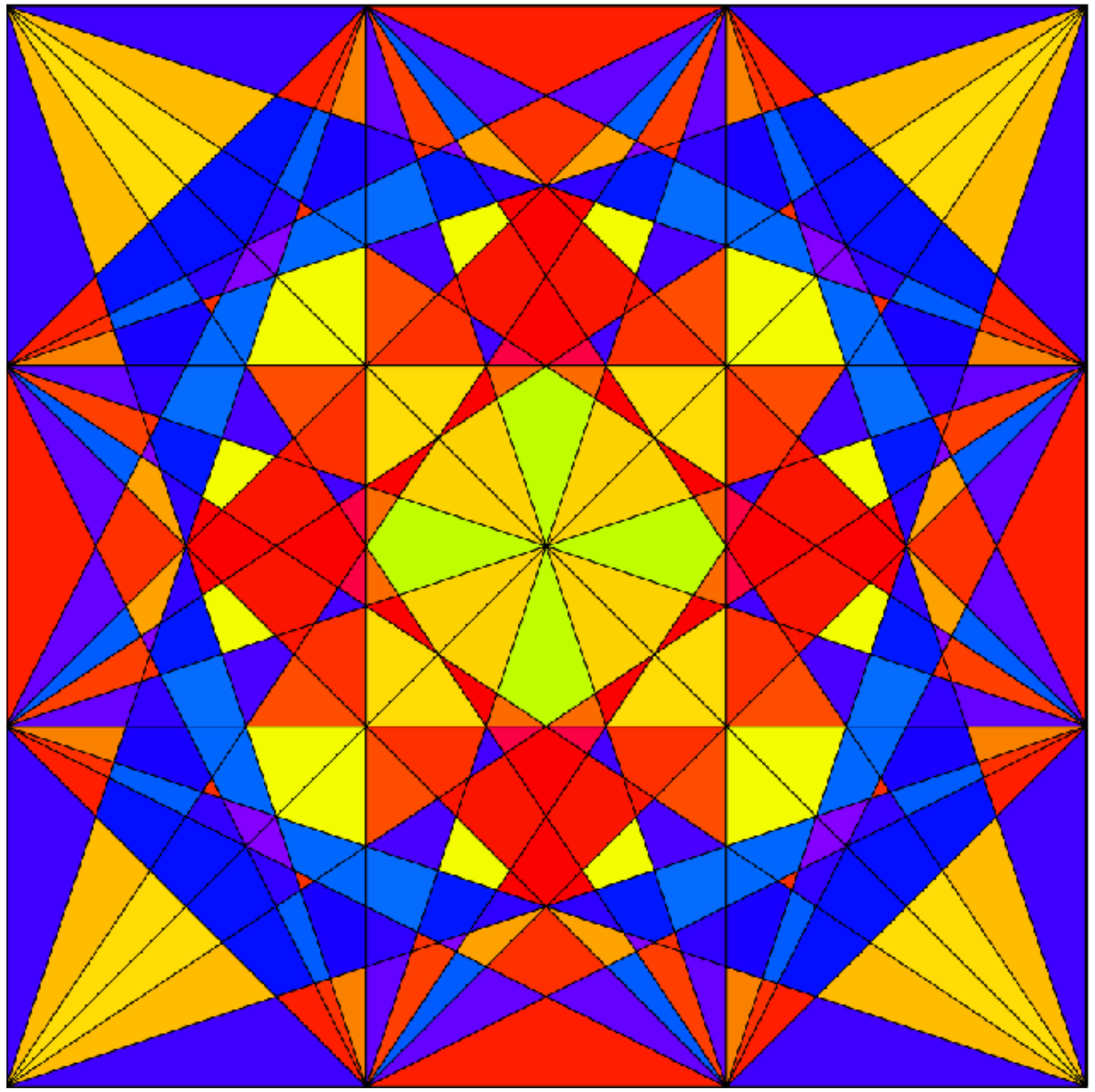
Take m x n grid of squares
or (m+1) x (n+1) grid of points

Join each pair of boundary points
by a chord

In resulting graph, count
vertices, edges, regions.

This is the graph BC(m,n)
“Boundary Chords”

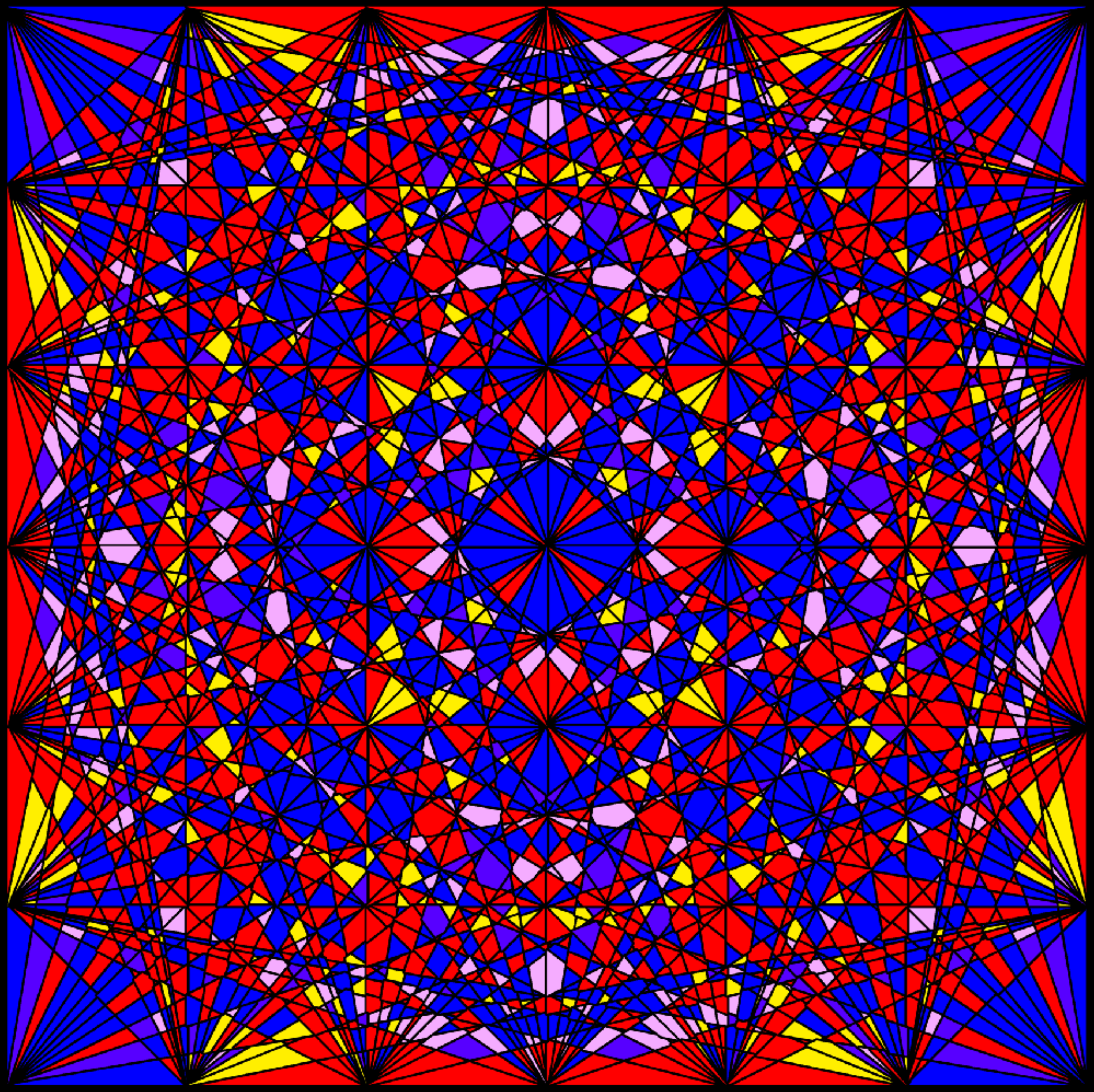
A331452 has many
pictures of BC(m,n) stained
glass windows.



