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FURTHER LATTICE PACKINGS IN HIGH DIMENSIONS

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Abstract. Barnes and Sloane recently described a "general construction" for lattice packings of equal spheres in Euclidean space. In the present paper we simplify and further generalize their construction, and make it suitable for iteration. As a result we obtain lattice packings in $\mathbb{R}^m$ with density Δ satisfying $\log_2 \Delta \sim -m \log_2^* m$, as $m \to \infty$, where $\log_2^* m$ is the smallest value of k for which the k-th iterated logarithm of m is less than 1. These appear to be the densest lattices that have been explicitly constructed in high-dimensional space. New records are also established in a number of lower dimensions, beginning in dimension 96.

§1. *Introduction.* In the past 23 years a series of papers [**1**–**3**, **5**, **8**–**12**, **15**–**17**] have described a variety of methods for packing equal spheres in Euclidean space. The present paper continues the sequence by simplifying the "general construction" of Barnes and Sloane [**1**], and thereby eliminating the requirement that the initial lattice be generated by its minimal vectors. This enables us to iterate the construction, obtaining lattice packings in $\mathbb{R}^m$ with density Δ satisfying

$$\log_2 \Delta \sim -m \log_2^* m, \qquad \text{as } m \to \infty, \tag{1}$$

where $\log_2^* m$ is the smallest value of k for which the k-th iterated logarithm of m is less than 1. These appear to be the densest lattice packings that have been explicitly constructed in high-dimensional space. Non-lattice packings with $\log_2 \Delta > -6m + o(m)$ were constructed in [**15**] (and non-lattice packings satisfying (1) in [**3**]), but there is still room for improvement since the best lattice or non-lattice packings are known to lie in the range

$$-m < \log_2 \Delta < -0{\cdot}599m + o(m), \quad \text{as } m \to \infty \tag{2}$$

(see [**4**, **7**, **14**, **18**]). The new construction also generalizes that given in [**1**] so as to use codes over other alphabets.

The construction is described in Section 2, and Section 3 contains a number of applications. A selection of the best packings known in dimensions up to 2^{20} will be found in Table 1.

Notation. The norm of a vector x is its squared length $x \cdot x$. If L_m is a lattice in $\mathbb{R}^m$ (usually the subscript indicates the dimension), its minimum norm M is $\min\{x \cdot x : x \in L_m, x \neq 0\}$, its determinant $\det L_m$ is the volume of a fundamental region, and its density Δ and centre density δ are given by

$$\Delta = \frac{V_m(\sqrt{M/2})^m}{\det L_m}, \qquad \text{and } \delta = \frac{\Delta}{V_m},$$

where V_m is the volume of an m-dimensional sphere of unit radius. In Section 3 we also use the parameters

$$\mu = \tfrac{1}{2}m, \quad \gamma = \log_2 \delta .$$

All the packings mentioned in this paper are lattice packings.

§2. *The Construction.* The ingredients for the construction are a lattice Λ in $\mathbb{R}^m$, an endomorphism D of Λ that satisfies certain conditions, one of which is that $\Lambda/D\Lambda$ is an elementary abelian group E of order p^b, say, and a family of codes $C_0 \supseteq C_1 \supseteq \ldots \supseteq C_a$ of length n over E; the result is a family $L_0 \subseteq L_1 \subseteq \ldots \subseteq L_a$ of lattice packings in $\mathbb{R}^{mn}$.

Hypotheses.

(i) Let Λ be a lattice in $\mathbb{R}^m$ with minimum non-zero norm M.

(ii) Let D be an endomorphism of Λ which is also a similarity (*i.e.* a constant times an orthogonal transformation) and which satisfies

$$pD^{-1} = \sum_{i=0}^{r} a_i D^i \tag{3}$$

for integers $p \geqslant 1, r \geqslant 0, a_0, \ldots, a_r$. Let $T = D^{-1}$, so that, from (3), pT is also an endomorphism of Λ.

