

# The Unimodular Lattices of Dimension up to 23 and the Minkowski–Siegel Mass Constants

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In an earlier paper we enumerated the integral lattices of determinant one and dimension not exceeding 20. The present paper extends this enumeration to dimension 23, finding 40 lattices of dimension 21, 68 of dimension 22, and 117 of dimension 23. We also give explicit formulae for the Minkowski–Siegel mass constants for unimodular lattices (apparently not stated correctly elsewhere in the literature) and an exact table of the mass constants up to 32 dimensions, which provided a valuable check on our enumeration.

## 1. INTRODUCTION

The norm of a vector  $x$  is its squared length  $x \cdot x$ , and is written  $N(x)$ . An integral lattice  $A$  in real  $n$ -dimensional Euclidean space  $\mathbb{R}^n$  is said to be unimodular if it has determinant  $\pm 1$ , and an integral unimodular lattice is *even*, or of genus II, if the norm of every lattice vector is an even integer; otherwise it is *odd*, or of genus I. Even unimodular lattices exist only in dimensions which are multiples of eight, whereas odd lattices exist in all dimensions greater than zero. The history of the enumeration of integral unimodular lattices is described briefly in our earlier paper [9]. It is worth mentioning, however, that some of the recent interest in these lattices is due to their connections, as yet not fully understood, with the Leech lattice and with hyperbolic geometry (see [6, 10–12, 15, 28, 29] and Section 2.1 below). The even unimodular lattices in dimension 24 were found by Niemeier in 1968 [16], and in [9] we enumerated the odd lattices in up to 20 dimensions. In the present paper we complete the enumeration of the odd lattices in dimensions below 24 by deriving them from Niemeier’s lattices. We shall prove the following theorem.

**THEOREM 1: THE CLASSIFICATION.** *The unimodular lattices of dimension not exceeding 23 and containing no vectors of norm 1 are those shown in Table 1.*

TABLE 1

The unimodular lattices of dimension  $\leq 23$  and containing no vectors of norm 1. All are odd (or of genus I) except for four lattices: the empty lattice,  $E_8$ ,  $E_8^2$ , and  $D_{16}$ , which are even (or of genus II). The notation is explained in Section 2.8

dim	$N$	$w$	$V$	$c(w)$	$g_{182}$	$t_2$
0	$D_{24}$	–	$\emptyset$	1	1	0
8	$D_{16}E_8$	–0	$E_8$	1	1	240
12	$D_{12}^2$	–0	$D_{12}$	2	1	264
14	$D_{10}E_7^2$	–00	$E_7^2$	1	2	252
15	$A_{15}D_9$	0–	$A_{15}$	1	2	240

TABLE 1 (cont.)

dim	<i>N</i>	<i>w</i>	<i>V</i>	<i>c(w)</i>	<i>g</i> <sub>1</sub> <i>g</i> <sub>2</sub>	<i>t</i> <sub>2</sub>
16	$E_8^3$	-00	$E_8^2$	3	2	480
16	$D_{16}E_8$	0-	$D_{16}$	1	1	480
16	$D_8^3$	-00	$D_8^2$	3	2	224
17	$A_{11}D_7E_6$	0-0	$A_{11}E_6$	1	2	204
18	$A_{17}E_7$	0-	$A_{17}A_1$	1	2	308
18	$D_{10}E_7^2$	0-0	$D_{10}E_7A_1$	2	1	308
18	$D_6^4$	-000	$D_6^3$	4	6	180
18	$A_9^2D_6$	00-	$A_9^2$	1	4	180
19	$E_6^4$	-000	$E_6^3O_1$	4	12	216
19	$A_{11}D_7E_6$	00-	$A_{11}D_7O_1$	1	2	216
19	$A_7^2D_5^2$	00-0	$A_7^2D_5$	2	4	152
20	$D_{24}$	+	$D_{20}$	1	1	760
20	$D_{16}E_8$	+0	$D_{12}E_8$	1	1	504
20	$D_{12}^2$	+0	$D_{12}D_8$	2	1	376
20	$D_{10}E_7^2$	+00	$E_7^2D_6$	1	2	312
20	$A_{15}D_9$	0+	$A_{15}D_5$	1	2	280
20	$D_8^3$	+00	$D_8^2D_4$	3	2	248
20	$A_{11}D_7E_6$	0+0	$A_{11}E_6A_3$	1	2	216
20	$D_6^4$	+000	$D_6^3A_1^2$	4	6	184
20	$A_9^2D_6$	00+	$A_9^2A_1^2$	1	4	184
20	$A_7^2D_5^2$	00+0	$A_7^2D_5O_1$	2	4	152
20	$D_4^6$	+0 <sup>5</sup>	$D_4^5$	6	120	120
20	$A_5^4D_4$	0 <sup>4</sup> +	$A_5^4$	1	16	120
21	$A_{24}$	+	$A_{20}O_1$	1	2	420
21	$A_{17}E_7$	+0	$A_{13}E_7O_1$	1	2	308
21	$A_{15}D_9$	+0	$A_{11}D_9O_1$	1	2	276
21	$A_{12}^2$	+0	$A_{12}A_8O_1$	2	2	228
21	$A_{11}D_7E_6$	+00	$D_7A_7E_6O_1$	1	2	212
21	$A_9^2D_6$	+00	$A_9D_6A_5O_1$	2	2	180
21	$A_8^3$	+00	$A_8^2A_4O_1$	3	4	164
21	$A_7^2D_5^2$	+000	$A_7D_5^2A_3O_1$	2	4	148
21	$A_6^4$	+000	$A_6^2A_2O_1$	4	6	132
21	$A_5^4D_4$	+0000	$A_5^3D_4A_1O_1$	4	12	116
21	$A_4^6$	+0 <sup>5</sup>	$A_4^5O_1$	6	40	100
21	$A_3^8$	+0 <sup>7</sup>	$A_3^7$	8	336	84
22	$D_{16}E_8$	**	$D_{14}E_7A_1$	1	1	492
22	$E_8^3$	**0	$E_8E_7^2$	3	2	492
22	$D_{12}^2$	**	$D_{10}A_1^2$	1	2	364
22	$A_{17}E_7$	**	$A_{15}D_6O_1$	1	2	300
22	$D_{10}E_7^2$	0**	$D_{10}D_6^2$	1	2	300
22	$D_{10}E_7^2$	**0	$D_8E_7D_6A_1$	2	1	300
22	$A_{15}D_9$	**	$A_{13}D_7A_1O_1$	1	2	268
22	$D_8^3$	**0	$D_8D_6^2A_1^2$	3	2	236
22	$A_{12}^2$	**	$A_{10}^2O_2$	1	4	220
22	$E_6^4$	**00	$E_6^2A_5^2$	6	8	204
22	$A_{11}D_7E_6$	0**	$A_{11}D_5A_5A_1$	1	2	204
22	$A_{11}D_7E_6$	*0*	$A_9D_7A_5O_1$	1	2	204
22	$A_{11}D_7E_6$	**0	$A_9E_6D_5A_1O_1$	1	2	204
22	$D_6^4$	**00	$D_6^2D_4^2A_1^2$	6	4	172
22	$A_9^2D_6$	**0	$A_7^2D_6O_2$	1	4	172

