

TWO-GRAPHS, SWITCHING CLASSES AND EULER GRAPHS ARE EQUAL IN NUMBER*

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Abstract. Seidel has shown that the number t_n of two-graphs on n nodes is equal to the number of switching classes of graphs on n nodes. Robinson, and independently Liskovec, have given an explicit formula for the number e_n of Euler graphs on n nodes. It is shown here that $t_n = e_n$ for all n .

1. Introduction.

DEFINITION 1. A *two-graph* [9] consists of a set Ω of nodes together with a collection of triples of elements of Ω such that each 4-element subset of Ω contains an even number of these triples.

DEFINITION 2. Let G be an (ordinary) graph with node set Ω (containing no loops or multiple edges). Let ω be a node in Ω and suppose ω is adjacent in G to the (possibly empty) subset $X \subseteq \Omega$. The operation of *switching* at node ω ([5], [9]) transforms G by deleting the edges from ω to X , and adding edges from ω to $\Omega - X$. In other words, take the complement of the edges incident with node ω .

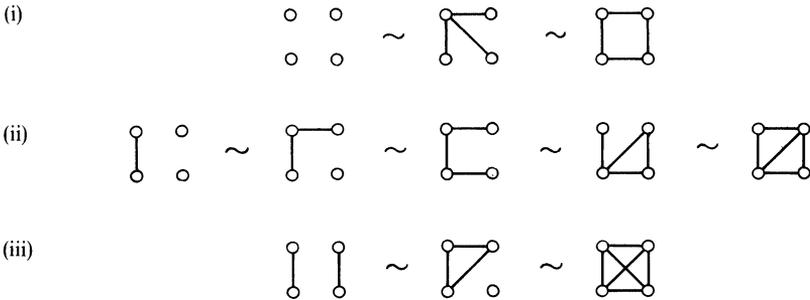


FIG. 1. The three switching classes for $n = 4$

We define an equivalence relation on graphs by saying that G' is equivalent to G if there exists a sequence of nodes $\omega_1, \omega_2, \dots$ such that G' is obtained from G by switching at $\omega_1, \omega_2, \dots$. Figure 1 shows the three equivalence classes (or *switching classes*) of graphs with $n = 4$ nodes.

Seidel [9] showed that the number t_n of switching classes of graphs with n nodes is equal to the number of two-graphs with n nodes. The values of t_1, \dots, t_7 were given in [5], and of t_8, t_9 (found by F. C. Bussemaker) in [9].

DEFINITION 3. An *Euler graph* ([7], [11, p. 20]) is a graph in which every node has even degree.

Robinson [8] and independently Liskovec [6] have given formulas for the number e_n of Euler graphs on n nodes. Robinson's formula is

$$(1) \quad e_n = \sum_{(\sigma)} \frac{2^{v(\sigma) - \lambda(\sigma)}}{\prod_i i^{\sigma_i} \sigma_i!},$$

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the sum being over all ordered n -tuples $\sigma = (\sigma_1, \dots, \sigma_n)$ such that $n = \sum_i i\sigma_i$, where

$$(2) \quad v(\sigma) = \sum_{i < j} \sigma_i \sigma_j (i, j) + \sum_i i \left(\sigma_{2i} + \sigma_{2i+1} + \binom{\sigma_i}{2} \right),$$

$$(3) \quad \lambda(\sigma) = \sum_i \sigma_i - \operatorname{sgn} \left(\sum_i \sigma_{2i+1} \right),$$

$$\operatorname{sgn}(x) = 0 \quad \text{if } x = 0, \quad = 1 \text{ if } x > 0,$$

and (i, j) is the greatest common divisor of i and j . Both [8] and [6] give the values of e_1, \dots, e_8 . (The value of e_8 is given incorrectly in [4, p. 117].) Values of e_n for $1 \leq n \leq 21$, obtained from (1), are given in Table 1.

TABLE 1
The number t_n of two-graphs, switching classes, or Euler graphs
on n nodes

n	t_n
1	1
2	1
3	2
4	3
5	7
6	16
7	54
8	243
9	2,038
10	33,120
11	1,182,004
12	87,723,296
13	12,886,193,064
14	3,633,057,074,584
15	1,944,000,150,734,320
16	1,967,881,448,329,407,496
17	3,768,516,017,219,786,199,856
18	13,670,271,807,937,483,065,795,200
19	94,109,042,015,724,412,679,233,018,144
20	1,232,069,666,043,220,685,614,640,133,362,240
21	30,739,974,599,837,035,594,494,908,346,443,726,272

In collecting data for a supplement to [12] it was observed that the numbers e_n and t_n agree for $1 \leq n \leq 8$. It is the object of this note to prove the following.