(iii) Assume that $\Lambda/D\Lambda$ has the structure of an elementary abelian group E of order p^b for some integer $b \geqslant 1$. This implies that there is a b-dimensional sublattice $K \subseteq \Lambda$, spanned, say, by vectors $v_1, \ldots, v_b \in \Lambda$, such that

$$K/(D\Lambda \cap K) \cong E .$$

The assumptions also imply that

$$pK \subseteq K \subseteq \Lambda , \tag{4}$$

$$pK \subseteq p\Lambda \subseteq D\Lambda \subseteq \Lambda , \tag{5}$$

and therefore that

$$K/pK = K/(D\Lambda \cap K) \cong E . \tag{6}$$

Note that (4) and (5) imply $pK \subseteq D\Lambda \cap K$, so $K/pK \supseteq K/(D\Lambda \cap K)$. But both sides have order p^b and so must be equal, and (6) follows. Furthermore

$$|\det T| = \frac{1}{p^b} = t^m, \text{ say}, \tag{7}$$

and T multiplies norms by t^2.

(iv) Assume that all $p^b - 1$ non-zero congruence classes of $T\Lambda/\Lambda$ have minimum norm at least $t^2 M$.

(v) Let ϕ denote the natural map from $\mathbb{Z}/p\mathbb{Z}$ to $\mathbb{Z}$ which takes the congruence class $\bar{x}$ to x, for $x \in \{0, 1, ..., p-1\}$. The elements of E may be identified with the b-tuples $\bar{X} = (\bar{x}_1, ..., \bar{x}_b)$, where all $\bar{x}_i \in \mathbb{Z}/p\mathbb{Z}$. Then

$$\bar{X} \to V(\bar{X}) = \phi(\bar{x}_1)v_1 + ... + \phi(\bar{x}_b)v_b$$

$$= x_1 v_1 + ... + x_b v_b$$

is a map from E into Λ, and

$$\bar{\mathbf{X}} = (\bar{X}_1, ..., \bar{X}_n) \to V(\bar{\mathbf{X}}) = (V(\bar{X}_1), ..., V(\bar{X}_n))$$

maps E^n into Λ^n.

(vi) Let $C_0 \supseteq C_1 \supseteq ... \supseteq C_a$ be additive codes† over E of length n, where C_i contains p^{bk_i} codewords and has minimum distance d_i (we indicate this by saying that C_i has parameters $[n, k_i, d_i]$), and suppose that C_0 is the trivial $[n, n, 1]$ code. Let $\mathbf{c}_1, ..., \mathbf{c}_{bk_a} \in E^n$ be chosen so that a typical codeword of C_i can be written as

$$\sum_{j=1}^{bk_i} \bar{x}_j \mathbf{c}_j, \quad \bar{x}_j \in \mathbb{Z}/p\mathbb{Z},$$

for $i = 1, ..., a$.

The New Lattices. Let $L_0 = \Lambda^n$ and, for $i = 1, 2, ..., a$, define

$$L_i = \bigcup_{x_1, ..., x_{bk_i}} \left\{ L_{i-1} + \sum_{j=1}^{bk_i} x_j T^i V(\mathbf{c}_j) \right\}, \tag{8}$$

where $x_1, ..., x_{bk_i} \in \{0, 1, ..., p-1\}$. By abuse of notation we shall also use D and T to denote the maps $(D, D, ..., D)$ and $(T, T, ..., T)$ acting on $\mathbb{R}^{mn}$. It is clear that L_0 is a lattice and that D and pT are endomorphisms of L_0.

THEOREM 1. *For* $i = 1, ..., a$,

(a) *L_i is a lattice, and in fact*

$$L_i = \mathbb{Z}\langle L_{i-1}, T^i V(\mathbf{c}_1), ..., T^i V(\mathbf{c}_{bk_i})\rangle; \tag{9}$$

† *I.e.* abelian subgroups of E^n (compare Delsarte [6]).