TABLE 1 (cont.)

dim	N	w	V	c(w)	$\delta_1 \delta_2$	$t_2$
22	$A_9 D_6$	*0*	$A_9 A_7 D_4 A_1 O_2$	2	2	172
22	$A_8^3$	**0	$A_8 A_6^2 O_2$	3	4	156
22	$A_7^2 D_5^2$	00**	$A_7^2 A_3^2 A_1^2$	1	8	140
22	$A_7^2 D_5^2$	**00	$D_5^2 A_5^2 O_2$	1	8	140
22	$A_7^2 D_5^2$	*0*0	$A_7 D_5 A_5 A_3 A_1 O_1$	4	2	140
22	$A_6^4$	**00	$A_6^2 A_4^2 O_2$	6	4	124
22	$D_4^6$	**0 <sup>4</sup>	$D_4^4 A_1^6$	15	144	108
22	$A_5^4 D_4$	*000*	$A_5^3 A_3 A_1^3 O_1$	4	12	108
22	$A_5^4 D_4$	**000	$A_5^2 D_4 A_3^2 O_2$	6	8	108
22	$A_4^6$	**0 <sup>4</sup>	$A_4^4 A_2^2 O_2$	15	16	92
22	$A_3^8$	**0 <sup>6</sup>	$A_3^6 A_1^2 O_2$	28	96	76
22	$A_2^{12}$	**0 <sup>10</sup>	$A_2^{10} O_2$	66	2880	60
22	$A_1^{24}$	**0 <sup>22</sup>	$A_1^{22}$	276	887040	44
23	$D_{16} E_8$	10	$A_{15} E_8$	1	2	480
23	$A_{24}$	5	$A_{19} A_4$	2	2	400
23	$D_{12}^2$	12	$D_{11} A_{11} O_1$	2	2	352
23	$A_{17} E_7$	60	$A_{11} E_7 A_5$	2	2	288
23	$D_{10} E_7^2$	110	$A_9 E_7 E_6 O_1$	2	2	288
23	$D_{10} E_7^2$	211	$D_9 E_6^2 O_2$	1	4	288
23	$A_{17} E_7$	31	$A_{14} E_6 A_2 O_1$	2	2	288
23	$A_{15} D_9$	80	$D_9 A_7^2$	1	4	256
23	$A_{15} D_9$	21	$A_{13} A_8 A_1 O_1$	2	2	256
23	$A_{15} D_9$	42	$A_{11} D_8 A_3 O_1$	2	2	256
23	$D_8^3$	033	$D_8 A_7^2 O_1$	3	4	224
23	$D_8^3$	122	$D_7^2 A_7 O_2$	3	4	224
23	$A_{12}^2$	15	$A_{11} A_7 A_4 O_1$	4	2	208
23	$A_{12}^2$	32	$A_{10} A_9 A_2 A_1 O_1$	4	2	208
23	$E_6^4$	0111	$E_6 D_5^3 O_2$	8	12	192
23	$A_{11} D_7 E_6$	620	$E_6 D_6 A_5^2 O_1$	1	4	192
23	$A_{11} D_7 E_6$	401	$D_7 A_7 D_5 A_3 O_1$	2	2	192
23	$A_{11} D_7 E_6$	330	$A_8 E_6 A_6 A_2 O_1$	2	2	192
23	$A_{11} D_7 E_6$	111	$A_{10} A_6 D_5 O_2$	2	2	192
23	$A_{11} D_7 E_6$	222	$A_9 D_6 D_5 A_1 O_2$	2	2	192
23	$D_6^4$	2222	$D_5^4 O_3$	1	48	160
23	$A_9^2 D_6$	501	$A_9 A_5 A_4^2 O_1$	2	4	160
23	$D_6^4$	0123	$D_6 D_5 A_5^2 O_2$	12	4	160
23	$A_9^2 D_6$	240	$A_7 D_6 A_5 A_3 A_1 O_1$	4	2	160
23	$A_9^2 D_6$	312	$A_8 A_6 D_5 A_2 O_2$	4	2	160
23	$A_9^2 D_6$	121	$A_8 A_7 A_5 A_1 O_2$	4	2	160
23	$A_8^3$	036	$A_8 A_5^2 A_2^2 O_1$	6	4	144
23	$A_8^3$	411	$A_7^2 A_4 A_3 O_2$	6	4	144
23	$A_8^3$	177	$A_7 A_6^2 A_1^2 O_2$	6	4	144
23	$A_7^2 D_5^2$	4400	$D_5^2 A_3^4 O_1$	1	16	128
23	$A_7^2 D_5^2$	4022	$A_7 D_4^2 A_3^2 O_2$	2	8	128
23	$A_7^2 D_5^2$	2031	$A_7 A_4 A_2^2 A_1 O_2$	4	4	128
23	$A_7^2 D_5^2$	2220	$D_5 A_5^3 D_4 A_1^2 O_2$	4	4	128
23	$A_7^2 D_5^2$	1112	$A_6^2 D_4 A_4 O_3$	4	4	128