3 x 3 grid of squares

**BC(3,3)**

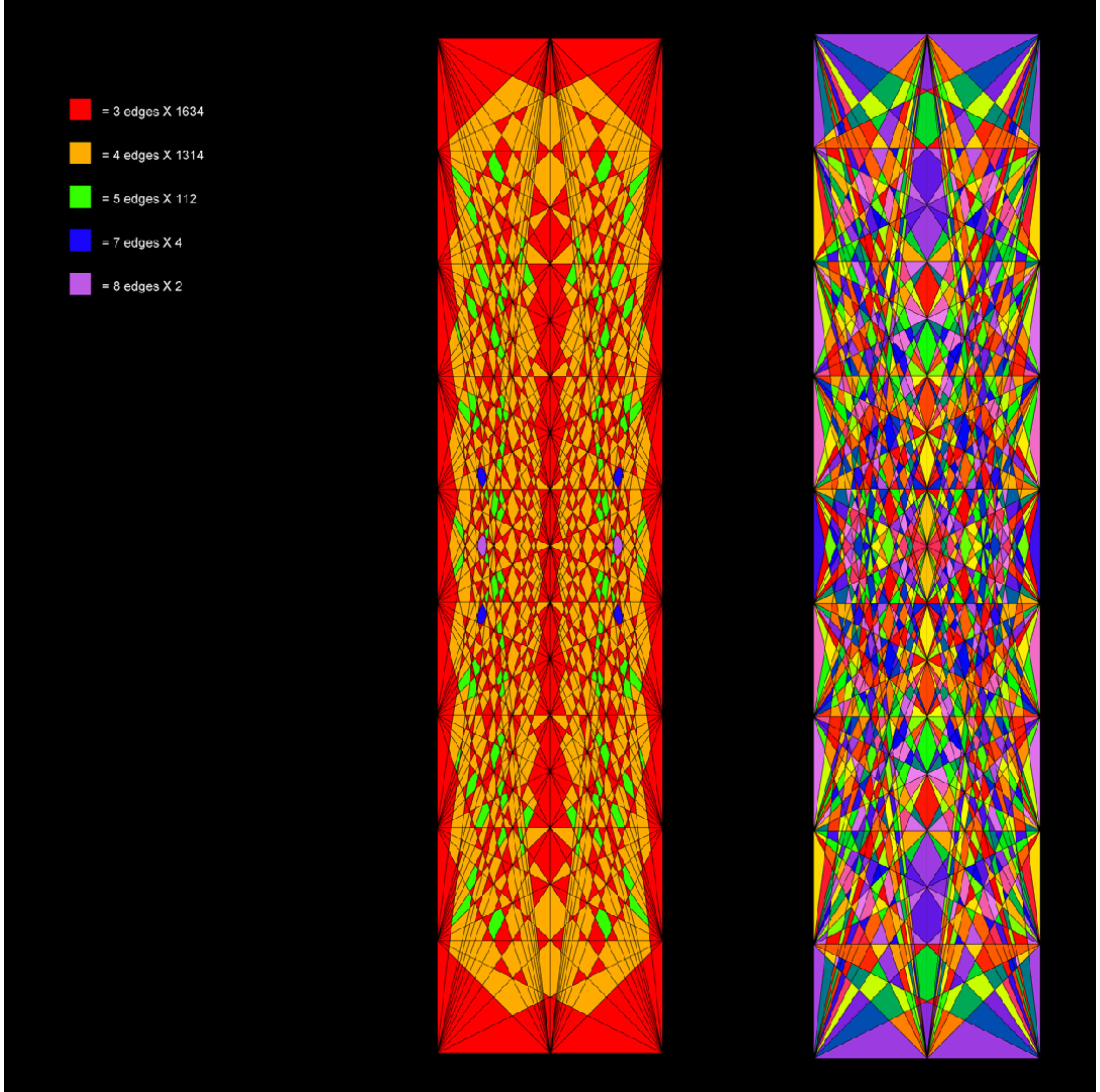
See: Blomberg, Shannon, NJAS,
Graphical enumeration and stained
glass windows I, Integers, 2022.



