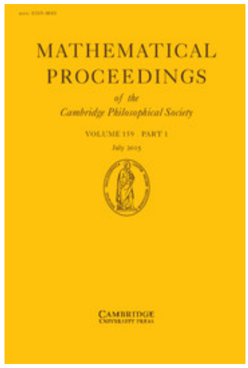

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The Coxeter–Todd lattice, the Mitchell group, and related sphere packings

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Abstract

This paper studies the Coxeter–Todd lattice Λ_6^ω, its automorphism group (which is Mitchell's reflection group $6\cdot PSU(4,3)\cdot 2$), and the associated 12-dimensional real lattice K_{12}. We give several constructions for Λ_6^ω, which is a $\mathbf{Z}[\omega]$-lattice where $\omega = e^{2\pi i/3}$; enumerate the congruence classes of $\Lambda_6^\omega/2\Lambda_6^\omega$ and $\Lambda_6^\omega/\theta\Lambda_6^\omega$, where $\theta = \omega - \bar{\omega}$; prove the lattice is unique; determine its covering radius and deep holes; and study its connections with the lattice E_6 and the Leech lattice. A number of new dense lattices in dimensions up to about 10^7 are constructed. We also give an explicit basis for the invariants of the Mitchell group. The paper concludes with an extensive bibliography.

1. *Introduction*

This paper studies the Coxeter–Todd lattice Λ_6^ω in six complex dimensions (23), and its automorphism group $G_0 = \mathrm{Aut}(\Lambda_6^\omega)$, which is Mitchell's (46) complex reflection group of order $108\cdot 9!$, isomorphic† to $6\cdot P\Omega^-(6,3)\cdot 2$ and to $6\cdot PSU(4,3)\cdot 2$. We simultaneously investigate the corresponding real lattice K_{12}, which is the densest 12-dimensional sphere packing known.

We begin (in Section 2) by giving four constructions for Λ_6^ω, each referring to a different base; the four versions are denoted by $\Lambda^{(2)}$, $\Lambda^{(3)}$, $\Lambda^{(4)}$ and $\Lambda^{(7)}$ (see Tables 1–3). Each version makes a different subgroup of G_0 visible. Λ_6^ω (by any of these definitions) is a $\mathbf{Z}[\omega]$-lattice, where $\omega = e^{2\pi i/3}$ (cf. (17)), and in Section 3 we describe the congruence classes of $\Lambda_6^\omega/2\Lambda_6^\omega$ and $\Lambda_6^\omega/\theta\Lambda_6^\omega$, where $\theta = \omega - \bar{\omega} = \sqrt{-3}$. The knowledge of these classes is used repeatedly in the rest of the paper. The first application is found in Section 4, where we give two proofs that Λ_6^ω is unique. The first proof (Theorem 1) characterizes Λ_6^ω by the number of vectors of norms 1, 2, 3 and 4, and establishes that $\Lambda^{(2)}$, $\Lambda^{(3)}$, $\Lambda^{(4)}$ and $\Lambda^{(7)}$ are isomorphic. The second proof (Theorem 3) characterizes Λ_6^ω as the unique unimodular $\mathbf{Z}[\omega]$-lattice in dimension < 12 that contains no vector of norm 1, a result originally obtained by Feit (29) via direct enumeration.

In Section 5 the notion of a congruence base for a lattice is introduced. When combined with the information about congruence classes, this makes it easy to determine the order of the automorphism group. Sections 6 and 7 describe the strong connections

† Other names for this group are $6\cdot U_4(3)\cdot 2$, $6\cdot HO(4, 3^2)\cdot 2$ (Dickson (25)), $[2\ 1;\ 3]^2$ (Shephard (49)), $[3\ 2\ 1]^3$ (Coxeter, Benard (2)), and $W(K_6)$ (Cohen (8)). It is number 34 on Shephard and Todd's list (50).

between Λ_6^ω, the lattice E_6, and the Leech lattice Λ_{24}. Theorem 4 for example shows that Λ_6^ω is the $\mathbf{Z}[\omega]$-span of E_6 and θE_6^* (the star denoting the dual lattice). The Leech lattice contains two orthogonal copies of Λ_6^ω, one in the form $\Lambda^{(2)}$, the other $\Lambda^{(4)}$. Theorem 5 describes the reverse process, by which $\Lambda^{(2)}$ and $\Lambda^{(4)}$ are glued together to form Λ_{24}. In Section 8 we determine the covering radius of Λ_6^ω and K_{12}, and show that there is a unique type of deep hole (cf. (13)) in these lattices. As a corollary it follows that $K_{13}, \ldots, K_{18}$ (see (41)) are densest possible lattices containing K_{12}.