23	$A_7^2 D_5^2$	1303	$A_6 D_5 A_4^2 A_2 O_2$	8	2	128
23	$A_6^4$	5111	$A_5^2 A_4 A_1 O_3$	8	6	112
23	$A_6^4$	0124	$A_6 A_5 A_4 A_3 A_2 A_1 O_2$	24	2	112
23	$D_4^6$	002332	$D_4^4 A_3^4 O_3$	45	96	96
23	$A_5^4 D_4$	00331	$A_5^2 A_3 A_2^2 O_2$	6	16	96
23	$A_5^4 D_4$	02220	$A_5 D_4 A_3^3 A_1^3 O_2$	8	12	96
23	$A_5^4 D_4$	31110	$D_4 A_4^3 A_2^2 O_3$	8	12	96
23	$A_5^4 D_4$	04111	$A_5 A_4^2 A_3 A_1 O_3$	24	4	96

TABLE 1 (cont.)

dim	N	w	V	c(w)	g <sub>1</sub> g <sub>2</sub>	t <sub>2</sub>
23	A <sub>4</sub> <sup>6</sup>	0111111	A <sub>4</sub> A <sub>3</sub> <sup>5</sup> O <sub>4</sub>	12	40	80
23	A <sub>4</sub> <sup>6</sup>	001234	A <sub>4</sub> <sup>2</sup> A <sub>3</sub> <sup>2</sup> A <sub>2</sub> <sup>2</sup> A <sub>1</sub> <sup>2</sup> O <sub>3</sub>	60	8	80
23	A <sub>3</sub> <sup>8</sup>	0 <sup>4</sup> 2 <sup>4</sup>	A <sub>3</sub> <sup>4</sup> A <sub>1</sub> <sup>8</sup> O <sub>3</sub>	14	384	64
23	A <sub>3</sub> <sup>8</sup>	0 <sup>3</sup> 2 <sup>1</sup> 3 <sup>3</sup>	A <sub>3</sub> <sup>3</sup> A <sub>2</sub> <sup>4</sup> A <sub>1</sub> <sup>2</sup> O <sub>4</sub>	112	48	64
23	A <sub>2</sub> <sup>12</sup>	0 <sup>6</sup> 1 <sup>6</sup>	A <sub>2</sub> <sup>6</sup> A <sub>1</sub> <sup>6</sup> O <sub>5</sub>	264	1440	48
23	A <sub>1</sub> <sup>24</sup>	0 <sup>16</sup> 1 <sup>8</sup>	A <sub>1</sub> <sup>16</sup> O <sub>7</sub>	759	645120	32
23	A <sub>24</sub>	min	O <sub>23</sub>	196560	84610842624000	0

The lattices of minimum norm 1 are easily determined, as we shall see in Section 2, and the numbers of lattices in each dimension may be found in Table 6 below. The entries in Table 1 are explained in Section 2.8. The proof of Theorem 1 is based on the remark (justified in Section 2.3) that any such lattice is associated with a certain Niemeier lattice, and consists in finding all lattices that can be obtained from the Niemeier lattices. As a valuable check we determined the automorphism groups of the lattices in Table 1, and verified that the sum of the reciprocals of the group orders of the lattices of each dimension is equal to the Minkowski–Siegel mass constant. Since we have not been able to find the mass constants stated correctly anywhere in the literature we give them here. (There are unfortunately still three errors even in the formulae as given in [26].) For any undefined terms the reader is referred to [3, 14, 17, 22].