# Planar Graphs and Stained Glass Windows (6)

BC(6,6)
6X6 grid of squares
Join every pair of boundary points by a chord.
6264 = A265011(6) regions
4825 = A331449(6) vertices
No formulas known.

Source: Blomberg, Shannon, NJAS, Graphical enumeration and stained glass windows I, Integers, 2022.




# BC(9,2)

**Left: Color-coded to show number of sides: 3 (red), 4 (orange), 5 (green), 7 (blue), 8 (purple)**

**Right: Same graph, colored using our special algorithm.**

# Answers are known for BC(1,n)

**Theorem** **(Stéphane Legendre (2009) and Martin Griffiths (2010))**

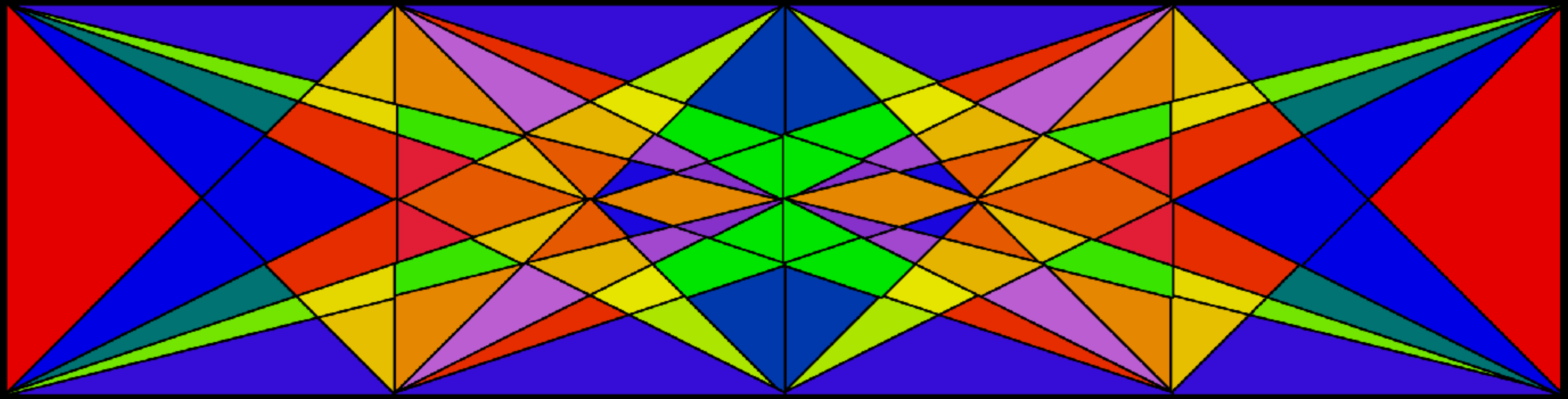
**Define** $$V(m,n,q) = \sum_{a=1..m} \sum_{\substack{b=1..n \\ \gcd\{a,b\}=q}} (m+1-a)(n+1-b)$$

**Nodes in BC(1,n):** $$2(n+1) + V(n,n,1) - V(n,n,2)$$

**Cells in BC(1,n):** $$n^2 + 2n + V(n,n,1)$$

P.G. and S.G.W (10)

# BC(1, 4)



104 cells (70 triangles, 34 quadrilaterals)
but no pentagons or hexagons - why?!

P.G. and S.G.W (11)