Several useful maps are defined in this paper. For example equation (18) embeds E_6 in $\Lambda^{(4)}$. The maps σ and τ (equations (12), (13) are explicit isomorphisms from $\Lambda^{(2)}$ to $\Lambda^{(3)}$ and $\Lambda^{(2)}$ to $\Lambda^{(4)}$, and the gluing map Δ_w (30) is a norm-doubling homomorphism from $\Lambda^{(4)}$ to $\Lambda^{(2)}$. Thus the composition $\lambda = \Delta_w \circ \tau$ (43) is a norm-doubling endomorphism of Λ_6^ω, and Θ (44) is a norm-trebling endomorphism of Λ_6^ω. In Section 9 we use λ to construct new lattice packings, that are denser than any previously known in dimensions 228, 240, 252, ..., 780. Similarly a norm-trebling endomorphism of Λ_{24} leads to new records in dimensions $24(2^{12}+1) < 24n \leqslant 24(3^{12}+1)$ (see the example at the end of Section 9).

Finally, in Section 10, we discuss the history of the invariants of G_0, and in Theorem 10 present a basis for these invariants.

We shall describe algorithms for decoding Λ_6^ω and K_{12} (i.e. finding the closest lattice point to an arbitrary point of $\mathbf{C}^6$ or $\mathbf{R}^{12}$, cf. (16)), as well as properties of their Voronoi regions, in a sequel (19) to this paper.

The lattice Λ_6^ω and the group G_0 have a long history. The group and its associated 5-dimensional collineation group G_1 (obtained by factoring out the centre of order 6) were discovered by Mitchell (46) in 1914. The conjugacy classes of G_1 were enumerated by Hamill (37), and the character tables of G_1 and G_0 were given by Todd (59) and Benard (2). The subgroups and their associated geometrical configurations were extensively studied by Hartley (38), (39), Todd (59), Hamill (37) and Edge (27), (28). G_0 has received attention recently because of its connection with certain sporadic simple groups: see Conway (11), Gorenstein (35), (36), Kantor (40), Wilson (60), (61). For the isomorphisms $G_0 \cong 6\cdot P\Omega^-(6, 3)\cdot 2 \cong 6\cdot PSU(4, 3)\cdot 2$ see Tits (57), Dieudonné (26), Carter (6), and Bruen and Hirschfeld (5). See also Coxeter (21) and Lindsey (42), (43).

The lattice Λ_6^ω, which may be regarded as being generated by the centres of the homologies of G_1, was first explicitly described by Coxeter and Todd in 1954 (23). The 756 minimal vectors are the vertices of the complex uniform polytope $(2\ 1;\ 3_3)^3$, and Λ_6^ω itself is the degenerate polytope $(2\ 1;\ 4_4)^3$ (see (49), pp. 380–381). The theta series is given in equation (14) below, and more fully in ((54), table VII).

The name Λ_6^ω for this lattice is explained by the following characterization, established in (18). Starting with the 0-dimensional 1-point lattice Λ_0^ω, let us define n-dimensional lattices Λ_n^ω inductively by: (i) each Λ_n^ω is an integral $\mathbf{Z}[\omega]$-lattice of minimal norm 2, (ii) each Λ_n^ω contains at least one Λ_{n-1}^ω, and (iii) the Λ_n^ω have the smallest possible determinant subject to (i) and (ii). Then there is a unique Λ_6^ω, the Coxeter–Todd lattice. Other references to this lattice and its group will be found in the body of the paper.

Definitions and notation (see also (17), (18)). An n-dimensional $\mathbf{Z}[\omega]$-lattice L_n is a free $\mathbf{Z}[\omega]$-module in $\mathbf{C}^n$ (usually the subscript gives the dimension). The dual lattice $L_n^* = \{x \in \mathbf{C}^n : x\cdot\bar{y} \in \mathbf{Z}[\omega] \text{ for all } y \in L_n\}$. L_n is integral if $L_n \subseteq L_n^*$, and unimodular if

$L_n = L_n^*$. The norm of a vector $x = (x_1, \ldots, x_n) \in \mathbf{C}^n$ is $N(x) = x \cdot \bar{x} = \Sigma |x_i|^2$, and $(x_1, \ldots, x_n)_c$ is an abbreviation for $\alpha^{-1}(x_1, \ldots, x_n)$, where α is an appropriate constant of norm c. The number of vectors of norm i in L_n is denoted by u_i. The automorphism group $\mathrm{Aut}(L_n)$ is the subgroup of the unitary group $U(n, \mathbf{C})$ fixing L_n. Two lattices L_n and M_n are isomorphic (written $L_n \cong M_n$) if they differ by an element of $U(n, \mathbf{C})$ and possibly a change of scale. Finally $\mathbf{H}$ denotes the skew field of quaternions.