**THEOREM 2: THE MINKOWSKI-SIEGEL MASS CONSTANTS.** *Let  $\Phi_X(n)$  denote the family of all inequivalent lattices of dimension  $n$  and genus  $X$ . Then*

$$\sum_{\Lambda \in \Phi_X(n)} \frac{1}{|\text{Aut}(\Lambda)|}$$

is given by

$$\begin{aligned} & \frac{(1-2^{-k})(1+2^{1-k})}{2 \cdot k!} |B_k \cdot B_2 B_4 \cdots B_{2k-2}|, \text{ if } n = 2k \equiv 0 \pmod{8}, \\ & \frac{2^k + 1}{k! 2^{2k+1}} |B_2 B_4 \cdots B_{2k}|, \text{ if } n = 2k + 1 \equiv \pm 1 \pmod{8}, \\ & \frac{1}{(k-1)! 2^{2k+1}} |E_{k-1} \cdot B_2 B_4 \cdots B_{2k-2}|, \text{ if } n = 2k \equiv \pm 2 \pmod{8}, \\ & \frac{2^k - 1}{k! 2^{2k+1}} |B_2 B_4 \cdots B_{2k}|, \text{ if } n = 2k + 1 \equiv \pm 3 \pmod{8}, \\ & \frac{(1-2^{-k})(1-2^{1-k})}{2 \cdot k!} |B_k \cdot B_2 B_4 \cdots B_{2k-2}|, \text{ if } n = 2k \equiv 4 \pmod{8}. \end{aligned}$$

Here  $B_k$  and  $E_k$  are the  $k$ -th Bernoulli and Euler numbers, respectively:  $B_0 = 1, B_1 = -1/2, B_2 = 1/6, \dots, E_0 = 1, E_2 = -1, E_4 = 5, \dots$  [1, p. 810]. Also

$$\sum_{\Lambda \in \Phi_{II}(n)} \frac{1}{|\text{Aut}(\Lambda)|} = \frac{|B_k|^{k-1}}{2k} \prod_{j=1}^{k-1} \frac{|B_{2j}|}{4j},$$

if  $n = 2k \equiv 0 \pmod{8}$ .

(The formulae for genus I may be derived from [18–21, 23–25], while the formula for genus II is given in [22].) The numerical values of these expressions for  $n \leq 32$  are shown in Tables 2–4.

## 2. THE PROOF OF THEOREM 1

**2.1. NIEMEIER'S RESULT.** Niemeier [16] showed that there are exactly 24 even unimodular lattices in  $\mathbb{R}^{24}$ , 23 of which have minimum norm 2, and one, the Leech lattice  $A_{24}$ , having minimum norm 4. We shall refer to these 24 as the *Niemeier lattices*.

The Niemeier lattices of minimum norm 2 (and more generally any integral lattice of minimum norm 1 or 2) are composed of various *Witt components* held together by *glue*. This glueing theory has been adequately described elsewhere ([8, 9]), and so we shall just recall that the Witt component lattices are taken from the list  $O_n (n \geq 1)$ , an empty component;  $\mathbb{Z}$ , the one-dimensional lattice of integers; and the root lattices  $A_n (n \geq 1)$ ,  $D_n (n \geq 1)$ ,  $E_6$ ,  $E_7$  and  $E_8$ . The subscript on a lattice indicates its dimension. We sometimes use  $I_m$  to denote the  $m$ -dimensional integer lattice  $\mathbb{Z}^m$ . The enumeration of these lattices is usually carried out up to equivalence, two lattices being called *equivalent* if one can be obtained from the other by a rotation and change of scale.

Alternative proofs of Niemeier's result have been given by Venkov [27] and in our earlier paper [9]. It is perhaps worth outlining a fourth proof, that uses hyperbolic geometry and throws some light on the connections between [7], [10] and [16].

Let  $G$  denote the group of all autochronous automorphisms of the even unimodular 26-dimensional Lorentzian lattice  $II_{25,1}$ , and let  $H$  be the reflection subgroup of  $G$ . The groups  $G$  and  $H$  were found in [6], where in particular it was shown that  $H$  is a Coxeter group whose graph is isomorphic to the Leech lattice. From Vinberg's work [28] it follows that the even unimodular 24-dimensional Euclidean lattices of minimum norm 2 are described by those maximal subdiagrams of the Leech lattice that are unions of the extended Coxeter–Dynkin diagrams  $A_n (n \geq 1)$ ,  $D_n (n \geq 4)$ ,  $E_6$ ,  $E_7$  and  $E_8$ . But in [7] it was shown that there are precisely 23 such subdiagrams, which are exactly those corresponding to the Niemeier lattices of minimum norm 2. This result, plus the fact [4] that the Leech lattice is the unique even unimodular lattice of minimum norm  $\geq 4$ , provides another proof of Niemeier's result. It also clarifies the one-to-one correspondence between the deep holes in the Leech lattice [7] and the Niemeier lattices. We must emphasize, however, that in no sense is this a short-cut to Niemeier's result, for the proofs in [7] required extensive computations.