# Interior Nodes in BC(1,n)

**It appears that most interior nodes in BC(1,n) are "simple", i.e. are where just two chords cross.**

**For n = 1, 2, 3, … the numbers of simple interior nodes are**

**1, 6, 24, 54, 124, 214, 382, 598, 950, 1334, …**

**A334701 has first 500 terms!**

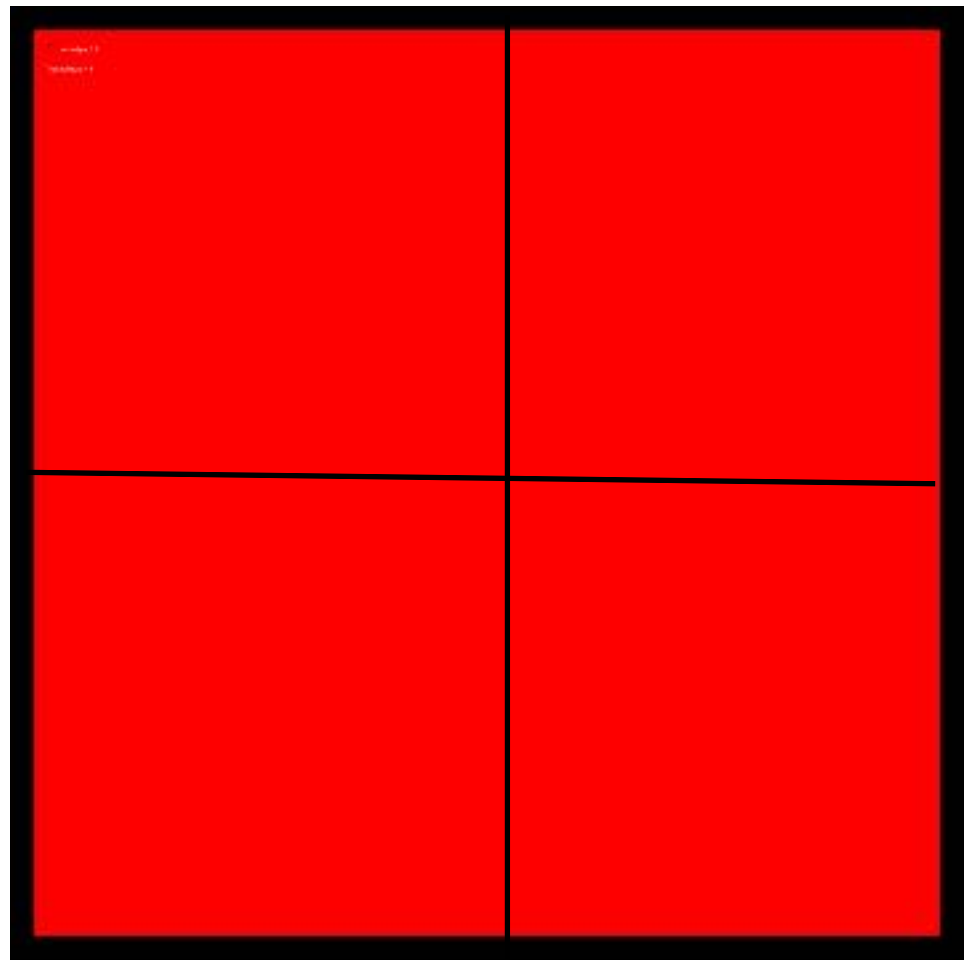
**Open Problem : Find a formula.**

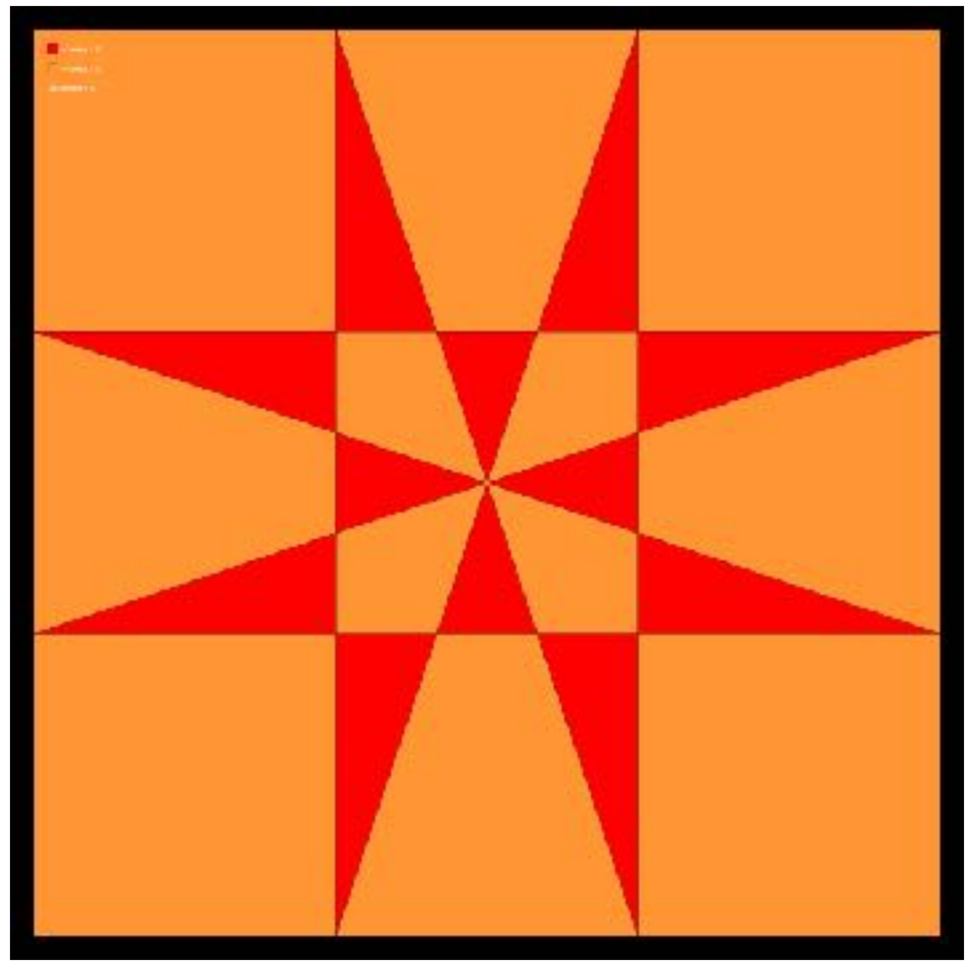
**This is a frequent problem: we have hundreds of terms of a sequence with a simple definition; the OEIS has 365,000 entries: need a smarter guessing program.**