(b) *D maps L_i into L_{i-1}; and*

(c) *pT is an endomorphism of L_i.*

Proof. The proof is by induction on i, the results for $i = 0$ having already been mentioned. (a) To show that L_i is a lattice, we write any integer combination of the elements on the right-hand side of (8) as

$$l+p \sum_{j=1}^{bk_i} y_j T^i V(\mathbf{c}_j) + \sum_{j=1}^{bk_i} z_j T^i V(\mathbf{c}_j), \tag{10}$$

where $l \in L_{i-1}$ and $0 \leqslant z_j \leqslant p-1$. But

$$pT^i V(\mathbf{c}_j) = pT \,.\, T^{i-1} V(\mathbf{c}_j) \in L_{i-1}$$

by the induction hypothesis, so (10) becomes

$$l' + \sum_{j=1}^{bk_i} z_j T^i V(\mathbf{c}_j)$$

where $l' \in L_{i-1}$, which by (8) is in L_i. Thus L_i is a lattice, and therefore can be defined by the right-hand side of (9). Now (b) follows from

$$DT^i V(\mathbf{c}_j) = T^{i-1} V(\mathbf{c}_j) \in L_{i-1}\,.$$

From (3) and (b) we have

$$pT(L_i) = pD^{-1}(L_i) = \sum_{j=0}^{r} a_j D^j(L_i) \subseteq L_i\,,$$

which is (c).

THEOREM 2. *For $i = 0, 1, \ldots, a$, the determinant, minimum norm and centre density of L_i are given by*

$$\det L_i = \frac{(\det \Lambda)^n}{p^{b \sum_{j=1}^{i} k_j}}, \tag{11}$$

$$\bar{M} = \min\{M, d_j t^{2j} M \text{ for } j = 1, \ldots, i\}, \tag{12}$$

and

$$\delta = \bar{M}^{mn/2}/2^{mn} \det L_i \tag{13}$$

respectively.

Proof. Equation (11) is immediate from (8), and (12) follows from the definition of L_i and the fact that the minimum distance of C_i is d_i. For more details compare the proof of Theorem 3 in [**1**].

Usually we are only interested in the finest lattice L_a. The construction may now be applied to L_a, since it inherits D and T from Λ, and (3) still holds. The values of p and t are unchanged, while b becomes nb.

§3. *Examples.* The above construction includes some of the most important special cases of Constructions A and B of [12] (see [1]), as well as the constructions of [1], and so a large number of examples are already known. In this section we give several additional examples of some interest.

If D is a norm-doubler. In all these examples D will be a norm-doubler, so that $t = 1/\sqrt{2}$, $p = 2$ and $b = m/2$. We shall take the codes $C_i, i \geqslant 1$, to be maximal distance separable codes over the field $GF(2^b)$, with parameters $[n, k_i = n-2^i+1, d_i = 2^i]$. In general, the largest n for which such codes are presently known to exist is 2^b+1 (see [13, Chapter 11]). When $n = 2^b+1$, $C_i, i \geqslant 1$, can be taken to be the cyclic code with generator polynomial

$$g(x) = \prod_{j=-s}^{s} (x+\xi^j),$$

where $\xi \in GF(2^{2b})$ is a primitive n-th root of unity and $s = 2^{i-1}-1$ (see the proof of Theorem 9, Chapter 11 of [13])†. Besides being additive, these codes are also closed under multiplication by elements of $GF(2^b)$. However we make no use of this multiplicative structure in our construction. Clearly $C_0 \supseteq C_1 \supseteq C_2 \supseteq \ldots$. For n less than 2^b+1 the codes are shortened by setting the appropriate number of information symbols equal to zero. If n lies in the range $2 \leqslant n \leqslant 2^{b+1}$, we use the codes $C_1, C_2, \ldots, C_a$, where a is determined by $2^a \leqslant n < 2^{a+1}$. Then the value of $\sum_{j=1}^{a} k_j$ for use in (11) is

$$an-2^{a+1}+a+2. \tag{14}$$

Since $d_j t^{2j} = 1$, the minimum norm is unchanged (see (12)). This version of the construction therefore converts a lattice Λ in $\mathbb{R}^m$ with centre density $\delta = 2^\gamma$ (say) into a lattice $\Lambda' = L_a$ in $\mathbb{R}^{mn}$ with centre density $\delta' = 2^{\gamma'}$, where from (11), (13), (14)

$$\gamma' = n\gamma + \frac{m}{2}(an-2^{a+1}+a+2). \tag{15}$$

Our examples are all descendants of the three lattices $\mathbb{Z}^2$, Λ_{24} and $\mathbb{P}48q$.