**REMARK.** The notation used in this paper to specify the components of a lattice differs slightly from that used in [9]. For example the lattice  $E_6^3 O_1$  was called  $E_6^3[3]$  in reference [9]. The latter notation is more informative but less general (it fails when the empty component has dimension greater than one), and somewhat confusing, since [3] is also the name of a glue digit. We therefore recommend the  $\dots O_n$  notation for general use.

**2.2. CHAINS OF LATTICES.** Every unimodular lattice  $\Lambda$  appears in a uniquely determined chain of the form

$$\Lambda_n, \Lambda_n \oplus I_1, \Lambda_n \oplus I_2, \dots, \Lambda = \Lambda_n \oplus I_m, \dots$$

where the initial lattice  $\Lambda_n$  does not represent 1. We call  $\Lambda_n$  the *reduced* version of  $\Lambda$ . The summand  $I_m$  is the sublattice of  $\Lambda$  generated by vectors of norm 1, and  $\Lambda_n$  is its orthogonal complement. In these circumstances we have

$$|\text{Aut}(\Lambda_n \oplus I_m)| = |\text{Aut}(\Lambda_n)| \cdot 2^m m!$$

TABLE 2  
 Exact values of the Minkowski–Siegel mass constants for lattices of genus I

$n$	mass
1	1
2	$\frac{1}{8}$
3	$\frac{1}{48}$
4	$\frac{1}{384}$
5	$\frac{1}{3840}$
6	$\frac{1}{46080}$
7	$\frac{1}{645120}$
8	$\frac{1}{10321920}$
9	$\frac{17}{2786918400}$
10	$\frac{1}{2229534720}$
11	$\frac{31}{735746457600}$
12	$\frac{31}{5885971660800}$
13	$\frac{691}{765176315904000}$
14	$\frac{42151}{192824431607808000}$
15	$\frac{29713}{385648863215616000}$
16	$\frac{505121}{12340763622899712000}$
17	$\frac{642332179}{18881368343036559360000}$
18	$\frac{692319119}{15105094674429247488000}$
19	$\frac{8003636403977}{77489135679822039613440000}$
20	$\frac{248112728523287}{619913085438576316907520000}$
21	$\frac{593468652605200909}{216969579903501710917632000000}$
22	$\frac{50904295073459007001}{1507367607750643465322496000000}$
23	$\frac{1015740532498234470066371}{1317439289174062388691861504000000}$

TABLE 2 (cont.)

<i>n</i>	mass
24	<u>701876707956280018815862361</u> 21079028626784998219069784064000000
25	<u>84715059480304651623612272842147</u> 3046539608000631801426732908544000000
26	<u>14616335635894388876188472684851927</u> 31871491283698917307233513504768000000
27	<u>1894352751772146867430486995462923265007</u> 1242988160064257774982107026685952000000
28	<u>10345060377427694043037889482223023950203227</u> 99439052805140621998568562134876160000000
29	<u>4285009823959590682115628739356169586687220752159</u> 2883732531349078037958488301911408640000000
30	<u>156429914319579070270102710292957201465725850451195039</u> 34604790376188936455501859622936903680000000
31	<u>447543572700878404232772149927275573042725639059585587715489</u> 150184790232659984216878070763546161971200000000
32	<u>416022631331982837344253774635747627157753429574979915187099394241</u> 961182657489023898988019652886695436615680000000

TABLE 3  
Decimal expansions of the mass constants for lattices of genus I

<i>n</i>	mass	<i>n</i>	mass	<i>n</i>	mass	<i>n</i>	mass
1	1.0	9	6.100×10 <sup>-9</sup>	17	3.402×10 <sup>-14</sup>	25	2.781×10 <sup>-6</sup>
2	1.25	10	4.485×10 <sup>-10</sup>	18	4.583×10 <sup>-14</sup>	26	4.586×10 <sup>-4</sup>
3	2.083×10 <sup>-2</sup>	11	4.213×10 <sup>-11</sup>	19	1.033×10 <sup>-13</sup>	27	1.524×10 <sup>-1</sup>
4	2.604×10 <sup>-3</sup>	12	5.267×10 <sup>-12</sup>	20	4.002×10 <sup>-13</sup>	28	1.040×10 <sup>2</sup>
5	2.604×10 <sup>-4</sup>	13	9.031×10 <sup>-13</sup>	21	2.735×10 <sup>-12</sup>	29	1.486×10 <sup>5</sup>
6	2.170×10 <sup>-5</sup>	14	2.186×10 <sup>-13</sup>	22	3.377×10 <sup>-11</sup>	30	4.520×10 <sup>8</sup>
7	1.551×10 <sup>-6</sup>	15	7.705×10 <sup>-14</sup>	23	7.710×10 <sup>-10</sup>	31	2.980×10 <sup>12</sup>
8	9.688×10 <sup>-8</sup>	16	4.093×10 <sup>-14</sup>	24	3.330×10 <sup>-8</sup>	32	4.328×10 <sup>16</sup>

Our enumeration process considers all lattices of a chain simultaneously. For some purposes the most appropriate lattice to consider is the initial lattice  $A_n$ , but for other purposes it is the 23-dimensional lattice  $A_n \oplus I_{23-n}$ .