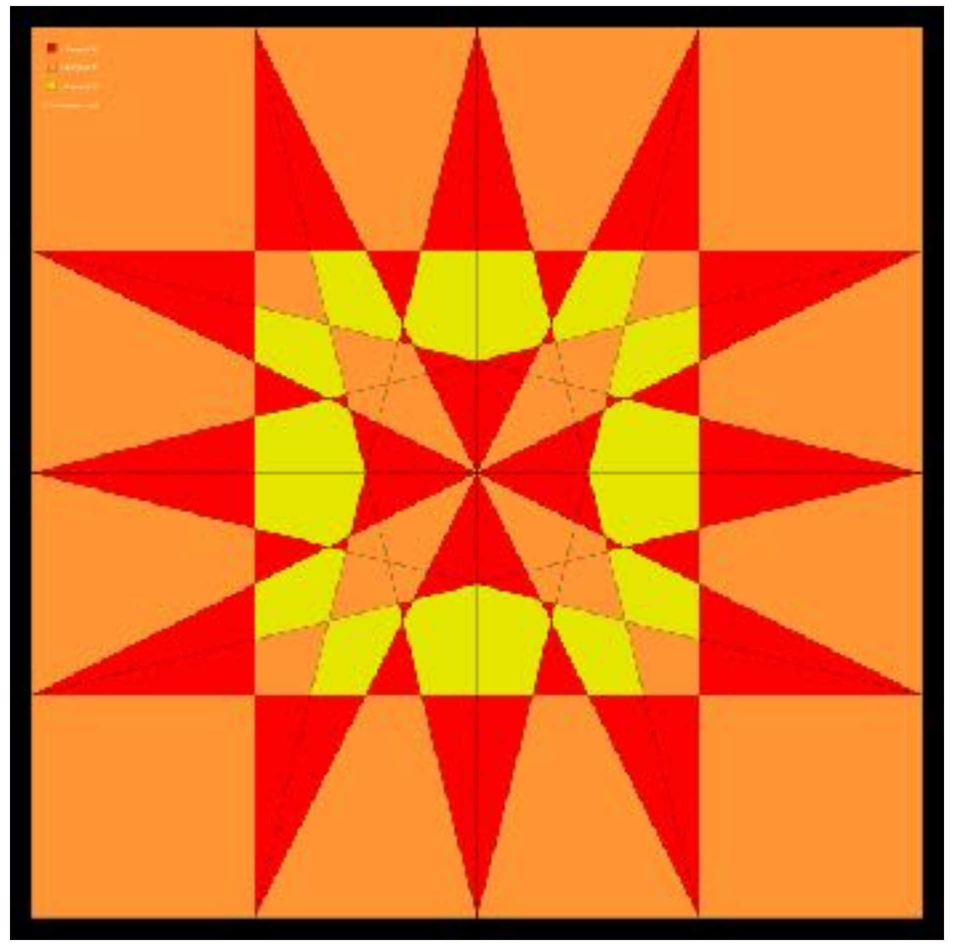
# Scott Shannon's Sequence A355798

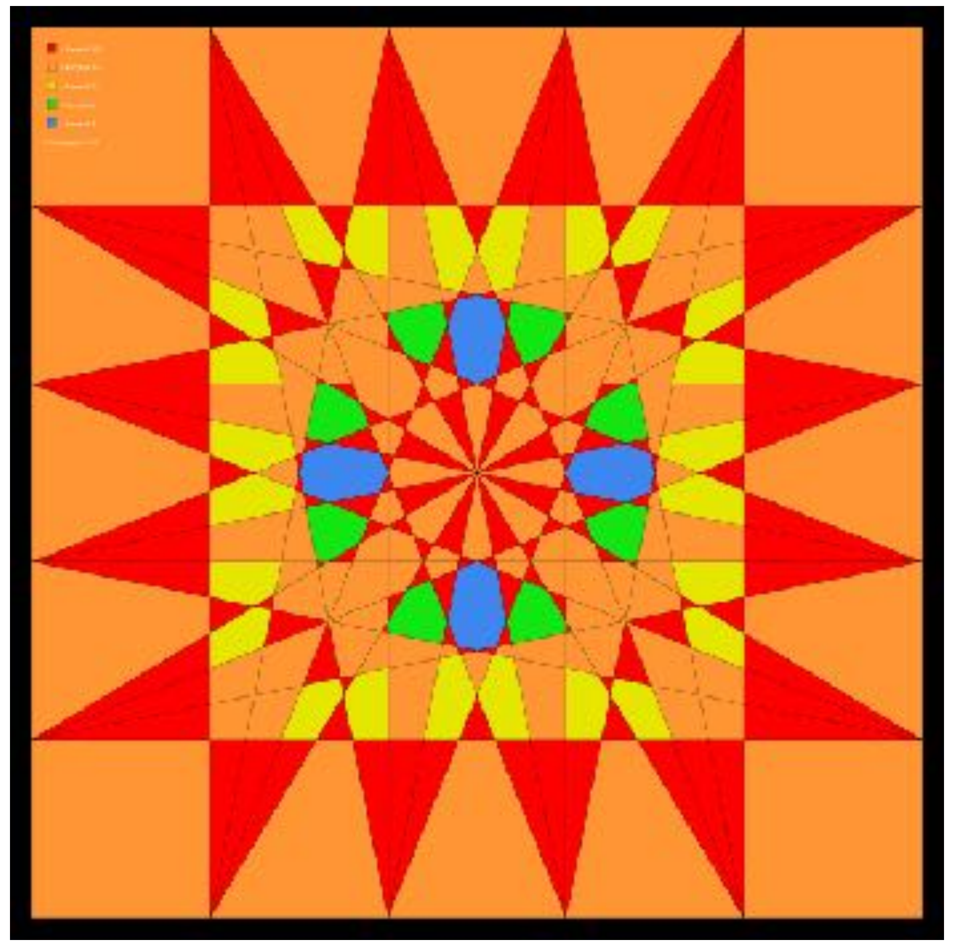
**Place n-1 points on each side of a square,**
**join each point to every point on the opposite side.**
**How many regions?**

