Pictures from 50 Years of the OEIS

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Tiling a Square with Dominoes

36 ways to tile a 4x4 square

\[ a(2) = 36 \]

\[ a(n) = \prod_{j=1}^{n} \prod_{k=1}^{n} \left( 4 \cos^2 \frac{j \pi}{2n+1} + 4 \cos^2 \frac{k \pi}{2n+1} \right) \]

1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000, ...

(A4003)

(Kastelyn, 1961)
In 2015, almost the same sequence arose:

Laura Florescu, Daniela Morar, David Perkinson, Nicholas Salter, Tianyuan Xu, Sandpiles and Dominoes, 2015

1, 2, 36, 6728, 12988816, 258584046368, 53060477521960000/5, ... !! (A256043)

Figure 1: Identity element for the sandpile group of the 400 × 400 sandpile grid graph.
Outline of talk

- Triangular numbers, Recamán’s sequence, graphs
- Cellular automata (O. Pol, D. Applegate)
- Coordination sequence (C. Goodman-Strauss)
- Ways to draw n circles (J. Wild)
- Cutting polygons into squares, monotiles (G. Theobald)
- Stained glass windows (L. Blomberg, S. Shannon)
- The scariest sequence in the world (S. Shannon)
The poster, on the OEIS Foundation web site, http://oeisf.org

A250120
Graphs

Triangular numbers,
Recaman’s sequence and others,
A Facial Recognition Project
The Triangular Numbers

\[ a_n = \frac{n(n+1)}{2} \]

0, 1, 3, 6, 10, 15, 21, 28, ...

A217

[Graph of a parabola]
Triangular Numbers

Anicius Manlius Severinus Boethius, De institutione arithmetica

Boethius (480-524 AD), De institutione arithmetica

VIII. Ad hunc modum infinita progressio est, omnesque ex ordine trianguli aequilateri procreabantur, primum omnium ponend quod ex unitate nasce tur ut haec vi sua triangulus sit, non tamen etiam opere atque actu. Nam si conctoruer matre est numerorum, quicquid in his, quae ab ea nascentur, numeris inventur, necesse est ut ipsa naturali quadam potestate continentet. Ehi huuii trianguli latus est unitatis. Ternarius vero, qui primus est opere et actu ipsi triangulis, crescente unitate binarium numerorum latus babebit. Vi enim et potestate primiti trianguli, id est unitatis, unitas latus est, actu vero et opere trianguli primi, id est ternarii, duobus, quam Graeci dyada vocant. Secundique vero trianguli, qui operis atque accet actu secundus est, id est senarii, crescente naturali numero in lateribus ternarii inventur; tertii vero, id est denarii, quaternarii latus continent.

1 Descripionem numeri -XXVIII.- om. d. 2 Titulam om. d. 3 XIII. om. f. 4 in infinita a, b, c, d; in infinitum l. 5 omniaque f. 6 ex om. a, d; f; supra versus r. 5 po nent f, s; in a tamen i rarius est deletum. Supra versus: In si, ponent id b; tibi d. 7 pl. f. 8 ad sum. c. 8 quai Consequas qui esse legendum. 11 ipse l.; ipso, o in a multato i; in c. || triangles e, s. 13 unitas supra versus a.
Recamán’s Sequence
Bernardo Recamán Santos, 1991

Subtract or add: 1, 2, 3, 4, 5, 6, …
No negative terms, no repeats
0, 1, 3, 6, 2, 7, 13, 20, 12, 21, 11, …
1 2 3 -4 5 6 7 -8 9 -10 …

A5132
Recamán’s Sequence (2)

Edmund Harriss,
First 62 terms drawn as a spiral

start at 0, R to 1, R to 3, R to 6,
L to 2, R to 7, ...
The Recamán Sequence (3)

oeis.org/A005132

Animation: Michael De Vlieger.

There is a longer version on YouTube - see https://oeis.org/A005132 for link
Recamán’s Sequence (4)

(Ben Chaffin’s log-log plot of $10^{230}$ terms)
The Big Question: Does every number appear?

After $10^{15}$ terms, $852655 = 5 \times 31 \times 5501$ was missing (Allan Wilks, 2001)

After $10^{230}$ terms, $852655$ is still missing (Ben Chaffin, 2018)

30 years ago I believed that every number would eventually appear. Today I think that there are infinitely many missing terms, and $852655$ just got lucky and is the first of many.
Graphs (continued)

Keyword “look” means an interesting graph:

Exotic Graphs

A002487
A285687
A063543
A229037
A063543(n)
A117966
A229037
A002487
A285687
A063543
Graphs (continued)

Typical Graphs

A064413

A285487

A247665

A280864

A055748

A137655
The OEIS contains over 365000 sequences, and each one has a graph (click the “graph” button).

Use facial recognition software to find closest matches to the graph of a sequence you are studying.