2.3. THE ASSOCIATED NIEMEIER LATTICE. We now describe how an odd unimodular lattice  $A_n$  of dimension  $n \leq 23$  is associated with a uniquely determined Niemeier lattice  $N$ . For brevity we omit the easy justifications of some statements, since in any case the mass formula provides an independent verification. Let  $m = 24 - n$ . Both  $A_n$  and  $I_m$  have sublattices of index 2 containing only vectors of even norm, denoted by  $A_n^0$  and  $I_m^0$ , respectively. The dual of  $A_n^0$  consists of  $A_n^0$  and three cosets, say  $A_n^1, A_n^2, A_n^3$ , and similarly the dual of  $I_m^0$  consists of the cosets  $I_m^0, I_m^1, I_m^2, I_m^3$ . The notation can be chosen

TABLE 4  
The mass constants for lattices of genus II

<i>n</i>	mass
0	1
8	$\frac{1}{696729600}$
16	$\frac{691}{277667181515243520000}$
24	$\frac{1027637932586061520960267}{129477933340026851560636148613120000000}$
32	$\frac{4890529010450384254108570593011950899382291953107314413193123}{121325280941552041649762780685623131486814208000000000}$

so that

$$\Lambda_n = \Lambda_n^0 \cup \Lambda_n^2,$$

$$I_m = I_m^0 \cup I_m^2,$$

and the set of all vectors

$$\{x + y : x \in \Lambda_n^k, y \in I_m^k, k = 0, 1, 2, 3\}$$

is an even unimodular 24-dimensional lattice. The latter is the Niemeier lattice *N* associated with  $\Lambda_n$ .

2.4. CONSTRUCTING  $\Lambda_n$  FROM *N*. The lattice  $I_m^0$  is better known as  $D_m$ , and seen from the point of view of *N* the process we have just described appears as follows. We look for a copy of the lattice  $D_m$  in *N* and observe that  $\Lambda_n$  is the set of vectors *x* in the orthogonal complement  $D_m^\perp$  for which either  $x + [0]$  or  $x + [2]$  is in *N*, where  $[0], [1], [2], [3]$  are the glue vectors for  $D_m$  in  $D_m^\perp$ ,  $[2]$  being a glue vector of norm 1. (For further information about these glue vectors see reference [9].)  $\Lambda_n$  will be the initial member of its chain if  $D_m$  is not contained in a  $D_{m'}$  in *N* with  $m' > m$  using the same glue vector  $[2]$ .

Alternatively we can refer to the 23-dimensional member of the chain, for which the appropriate  $D_m$  is a  $D_1$  generated by a single vector  $v = 2e$  of norm 4 and  $[2] = e$ . This 23-dimensional lattice is the set of vectors  $x \in e^\perp$  for which  $x + ne \in N$  for some integer *n*. The other members of the chain can be obtained by removing summands  $I_k$ .

Thus our process can be viewed from either end of the chain containing  $\Lambda_n$ . On the one hand we search for a maximal sublattice  $D_m$  (and glue vector  $[2]$ ) in the Niemeier lattice *N*, and locate the initial member of the chain in  $D_m^\perp$ . Alternatively we find all possible vectors  $v = 2e$  of norm 4 in *N* and locate the 23-dimensional member of the chain in the subspace  $v^\perp$ .

2.5. THE NIEMEIER LATTICES OF MINIMUM NORM 2. Let *N* be one of the 23 Niemeier lattices with minimum norm 2, and let  $W_1, W_2, \dots$  be its Witt components. The norm 4 vector *v* used in the construction may then be written as

$$v = v_1 + v_2 + \dots$$

where  $v_1, v_2, v_3, \dots$  are

- (1) minimal representatives of glue digits corresponding to a glue word of norm 4;



(2) minimal vectors  $r, s$  of norm 2 in two distinct Witt components  $W_i, W_j$ , and 0 in the other components; or

( $\geq 3$ ) 0 except for one  $v_i$  which is a vector of norm 4 in its Witt component  $W_i$ .

(Again we omit the easy proof of this statement.) The cases have been numbered so as to correspond with the value of  $m$  mentioned in Section 2.4.

In case (1) the  $D_1$  generated by  $v$  is maximal and we obtain a 23-dimensional lattice not representing 1. Our symbol for this case is the glue word mentioned.

In case (2) the  $D_2$  generated by  $r$  and  $s$  is maximal with  $[2] = \frac{1}{2}(r+s)$ , and so the reduced lattice is 22-dimensional. The symbol for this case is a word with \*'s in positions  $i$  and  $j$  and 0's elsewhere.