1, 4, 24, 104, 316, 712, 1588, 2816, 4940, 7672, …







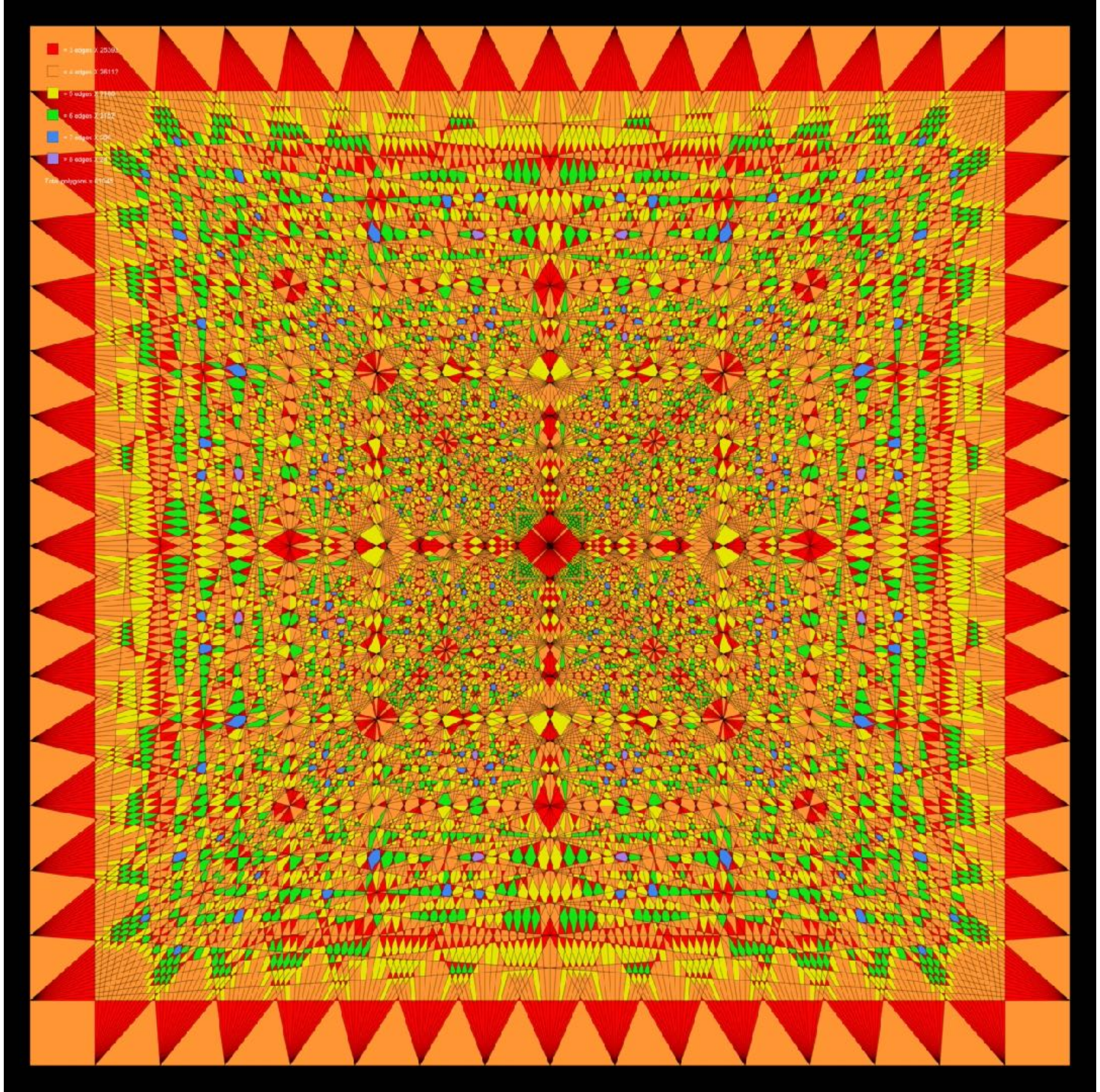


Open Problem: Have 40 terms, need a formula

Also A355799 (vertices) and A355800 (edges)

P.G. and S.G.W (13)

## Scott Shannon's Magic Carpet



n=16
61408 regions


= 3 edges X 25392
= 4 edges X 26112
= 5 edges X 7160
= 6 edges X 2152
= 7 edges X 208
= 8 edges X 24
Total polygons = 61048

## P.G. and S.G.W (15)

**Farey Tree of Order 6 (Scott Shannon & N.J.A.S., A358949(6) = 23770 vertices Dec 2022)**

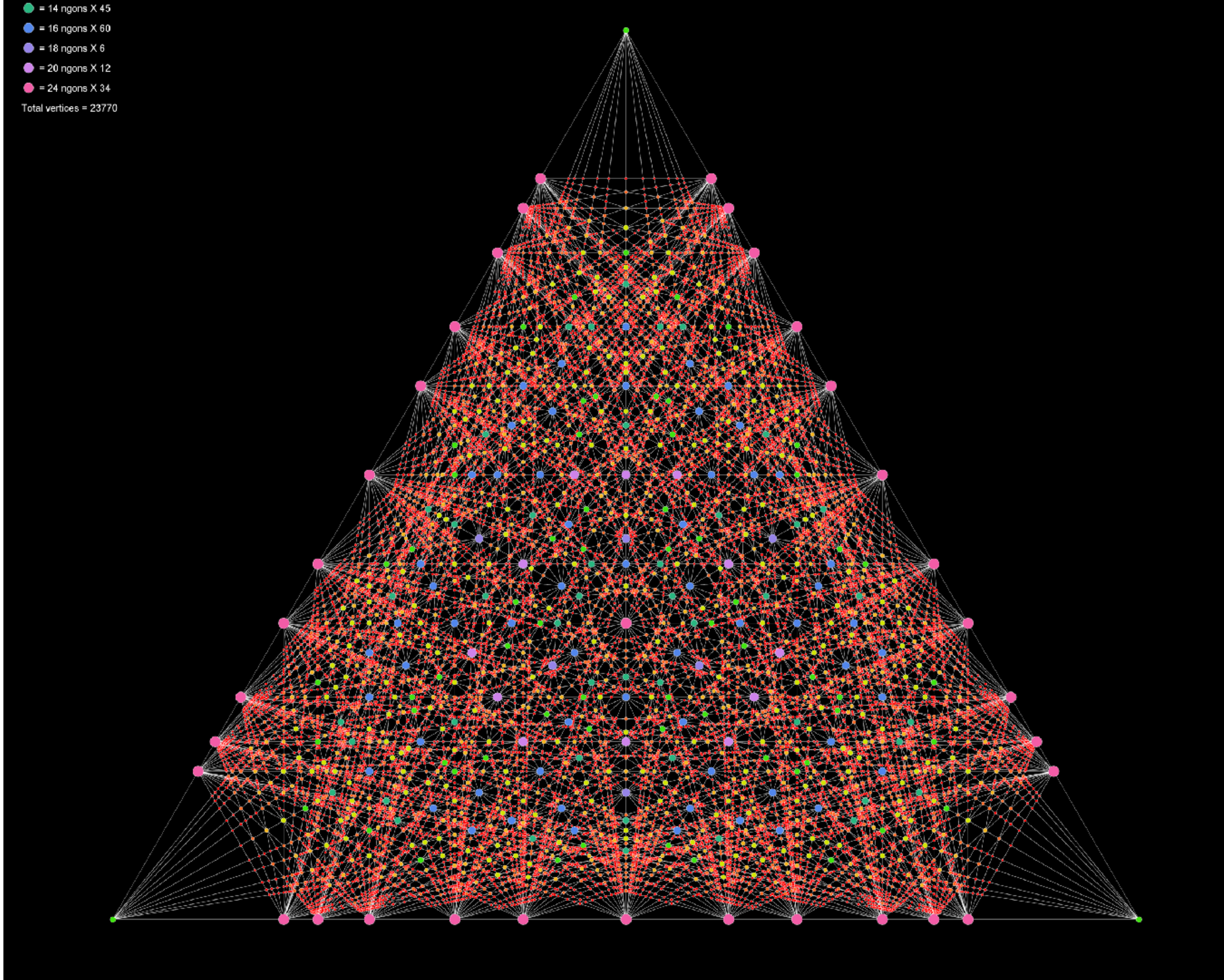



Image: Scott Shannon

# The Scariest Sequence
# in the World

The most horrible sequence in the world
A357082 (20K terms)