Packings constructed from the Lattice $\mathbb{Z}^2$. For the first example we take Λ to be the familiar two-dimensional square lattice $\mathbb{Z}^2$ with minimum norm $M = 1$ (see Figure 1). Although this is not a particularly dense packing, it has surprisingly good progeny. We let D map $(1,0)$ to $(1,1)$, and $(0,1)$ to $(1,-1)$, or in matrix notation, with D mapping v to vD,

$$D = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \quad T = D^{-1} = \frac{1}{2}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

so that (Eq. 3) reads $2D^{-1} = D$. Then $\Lambda/D\Lambda \cong \mathbb{Z}/2\mathbb{Z}$, $p^b = 2^1$, and we may take K to be the 1-dimensional lattice spanned by $v_1 = (1,0)$ (see Figure 1).

† That theorem is incorrect if the field size q is odd (see [13, p. xii]) but that does not concern us since here q is even.

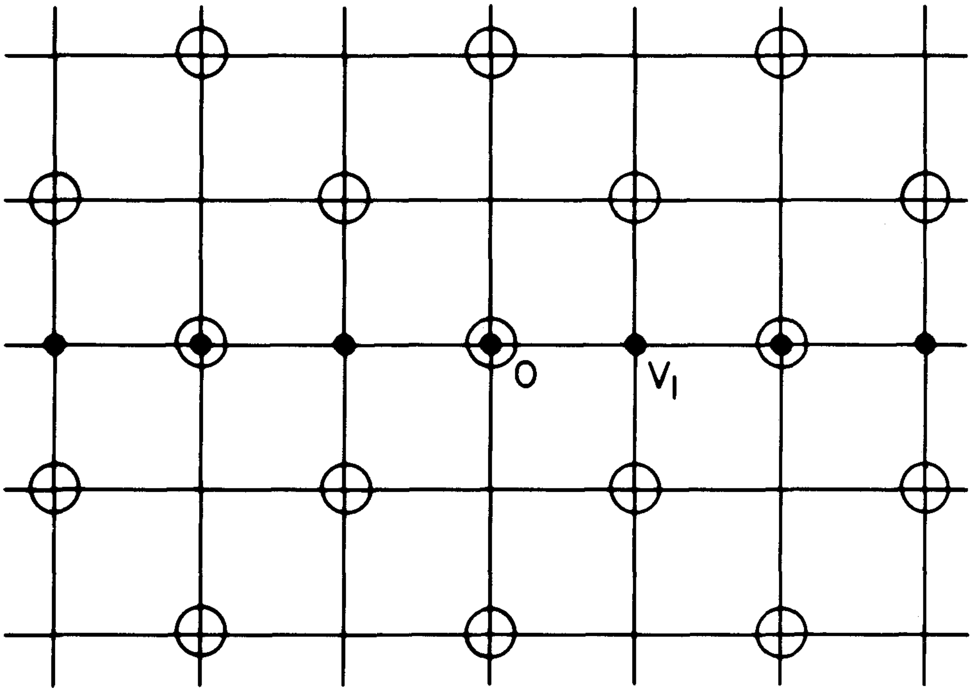

Figure 1. The lattice $\Lambda = \mathbb{Z}^2$ (represented by the underlying grid of points), and the sublattices $D\Lambda$ (small circles) and K (solid circles).

We apply our construction to $\mathbb{Z}^2$, with $n = 2$ and C_1 equal to the code $\{00, 11\}$ over $GF(2)$, and obtain the Schläfli lattice D_4, with $\gamma = -3$ (from (15)).

From now on we need not specify D, T or p^b, since these are determined (see the remarks at the end of Section 2). Applying the construction to D_4 with $n = 1, ..., 5$ we obtain the laminated lattices (see [5])

$$\Lambda_4 = D_4, \quad \Lambda_8 = E_8, \quad \Lambda_{12}, \quad \Lambda_{16}, \quad \Lambda_{20}$$

for which γ is

$$-3, -4, -5, -4, -3$$

respectively.