The cases ( $\geq 3$ ) can be further subdivided as follows. Case (3):  $v = v_i$  is the vector  $(1, 1, -1, -1, 0^{k-3})$  in a Witt component  $W_i = A_k$  for some  $k \geq 3$ . The maximal  $D_m$  is the sublattice  $A_3$  of  $A_k$  supported in the four non-zero coordinates of  $v_i$ . The reduced lattice is 21-dimensional, and the symbol for this case has a + in position  $i$  and 0's elsewhere.

Case (4):  $v = v_i$  is a vector of the form  $(\pm 1^4, 0^{k-4})$  in a component  $D_k$ . In this case the maximal  $D_m$  is the  $D_4$  supported in the four non-zero coordinates of  $v$ , and the reduced lattice is 20-dimensional. Again the symbol contains 0's except for a + in position  $i$ . We remark that the case  $k = 4$ , when there is a Witt component  $D_4$ , is rather special because all three non-trivial cosets of  $D_4$  in its dual contain vectors of norm 1 which can serve as the glue vector [2]. Fortunately, in both the cases  $N = D_4^6$  and  $N = A_5^4 D_4$ ,  $\text{Aut}(N)$  contains elements permuting all three cosets of any  $D_4$  component, so we may suppose  $v = (\pm 1^4, 0^{k-4})$ .

Cases ( $\geq 5$ ):  $v = v_i$  is the vector  $(2, 0^{k-1})$  in a  $D_k$  ( $k \geq 5$ ) or any norm 4 vector in  $E_6, E_7$  or  $E_8$ . The maximal  $D_m$  in these four cases is  $D_k, D_5, D_6$  or  $D_8$  respectively, and the symbol has a - in position  $i$  and 0's elsewhere.

2.6. INEQUIVALENT LATTICES. The above remarks make it easy to determine when two lattices constructed in this way are equivalent. Each lattice  $\Lambda_n$  is specified by a Niemeier lattice

$$N = W_1 W_2 \cdots W_a$$

and a symbol

$$w = d_1 d_2 \cdots d_a$$

which is either a glue word or a permutation of  $**0^{a-2}, +0^{a-1}$  or  $-0^{a-1}$ . Two lattices are equivalent if and only if they are specified by the same  $N$  and their symbols are equivalent under  $\text{Aut}(N)$ . The Witt components  $V_1, V_2, \dots$  of  $\Lambda_n$  are determined by the  $W_i$  and  $d_i$  as shown in Table 5.

2.7. THE AUTOMORPHISM GROUP OF  $\Lambda_n$ . To compute the order of the automorphism group of  $\Lambda_n$  we argue as follows. The lattices  $\Lambda_n, \Lambda_{n+1}, \dots$  of a chain whose reduced lattice is  $\Lambda_n$  have

$$|\text{Aut}(\Lambda_{n+m})| = |\text{Aut}(\Lambda_n)| \cdot 2^m m!,$$

so that it suffices to compute  $|\text{Aut}(\Lambda_{23})|$ . Now a 23-dimensional lattice  $\Lambda_{23}$  is completely determined by the associated Niemeier lattice  $N$  and either of the norm 4 vectors  $\pm v$  of  $\Lambda_{23}^\perp \cap N$ . Hence

$$|\text{Aut}(\Lambda_{23})| = \frac{2|\text{Aut}(N)|}{c(v)}, \tag{1}$$

where  $\text{Aut}(N)$  is given in [9] and  $c(v)$  is the number of images of  $v$  under  $\text{Aut}(N)$ . This number is easily computed as the product

$$c(v) = c(d_1)c(d_2) \cdots c(d_a)c(w), \tag{2}$$

where  $w = d_1d_2 \cdots d_a$  is the symbol for  $A_N$ ,  $c(w)$  is the number of images of  $w$  under  $\text{Aut}(N)$ , and  $c(d_i)$  is the number of choices for the component  $v_i$  of the vector  $v$  which would lead to the digit  $d_i$ . For example  $c(*)$  is the number of minimal vectors in  $W_i$ . The numbers  $c(d_i)$  can be found by elementary counting arguments and are given in Table 5. The values of  $c(w)$ , which are usually small, were found by careful consideration of the group action, and are given in Table 1.

TABLE 5

Each lattice  $A_n$  is specified by the Witt components  $W_1, W_2, \dots$  of a Niemeier lattice  $N$  and a certain symbol  $w = d_1, d_2, \dots$ . This table gives the Witt components  $V_1, V_2, \dots$  of  $A_n$  corresponding to each Witt component  $W$  of  $N$  and each digit  $d$  of  $w$ , as well as the numbers  $c(d)$  defined in Section 2.7. The four parts of the table correspond to (a)  $W = A_n$ , (b)  $W = D_4$ , (c)  $w = D_n (n \geq 5)$ , and (d)  $W = E_6, E_7$  or  $E_8$