Would be extremely useful.
Sequences from Cellular Automata
The Toothpick Problem

1 1 2 3 4 5 \( n \)
1 2 4 4 4 \( A139251 \)
1 3 7 11 15 \( A139250 \)

(Omar Pol, David Applegate, NJAS)
The Toothpick Sequence (2)

A139250

The first 32 generations of the toothpick structure. Notice that after every power of 2 generations, the number of new toothpicks added drops to 4. This is the key to finding a formula for the n-th term.

Animation created by David Applegate
The Toothpick Sequence (3)

4 free ends

64
The Toothpick Sequence (4)

(Turn ON if exactly one of your 4 neighbors is ON)
There is a simple formula for the number of ON cells
Another Ulam CA, on hexagonal grid:

A cell turns ON iff exactly one of its 6 neighbors is ON

No recurrence known!

(A151724)
The Toothpick Sequence (7)

Generations 1 (2), 2 (8), 3 (14), 4 (20), 5 (38), ... 32 (1124), ...
Sequence A161330.

The snowflake automaton

(trident)

Coordination Sequences

Joint work with Chaim Goodman-Strauss
Coordination Sequences
Joint work with Chaim Goodman-Strauss

With thanks to Jean-Guillaume Eon, Brian Galebach, Joseph Myers, Davide Proserpio, Rémy Sigrist, Allan Wechsler, and others

Definition. \( G = \text{graph}, \ P = \text{node}, \) the coordination sequence w.r.t \( P: \)
\( a(n) = \text{number of nodes at edge-distance } n \text{ from } P \)

\[ \text{CS is } \quad 1, 4, 8, 12, 16, 20, 24, 28, \ldots \]

\( \text{G.f. } = (1 + 2x + 2x^2 + 2x^3 + \ldots)^2 \)
The 3.3.3.3.6 uniform tiling (A250120)

Coordination sequence
1, 5, 9, 15, 19, ...

Conjecture
a(n+5) = a(n) + 24
for n > 2
Trunks and Branches for 2 of the 11 Uniform Tilings

3.3.4.3.4 (dual to Cairo), A219529

3.4.6.4, A8574 again!
Number of ways to draw \( n \) circles

Jonathan Wild
Music Department, McGill

A250001
\[a(2)=3\]
No. of arrangements of n circles in the plane

1, 3, 14, 173, 16951

Jonathan Wild

a(3) = 14:
Some of the 173 arrangements of 4 circles

Counted (and drawn) by Jon Wild
More of the 173 arrangements of 4 circles

Counted (and drawn) by Jon Wild
Two distinct arrangements with same “truth table” of intersections

Open Problem: How many ways to draw six circles?

Jonathan Wild
A subset: $n$ lines in general position

1, 1, 1, 1, 6, 43, 922, 38609

Wild and Reeves, 2004

$a(5)=6$
Dissecting Polygons into Squares, Rectangles, or Monotiles (Gavin Theobald)
Three fundamental sequences from geometry

\[ s(n) = \text{min number of pieces needed to dissect regular n-gon to a square} \]
\[ r(n) = \text{a rectangle} \]
\[ q(n) = \text{a monotile} \]

**n=3**

\[ s(3) = 4 ? \]
\[ r(3) = 2 \]
\[ q(3) = 1 \]

Rules: cuts are simple curves, turning over is allowed
Theorem \( s(3) > 2 \)

Oh, we're not bouncers. We just can't fit through the door.

Each piece can contain at most one vertex, so at least 3 pieces!
Three fundamental sequences from geometry (3)

$n = 5$

$s(5) \leq 6$

$r(5) \leq 4$

$q(5) = 2$

$n = 6$

$s(6) \leq 5$

$r(6) = 3$ ?

$q(6) = 1$
Three fundamental sequences from geometry (4)

n = 8

Octogon — Monotile
(2 pieces)

\[ s(8) \leq 5 \]
\[ r(8) \leq 4 \]
\[ q(8) = 2 \]
Three fundamental sequences from geometry (5)

\[ \begin{align*}
 n &= 9 \\
 s(9) &\leq 9 \\
 r(9) &\leq 7 \\
 q(9) &\leq 3
\end{align*} \]
Three fundamental sequences from geometry (6)

<table>
<thead>
<tr>
<th>n =</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
<th>10</th>
<th>11</th>
<th>12</th>
</tr>
</thead>
<tbody>
<tr>
<td>s(n) &lt;=</td>
<td>4</td>
<td>1</td>
<td>6</td>
<td>5</td>
<td>7</td>
<td>5</td>
<td>9</td>
<td>7</td>
<td>10</td>
<td>6</td>
</tr>
<tr>
<td>r(n) &lt;=</td>
<td>2</td>
<td>1</td>
<td>4</td>
<td>3</td>
<td>5</td>
<td>4</td>
<td>7</td>
<td>4</td>
<td>9</td>
<td>5</td>
</tr>
<tr>
<td>q(n) &lt;=</td>
<td>1</td>
<td>1</td>
<td>2</td>
<td>1</td>
<td>3</td>
<td>2</td>
<td>3</td>
<td>2</td>
<td>4</td>
<td>3</td>
</tr>
</tbody>
</table>