THEOREM 1. $e_n = t_n$ for all n .

This result had already been established by Seidel [10] for n odd. However, the result when n is even seems to lie much deeper. Indeed, when n is odd, it is not difficult to show that there is a unique Euler graph in each switching class, thus establishing a 1–1 correspondence between the two families. But for n even, we have been unable to find such a correspondence.

2. The labeled case. Before proving Theorem 1, we consider the corresponding problem when the nodes of the graph are labeled, say with the labels $\{1, 2, \dots, n\}$. Define

\mathcal{G}_n = set of all labeled graphs on n nodes,

\mathcal{E}_n = set of all Euler graphs in \mathcal{G}_n ,

\mathcal{T}_n = set of switching classes in \mathcal{G}_n ,

\mathcal{C}_n = set of all cocycles in \mathcal{G}_n ([3, p. 38], [1, p. 13]. In the notation of [11, p. 28],

\mathcal{C}_n = cutsets of the complete graph on $\{1, \dots, n\}$).

For $A, B \in \mathcal{G}_n$, define $A + B \in \mathcal{G}_n$ to be the graph containing the edges in either A or B but not both. With this definition of addition, $\mathcal{G}_n, \mathcal{E}_n, \mathcal{C}_n$ are elementary abelian groups of type $(1, 1, \dots, 1)$ of orders $G_n = 2^{\binom{n}{2}}, E_n = 2^{\binom{n-1}{2}}, C_n = 2^{n-1}$ respectively [1], [11].

Observe that two graphs are in the same switching class if and only if their sum is a cocycle. Hence $\mathcal{T}_n \cong \mathcal{G}_n/\mathcal{C}_n$, and has order $T_n = 2^{\binom{n}{2}}/2^{n-1} = 2^{\binom{n-1}{2}} = E_n$.

It is easy to construct a 1-1 correspondence between \mathcal{E}_n and \mathcal{T}_n . Let τ denote the tree with edges $12, 13, \dots, 1n$, and let the edges ij ($1 < i < j$) not in τ be called (in this section only) *chords*. As a basis for \mathcal{E}_n take the *fundamental circuits* [11,

p. 27] $\Phi_1, \dots, \Phi_d, d = \binom{n-1}{2}$, each consisting of a chord and the two co-

terminal edges of τ . Any element of \mathcal{E}_n can be written uniquely as $\sum_{i=1}^d a_i \Phi_i, a_i = 0$ or 1 . Each switching class contains exactly one *chordal* graph that has no edges of τ ; as a basis for these graphs we take the *unchordal* graphs ψ_1, \dots, ψ_d that each consist of a single chord. Then each chordal graph can be written uniquely as $\sum_{i=1}^d a_i \psi_i, a_i = 0$ or 1 . Thus we have proved the following.

THEOREM 2. *There is a 1-1 correspondence between the $2^{\binom{n-1}{2}}$ labeled Euler graphs and labeled switching classes on n nodes.*

Figure 2 shows the correspondence when $n = 4$.

3. The unlabeled case: Proof of Theorem 1. For a subset $C \subseteq \{1, \dots, n\}$, let $X(C)$ denote the operation of switching at all the nodes of C (in any order). $X(C)$ induces a permutation (which we also denote $X(C)$) on the set \mathcal{G}_n of all labeled graphs. Note that $X(C)$ and $X(\bar{C})$ are the same permutation (where \bar{C} is the complement of C).

The number t_n of switching classes is the number of equivalence classes in \mathcal{G}_n under the combined action of all $X(C)$ and all permutations π of the n nodes. The set of all $X(C)$ and all π generate a group of order $2^{n-1}n!$, consisting of pairs (π, C) . By Burnside's lemma [2],

$$(4) \quad t_n = \frac{1}{2^{n-1}n!} \sum_{\pi \in \mathcal{S}_n} \sum_{C \subseteq \{1, \dots, n\}} f(\pi, C),$$

where $f(\pi, C)$ is the number of graphs that are fixed under the operation of first permuting the nodes according to π and then switching at the nodes in C . We proceed to calculate $f(\pi, C)$.