$A_n$			$D_4$		
$d$	$V_1V_2 \cdots$	$c(d)$	$d$	$V_1V_2 \cdots$	$c(d)$
0	$A_n$	1	0	$D_4$	1
$i$	$A_{i-1}A_{n-i}$	$\binom{n+1}{i}$	1	$A_3$	8
*	$A_{n-2}$	$2\binom{n+1}{2}$	2	$A_3$	8
+	$A_{n-4}$	$6\binom{n+1}{4}$	3	$A_3$	8
			*	$A_1^3$	24
			+	-	24

$D_n (n \geq 5)$			$E_6$		$E_7$		$E_8$		
$d$	$V_1V_2 \cdots$	$c(d)$	$d$	$V_1V_2 \cdots$	$c(d)$	$V_1V_2 \cdots$	$c(d)$	$V_1V_2 \cdots$	$c(d)$
0	$D_n$	1	0	$E_6$	1	$E_7$	1	$E_8$	1
1	$A_{n-1}$	$2^{n-1}$	1	$D_5$	27	$E_6$	56		
2	$D_{n-1}$	$2n$	2	$D_5$	27				
3	$A_{n-1}$	$2^{n-1}$	*	$A_5$	72	$D_6$	126	$E_7$	240
			-	-	270	$A_1$	756	-	2160
*	$D_{n-2}A_1$	$4\binom{n}{2}$							
+	$D_{n-4}$	$16\binom{n}{4}$							
-	-	$2n$							

The orders of the automorphism groups of the lattices  $A_n$  in Table 1 were calculated from Equations (1) and (2). The lattices with minimum norm 1 were then found by forming direct sums  $A_n \oplus I_m$ , and the mass constants were checked against the formulae given in Theorem 2. (This completes the formal proof of Theorem 1.) The numbers of lattices found in each dimension are shown in Table 6.

TABLE 6  
The number of unimodular lattices in  $\mathbb{R}^n$  of genus I that do not represent 1 ( $\alpha_n$ ), the number of genus II ( $\beta_n$ ), and the total number of genus I and II ( $\gamma_n$ )

$n$	0	1	2	3	4	5	6	7	8	9	10	11	12
$\alpha_n$	0	0	0	0	0	0	0	0	0	0	0	0	1
$\beta_n$	1	0	0	0	0	0	0	0	1	0	0	0	0
$\gamma_n$	1	1	1	1	1	1	1	1	2	2	2	2	3
$n$	13	14	15	16	17	18	19	20	21	22	23	24	
$\alpha_n$	0	1	1	1	1	4	3	12	12	28	49	?	
$\beta_n$	0	0	0	2	0	0	0	0	0	0	0	24	
$\gamma_n$	3	4	5	8	9	13	16	28	40	68	117	?	

The purpose of the column headed  $g_1g_2$  in Table 1 is to provide the reader with an alternative and easier method of computing  $|\text{Aut}(A_n)|$ , via the formula

$$|\text{Aut}(A_n)| = \prod_{i=1}^b g_0(V_i) \cdot g_1(A_n)g_2(A_n). \tag{3}$$

Here  $V_1, \dots, V_b$  are the Witt components of  $A_n$  (given in the column headed  $V$  in Table 1), and the values of  $g_0(V_i)$  can be found in Table 7. The product  $g_1(A_n)g_2(A_n)$  is given in the column headed  $g_1g_2$  in Table 1. For example the penultimate line of Table 1 refers to the lattice  $A_1^{16}O_7$ , for which

$$|\text{Aut}(A_n)| = (2!)^{16} \cdot 1^7 \cdot 645120.$$

(The explanation for this decomposition of  $|\text{Aut}(A_n)|$  may be found in [8] or [9], but need not concern us here.) The values of  $g_1g_2$  were actually “back-computed” from Equations (1), (2) and (3) after the mass formulae check had been applied.

TABLE 7  
Values of  $g_0(W_i)$  for Witt components  $W_i$

$W_i$	$O_n$	$A_n$	$D_n$	$E_6$	$E_7$	$E_8$
$g_0(W_i)$	1	$(n+1)!$	$2^{n-1} \cdot n!$	$72 \cdot 6!$	$8 \cdot 9!$	$192 \cdot 10!$

2.8. DESCRIPTION OF TABLE 1. Table 1 shows the unimodular lattices  $A_n$  that contain no vectors of norm 1. The columns give the dimension  $n$ , the Witt components  $W_1, W_2, \dots$  of the corresponding Niemeier lattice  $N$ , the symbol  $w$ , the Witt components  $V = V_1, V_2, \dots$  of  $A_n$  itself, and the numbers  $c(w)$  and  $g_1(A_n)g_2(A_n)$  (see Section 2.7). The last column gives the number  $(t_2)$  of norm 2 vectors in  $A_n$ . We note that  $V$  identifies  $A_n$  uniquely.

2.9. THE LEECH LATTICE. It remains to consider the unique Niemeier lattice with minimum norm greater than 2, the Leech lattice. In this lattice there is just one orbit of norm 4 vectors ([5]). Our construction produces a certain 23-dimensional lattice with minimum norm 3, which may be called the *shorter Leech lattice*. It appears in the last line of Table 1, and is the only lattice in the table with minimum norm 3.

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