The three OEIS entries also have further illustrations
Square — 32-gon

(14 pieces)

\[ s(32) \leq 14 \] from Gavin Theobald’s
Geometric Dissections Database
Square — 50-gon
(20 pieces)

s(50) <= 20 from Gavin Theobald’s Geometric Dissections Database
Monotile dissections from Gavin Theobald

$q(12) \leq 3$

Heptadecagon — Monotile
(6 pieces)

$q(17) \leq 6$

Pentadecagon — Monotile
(5 pieces)

$q(15) \leq 5$

Enneadecagon — Monotile
(7 pieces)

$q(19) \leq 7$

See A362938
Graphical Enumeration and Stained Glass Windows

Lars Blomberg, Scott Shannon, and NJAS

Part 1 is on the arXiv and has been published in INTEGERS
Amiens, France

Rose window
Sainte-Chapelle, Paris
Planar Graphs and Stained Glass Windows (1)

Motivation

1. Extend work of Poonen-Rubinstein on $K_n$, and Legendre-Griffiths on $K_{n,n}$ to other families of graphs

2. Desire to create our own stained glass windows, in homage to Amiens, Sainte-Chapelle, Chartres, Strasbourg.

Our motto: “If you can’t solve it, make art”
Planar Graphs and Stained Glass Windows (2)

Complete graph $K_{23}$

9086 cells ($R$)
8878 nodes ($V$)
17963 edges ($E$)

Solved by Poonen and Rubinstein 1998

Euler says
$E = R + V - 1$.

$R$ and $V$ about equal

tells us most crossings
are simple.

Here $n$ is odd, so all crossings are simple.
Complete graph $K_{23}$ with 9086 cells. Colored by our special algorithm.
The Two Known Results

1. Poonen and Rubinstein, 1998: Number of nodes and cells in $K_n$:
   
   Basically $\binom{n}{4}$ minus complicated correction terms.

2. Legendre (2009), Griffiths (2010), ditto for $K_{n,n}$.

or equivalently

$K_{4,4}$ = BC(1,3)
Planar Graphs and Stained Glass Windows (5)

Typical problem:

Take m x n grid of squares or (m+1) x (n+1) grid of points

Join each pair of boundary points by a chord

In resulting graph, count vertices, edges, regions.

This is the graph BC(m,n) “Boundary Chords”

A331452 has many pictures of BC(m,n) stained glass windows.

3 x 3 grid of squares

BC(3,3)

BC(6,6)
6X6 grid of squares
Join every pair of boundary points by a chord.
6264 = A265011(6) regions
4825 = A331449(6) vertices
No formulas known.

Left: Color-coded to show number of sides: 3 (red), 4 (orange), 5 (green), 7 (blue), 8 (purple)

Right: Same graph, colored using our special algorithm.

BC(9,2)
Answers are known for BC(1,n)

Theorem (Stéphane Legendre (2009) and Martin Griffiths (2010))

Define \[ V(m, n, q) = \sum_{a=1}^{m} \sum_{b=1}^{n} (m + 1 - a)(n + 1 - b) \]

\[ \text{gcd}\{a, b\} = q \]

Nodes in BC(1,n): \[ 2(n + 1) + V(n, n, 1) - V(n, n, 2) \]

Cells in BC(1,n): \[ n^2 + 2n + V(n, n, 1) \]
BC(1, 4)

104 cells (70 triangles, 34 quadrilaterals) but no pentagons or hexagons - why?!
Interior Nodes in BC(1,n)

It appears that most interior nodes in BC(1,n) are “simple”, i.e. are where just two chords cross.

For \( n = 1, 2, 3, \ldots \) the numbers of simple interior nodes are

1, 6, 24, 54, 124, 214, 382, 598, 950, 1334, \ldots

A334701 has first 500 terms!

Open Problem : Find a formula.

This is a frequent problem: we have hundreds of terms of a sequence with a simple definition; the OEIS has 365,000 entries: need a smarter guessing program.
Scott Shannon’s Sequence A355798

Place \(n-1\) points on each side of a square, join each point to every point on the opposite side. How many regions?

1, 4, 24, 104, 316, 712, 1588, 2816, 4940, 7672, …

Open Problem: Have 40 terms, need a formula

Also A355799 (vertices) and A355800 (edges)
Scott Shannon’s Magic Carpet

P.G. and S.G.W (13)

n=16
61408 regions
Farey Tree of Order 6
(Scott Shannon & N.J.A.S., A358949(6) = 23770 vertices
Dec 2022)

Image: Scott Shannon
The Scariest Sequence in the World
The most horrible sequence in the world
A357082 (20K terms)

Image: Scott Shannon
Acknowledgements and Credits

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Omar Pol, Scott Shannon, Gavin Theobald,
Michael De Vlieger, Jonathan Wild.

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Tripod vessel
Egypt, 3800 BC
Brooklyn Museum
(07.447.399)