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Note

Hamiltonian Cycles in a Graph of Degree 4

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The following theorem answers a question raised by S. Lin [2, p. 2268].

THEOREM 1. *Let G be a graph with n nodes P_0, P_1, P_2,..., P_{n-1} and 2n branches which can be partitioned into 2 sets of n branches, H_1 and H_2, each of which forms a Hamiltonian cycle. Then G contains a third Hamiltonian cycle H_3.*

The proof depends on two corollaries of:

THEOREM 2. (Smith and Tutte, [1, p. 6; 3]). *In any cubic graph G, the symmetric difference of all the Hamiltonian cycles of G, regarded as sets of branches, is empty.*

COROLLARY 3. *If a cubic graph has a Hamiltonian cycle, then it has at least three Hamiltonian cycles.*

COROLLARY 4. *If a cubic graph has a Hamiltonian cycle through a branch b, then it has at least one other Hamiltonian cycle through b.*

PROOF OF THEOREM 1: The nodes of G may be relabeled so that the Hamiltonian cycle H_1 is a polygon with branches P_0P_1, P_1P_2,..., $P_{n-1}P_0$, and the Hamiltonian cycle H_2 consists of chords of H_1. If n is even, a cubic graph G_1 containing the Hamiltonian cycle H_2 is obtained from G by deleting alternate branches of H_1. Applying Corollary 3 to G_1 gives the desired cycle H_3.

If n is odd there are two cases: (1) There is some node P_1 (say) such that its neighbors P_0 and P_2 on the polygon H_1 are not adjacent in H_2, or (2) for every i the interior Hamiltonian cycle H_2 contains a branch $P_{i-1}P_{i+1}$, (with subscripts taken modulo n).

CASE (1). Suppose chords P_iP_1 and P_jP_1 meet node P_1, i.e., suppose P_1 is adjacent to nodes P_i and P_j in H_2. A new graph G_2 with $n - 1$ nodes is formed from G by replacing branches P_0P_1 and P_1P_2 by a new branch P_0P_2, deleting branches P_iP_1 and P_jP_1, and deleting node P_1.

Let the Hamiltonian cycle H_2 of G be $P_iP_1P_jP_{a_1}P_{a_2} \cdots P_{a_{n-3}}P_i$. Branches P_iP_1 and P_1P_j have already been removed. Removing the further branches $P_{a_1}P_{a_2}, P_{a_3}P_{a_4}, \ldots, P_{a_{n-2}}P_{a_{n-3}}$ makes G_2 into a cubic graph, containing a Hamiltonian cycle $P_0P_2P_3 \cdots P_{n-1}P_0$ through P_0P_2. The cycle H_3 is then obtained using Corollary 4.

CASE (2). In this case the graph is completely determined and a third Hamiltonian cycle, $P_0P_1P_{n-1}P_{n-2} \cdots P_2P_0$, is found immediately.

The question is still open as to whether H_3 can always be chosen so that its complement is also a Hamiltonian cycle.

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3. W. T. TUTTE, On Hamiltonian Circuits, *J. London Math. Soc.* **21** (1946), 98–101.