Suppose π permutes the nodes in σ_i cycles (called *node-cycles*) of length i , for $1 \leq i \leq n$. Let $\beta_1, \beta_2, \dots, \beta_c$ be the (node-) cycles of π , with lengths

b_1, b_2, \dots, b_c respectively ($c = \sum \sigma_i, \sum b_j = n$), and let $\gamma(\beta_i, C)$ be the number of nodes of β_i that are in C .

Euler graph	Chordal representative of switching class	a_1	a_2	a_3
$\begin{matrix} 1 & \circ & \circ & 2 \\ 3 & \circ & \circ & 4 \end{matrix}$	$\begin{matrix} 1 & \circ & \circ & 2 \\ 3 & \circ & \circ & 4 \end{matrix}$	0	0	0
		1	0	0
		0	1	0
		0	0	1
		1	1	0
		1	0	1
		0	1	1
		1	1	1

FIG. 2. 1-1 correspondence between labeled Euler graphs and labeled switching classes on four nodes

Under the action of π , the edges of the complete graph are permuted in cycles (called edge-cycles). It is shown in [8] that the total number of edge-cycles is $v(\sigma)$.

(i) If $\beta = (a_1, \dots, a_{2i})$ is a node-cycle of length $b = 2i$, the edges $a_\mu a_\nu$ ($1 \leq \mu < \nu \leq 2i$) are permuted in $i - 1$ distinct edge-cycles of length $2i$, and one edge-cycle (the cycle of diameters) of length i . Then as a diameter traverses the edge-cycle of length i under repeated applications of (π, C) , it is switched $\gamma(\beta, C)$ times. Thus in order for it to be possible for a graph to be fixed under (π, C) , $\gamma(\beta, C)$ must be even. An edge which is not a diameter will be switched $2\gamma(\beta, C)$ times, which is always even. Thus if $\gamma(\beta, C)$ is even, there are 2^i ways of inserting the chords of β so that this subgraph is fixed under (π, C) ; if $\gamma(\beta, C)$ is odd, there are no such ways.

(ii) The edges joining the nodes of a node-cycle of length $2i + 1$ comprise i distinct edge-cycles each of length $2i + 1$. An edge in one of these cycles is switched

an even number of times during $2i + 1$ successive applications of (π, C) . Thus there are 2^i ways of inserting the chords of such a node-cycle, for any C .

(iii) Let $\beta = (a_1, \dots, a_b)$, $\beta' = (a'_1, \dots, a'_b)$ be distinct node-cycles. Let $t = (b, b')$, and write

$$b = rt, \quad b' = r't, \quad \text{where } (r, r') = 1.$$

The bb' edges $a_\mu a'_\mu$ are permuted in t distinct edge-cycles of length $rr't$ each.

As an edge traverses its edge-cycle, it is switched $p = r'\gamma(\beta, C) + r\gamma(\beta', C)$ times. If both b and b' are even, and an invariant subgraph is possible, both $\gamma(\beta, C)$ and $\gamma(\beta', C)$ must be even by (i). If b is even and b' odd, r and $\gamma(\beta, C)$ must be even. But if both b and b' are odd, so r, r' are odd, then for p to be even, $\gamma(\beta, C)$ and $\gamma(\beta', C)$ must both be even or both odd. Therefore if an invariant subgraph is to be possible, all node-cycles of odd length must contain an even number of points of C , or all must contain an odd number of points of C . If this holds, the number of ways of drawing an invariant subgraph of edges from β to β' is 2^t .

(iv) Putting all this together, the number of ways of drawing a graph that is invariant under (π, C) is $f(\pi, C) = 2^{v(\sigma)}\xi(\pi, C)$, where $\xi(\pi, C) = 0$ or 1 , and $= 1$ if and only if $\gamma(\beta_i, C)$ is even whenever b_i is even, and $\{\gamma(\beta_j, C) : b_j \text{ odd}\}$ are all even or all odd together. It follows that

$$\sum_C \xi(\pi, C) = 2^{n - \lambda(\sigma)}.$$

Hence from (4) and (1),

$$t_n = \sum_{\pi} 2^{v(\sigma) - \lambda(\sigma)}$$

which collapses to (1), and the theorem is proved.

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