E_8 (the first grandchild of $\mathbb{Z}^2$) was one of the examples studied in [1]†. It produces Λ_{16} again, packings $\bar{\Lambda}_{32}$ and $\bar{\Lambda}_{40}$ with the same density as the laminated lattice Λ_{32} and Λ_{40}, excellent packings in dimensions 80 to 136, but indifferent packings in dimensions 24 and 48 to 72.

Table 1 contains a selection of the best lattice packings known up to dimension 2^{20}. The second column gives $\gamma = \log_2 \delta$, while a parenthesized entry (Λ) in the third column indicates that the packing can be obtained by applying our construction to the lattice Λ. Those packings (in dimension $\geqslant 96$) for which no reference is given are believed to set new records for lattices.

The children of E_8 extend to dimension 136 in the table and then stop. Beyond this point the lattices obtained by applying the construction to Λ_{12} and Λ_{16} are roughly comparable, up to dimension $16(2^8+1)$. The children of $\Lambda_{20}, \bar{\Lambda}_{32}$ and $\bar{\Lambda}_{40}$

† The map D used in [1] differs from the one given here, but is equivalent to it. One can in fact show that there is essentially only one norm-doubling map for E_8 with the requisite properties.

Table 1. A selection of the densest known packings
n = dimension, $\gamma = \log_2$ (centre density)

n	γ	name	Ref.	n	γ	name	Ref.
0	0	Λ_0		492	683	(Λ_{12})	
4	-3	D_4		496	660	(Λ_{16})	
8	-4	E_8		500	665	(Λ_{20})	
16	-4	Λ_{16}	[2]	504	708	(Λ_{12})	
24	0	Λ_{24}	[9]	512	698	B_{512}	[1]
32	0	Λ_{32}	[2,12]	516	733	(Λ_{12})	
40	4	Λ_{40}	[12]	1024	1856	(Λ_{16})	
48	14·039	$\mathbb{P}48q$	[12]	1032	1896	(Λ_{24})	[1]
64	22	—	[1]	1040	1900	(Λ_{16})	
80	36	(E_8)	[1]	2040	4680	(Λ_{24})	[1]
96	52·078	$(\mathbb{P}48q)$		2048	4680	(Λ_{16})	
128	88	(E_8)	[1]	2064	4752	(Λ_{24})	[1]
136	100	(E_8)	[1]	4096	11344	(Λ_{16})	
160	112	(Λ_{16})		4104	11400	(Λ_{24})	[1]
168	120	(Λ_{24})	[1]	8208	26808	(Λ_{24})	[1]
176	132	(Λ_{16})		16392	61608	(Λ_{24})	[1]
192	156	(Λ_{24})	[1]	32784	139488	(Λ_{24})	[1]
256	250	B_{256}	[1]	65544	311496	(Λ_{24})	[1]
264	264	(Λ_{24})	[1]	131088	$6{\cdot}613\times10^5$	$(\mathbb{P}48q)$	
272	268	(Λ_{16})		262176	$1{\cdot}453\times10^6$	$(\mathbb{P}48q)$	
288	300	(Λ_{24})	[1]	524304	$3{\cdot}168\times10^6$	$(\mathbb{P}48q)$	
384	464	(Λ_{16})		1048608	$6{\cdot}861\times10^6$	$(\mathbb{P}48q)$	

are not quite as good as other known packings. However, examination of Table 1 (and of a more extensive table of which this is an extract) shows that at the moment one cannot completely ignore the children of any of these lattices.

Packings constructed from the Leech lattice Λ_{24}. The Leech lattice Λ_{24} is another example that was studied in [1], using the norm-doubling map $D = I - i$, where i is a certain automorphism of Λ_{24}. There is a second choice for D, essentially different from $I - i$, which may be defined as follows. Let the 24 coordinate positions be labelled $\infty, 0, 1, \ldots, 22$, and as usual (see [**9**, **12**]) let Λ_{24} be spanned by the vectors:—

$(2^{12}, 0^{12})$, 23 vectors, supported on a translate of $\{0\} \cup q$, where q denotes the set of non-zero quadratic residues modulo 23,

$(-3, 1^{23})$, a single vector, and

$(\pm 4^2, 0^{22})$, $4 \,.\binom{24}{2}$ vectors.