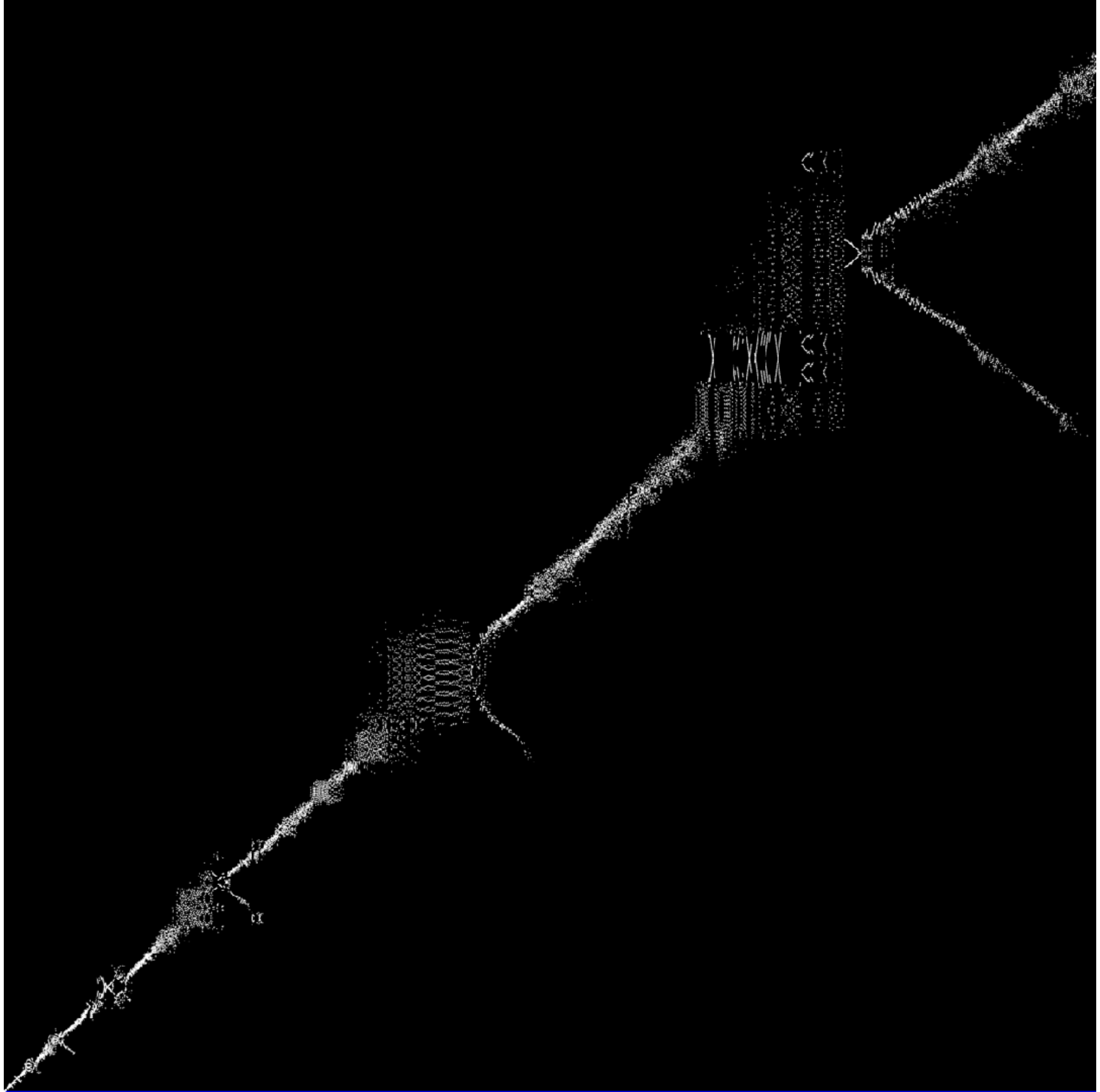
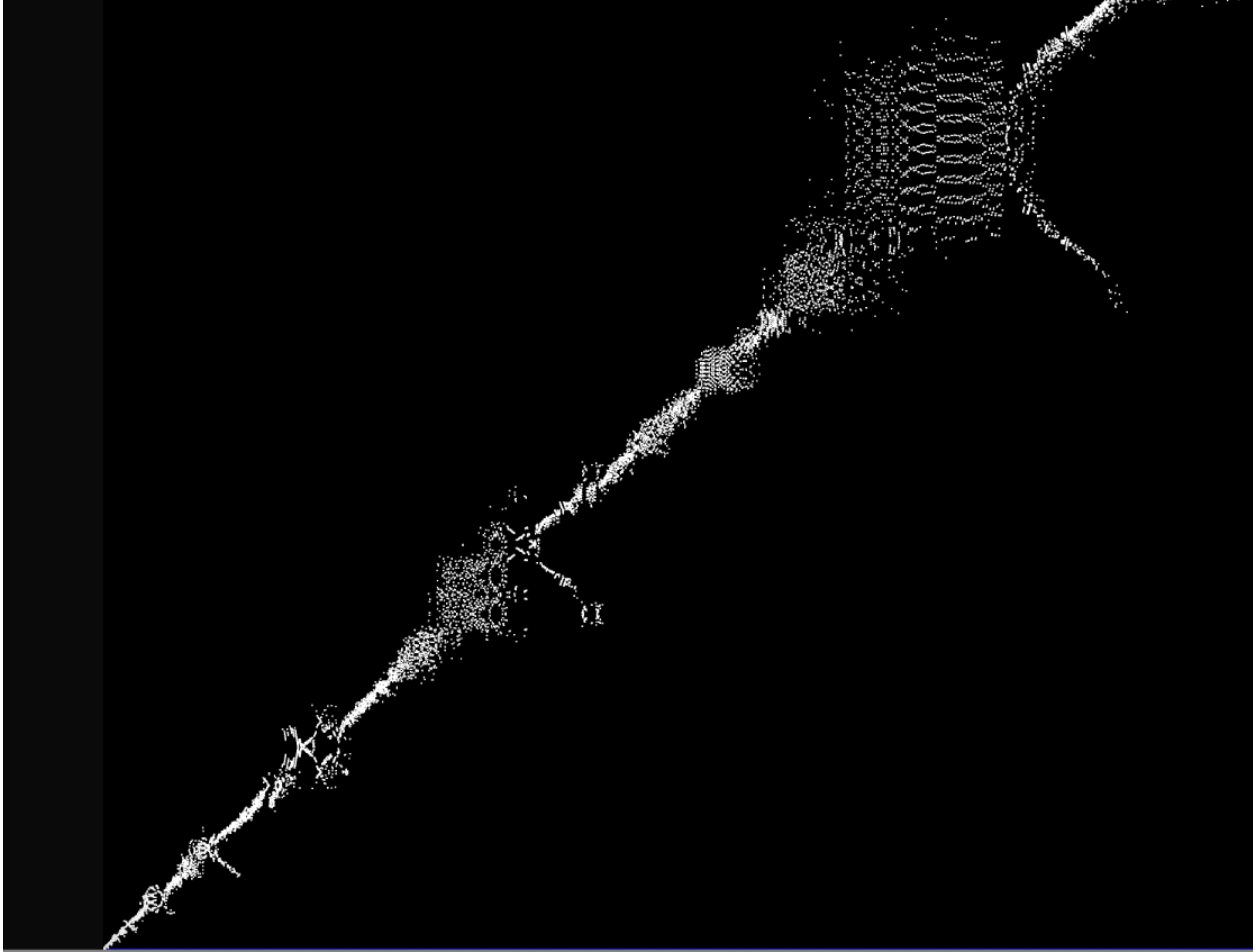


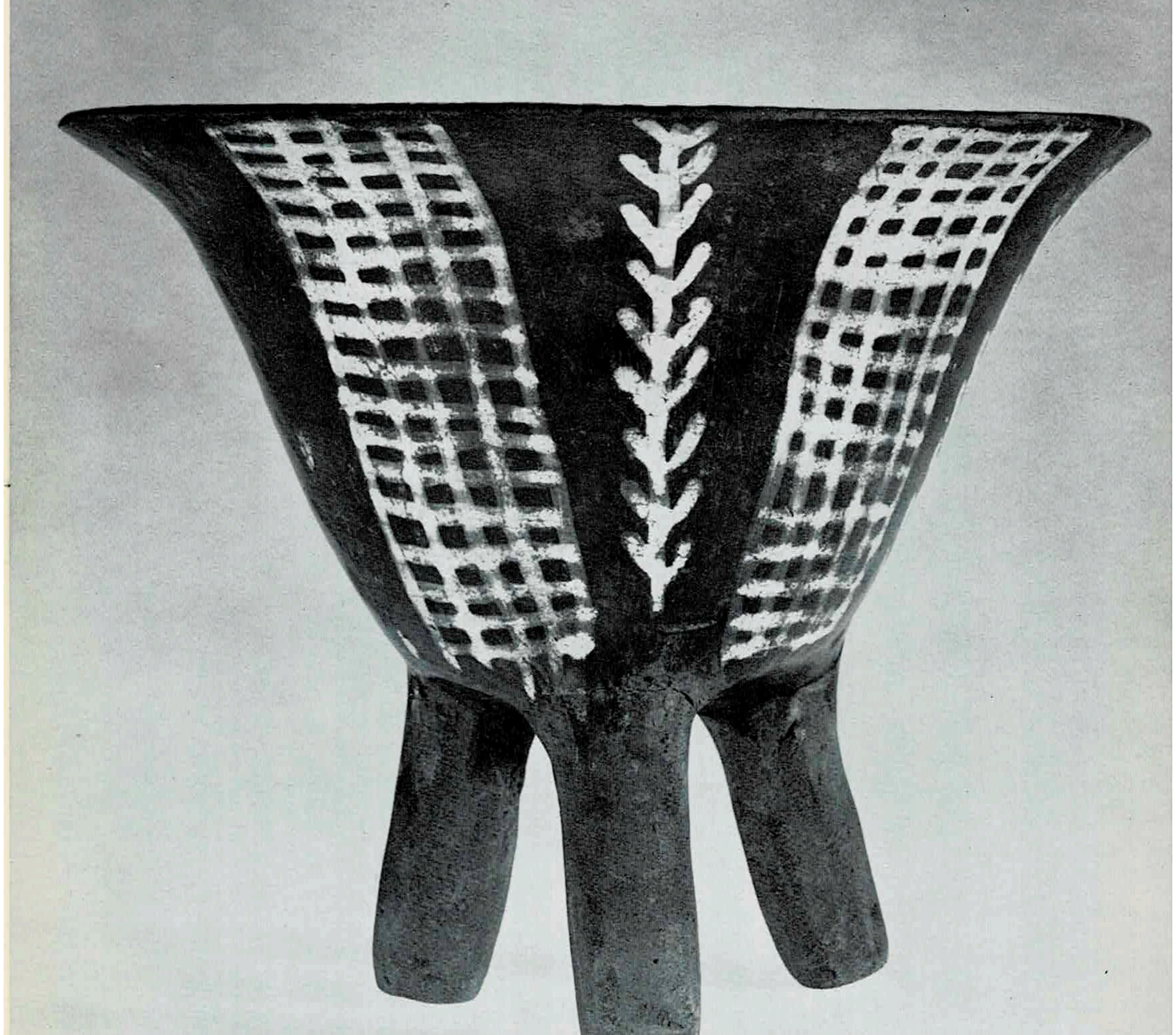
Image: Scott Shannon



# Acknowledgements and Credits

**Sources for images: David Applegate, Lars Blomberg, Ben Chaffin, Michael De Vlieger, Chaim Goodman-Strauss, Sébastien Palcoux, Omar Pol, Scott Shannon, Gavin Theobald, Michael De Vlieger, Jonathan Wild.**

**Thanks above all to the Editors, Trustees, Contributors, and Supporters of the OEIS and the OEIS Foundation who keep the OEIS running.**



**Tripod vessel**
**Egypt, 3800 BC**
**Brooklyn Museum**
**(07.447.399)**