Then we may take D to be represented by the matrix

$$\frac{1}{4}\left|\begin{array}{c|ccccccc} 3 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 \\ \hline -1 & -3 & 1 & 1 & 1 & 1 & \ldots & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & \ldots & -3 \\ -1 & 1 & 1 & 1 & -1 & 1 & \ldots & 1 \\ & & \ldots & & & & \ldots & \\ -1 & -1 & -3 & 1 & 1 & 1 & \ldots & -1 \end{array}\right|$$

where in the second row there is a -3 in position 0, 1's in the positions in q, and -1's elsewhere. Also $T = D^{-1} = \frac{1}{2}D$ and $p^b = 2^{12}$. Applying our construction to Λ_{24} using either version of D produces lattices which run neck and neck with Λ_{12} and Λ_{16} up to dimension 4096, and beyond this point are the best packings known up to dimension $24(2^{12}+1)$ (see Table 1 and [1]).

Packings constructed from $\mathbb{P}48q$. Our third starting point is the 48-dimensional lattice $\mathbb{P}48q$ constructed in [12], which has a norm-doubling map very similar to the second map found for Λ_{24}. If the coordinate positions are labelled $\infty, 0, 1, \ldots, 46$, we may take $\mathbb{P}48q$ to be spanned by the vectors:—

$(2^{24}, 0^{24})$, 47 vectors, supported on a translate of $\{0\} \cup q$, where q denotes the set of non-zero quadratic residues modulo 47,

$(-5, 1^{47})$, a single vector, and

$(\pm 6^2, 0^{46})$, $4 \cdot \binom{48}{2}$ vectors.

Then we may take D to be

$$\frac{1}{6}\left|\begin{array}{c|ccccccc} 5 & -1 & -1 & -1 & -1 & -1 & \ldots & -1 \\ \hline -1 & -5 & 1 & 1 & 1 & 1 & \ldots & -1 \\ -1 & 1 & 1 & 1 & 1 & -1 & \ldots & -5 \\ -1 & 1 & 1 & 1 & -1 & 1 & \ldots & 1 \\ & & & \cdots & & & \cdots & \\ -1 & -1 & -5 & 1 & 1 & 1 & \ldots & -1 \end{array}\right|$$

where in the second row there is a -5 in position 0, 1's in the positions in q, and -1's elsewhere. Also $T = D^{-1} = \frac{1}{2}D$. This gives excellent packings in dimensions 96 and from $24(2^{12}+1)$ to $48(2^{24}+1)$.

Higher dimensions. The construction may be applied repeatedly to any of these packings, producing an infinite tree of lattices. We may derive a lower bound to the density obtained in this way in high dimensions by the following argument. Let Λ be any of the lattices described in this section, with say dimension $m = 2\mu$, centre density $\delta = 2^{\gamma}$, and norm-doubling map D. We apply our construction using codes of length $n = 2^{\mu}$, obtaining a lattice $\Lambda'(= L_a)$ in dimension $m' = 2\mu'$ with centre density $\delta' = 2^{\gamma'}$, where, from (15)

$$\mu' = \mu 2^{\mu}, \tag{16}$$

$$\gamma' = 2^{\mu}\gamma + \mu(\mu-2)2^{\mu} + \mu(\mu+2). \tag{17}$$

It is simpler to work with the density Δ rather than the centre density δ, so let us define

$$\eta = -\frac{\log_2 \Delta}{m}.$$

(We know from (2) that for large dimensions m the best packings satisfy $0{\cdot}599 \leqslant \eta \leqslant 1$.) Now

$$\log_2 \Delta = \log_2 \delta - \mu \log_2 \mu + \mu \log_2 (\pi e) - \tfrac{1}{2}\log_2 (2\pi\mu) + o(1),$$

and so (17) becomes

$$\eta' = \eta + 1 - \frac{1}{4\mu}\log_2 (2\pi\mu) + o\left(\frac{1}{\mu}\right). \tag{18}$$

The solution of (16) and (18) is

$$\eta(m) = \log_2^* (m)$$

and therefore applying the construction repeatedly leads to lattices satisfying (1). Note that this asymptotic behavior is independent of the choice of the initial lattice.

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