2. *Four definitions for the lattice* Λ_6^ω

2.1. We begin by defining four lattices $\Lambda^{(2)}$, $\Lambda^{(3)}$, $\Lambda^{(4)}$, $\Lambda^{(7)}$, which in Section 4 will be shown to be isomorphic (to Λ_6^ω). The notation $\Lambda^{(a)}$ is suggested by the fact that all inner products in $\Lambda^{(a)}$ are divisible by a before scaling, and so, on the minimal scale at which it is an integral lattice, all the vectors of $\Lambda^{(a)}$ can be written in the form $(x_1, x_2, \ldots)_a$ with $x_1, x_2, \ldots \in \mathbf{Z}[\omega]$. We say that $\Lambda^{(2)}, \ldots, \Lambda^{(7)}$ show Λ_6^ω represented in the *2-base*, ..., *7-base* respectively.† A vector of norm $\leqslant 4$ in any of these lattices is called a *short* vector, and in Tables 1–3 we give lists of the short vectors in $\Lambda^{(2)} - \Lambda^{(4)}$.

2.2. *Definition of* $\Lambda^{(2)}$. The most concise description is to say that $\Lambda^{(2)}$ is obtained by applying construction A of (52) to the hexacode (13). More explicitly, the hexacode $\mathbf{C}_{\mathrm{hex}}$ (see (13), (14), (17), (44)) is the [6, 3, 4] code over GF(4) consisting of all vectors that can be obtained from the five vectors

$$\begin{array}{ccc} 01 & 01 & \omega\bar{\omega} \\ \omega\bar{\omega} & \omega\bar{\omega} & \omega\bar{\omega} \\ 00 & 11 & 11 \\ 11 & \omega\omega & \bar{\omega}\bar{\omega} \\ 00 & 00 & 00 \end{array}$$

by freely permuting the three pairs, reversing any even number of pairs, and scalar-multiplying by any power of ω. (The five words have 36, 12, 9, 6, 1 images respectively.) Also

$$\mathbf{Z}[\omega]/2\mathbf{Z}[\omega] \cong \mathrm{GF}(4), \tag{1}$$

and there is a natural map

$$\sigma: \mathbf{Z}[\omega] \to \mathbf{Z}[\omega]/2\mathbf{Z}[\omega] \overset{\approx}{\to} \mathrm{GF}(4).$$

Then

$$\Lambda^{(2)} = \{(x_1, \ldots, x_6)_2 : x_i \in \mathbf{Z}[\omega], (\sigma(x_1), \ldots, \sigma(x_6)) \in \mathbf{C}_{\mathrm{hex}}\}$$

((52), example 3). It is straightforward to show that $\Lambda^{(2)}$ is a unimodular $\mathbf{Z}[\omega]$-lattice.

The corresponding 12-dimensional real lattice K_{12} may be defined by

$$K_{12} = \{(\mathrm{Re}\,(x_1), \mathrm{Im}\,(x_1), \ldots, \mathrm{Re}\,(x_6), \mathrm{Im}\,(x_6)) : (x_1, \ldots, x_6)_2 \in \Lambda^{(2)}\}. \tag{2}$$

The short vectors in $\Lambda^{(2)}$ are listed in Table 1. The third column shows typical vectors of each shape and the fourth column the numbers of such vectors. For example the third line of the table refers to the vectors such as

$$\omega^\nu(0, \pm 1, 0, \pm 1, \pm \omega, \pm \bar{\omega})_2 \quad (\nu = 0, 1, 2)$$

† Mitchell (46), in his original construction of the group, used the 4-base to specify the centres of the homologies, and Todd (58) used all four bases (although without defining the hexacode). Coxeter and Todd (23) introduced the lattice in terms of the 3-base.

Table 1. $\Lambda^{(2)}$, *which is* Λ_8^{ω} *written in the* 2-*base*

(The visible group is $(2^6 \times 3)\,A_6$. The rows of the table give short vectors arranged in orbits under the visible group. The last two columns show how the congruence classes of $\Lambda^{(2)}/2\Lambda^{(2)}$ and $\Lambda^{(2)}/\theta\Lambda^{(2)}$ are divided among the rows.)

Norm of vector	Norm of coordinates	Coordinates	Number	Class mod 2	Class mod θ
0	(0^6)	$(0^6)_2$	1	1	1
2	$(4, 0^5)$	$(2, 0^5)_2$	$3\cdot 2^1\cdot 6$	2	3
	$(1^4, 0^2)$	$(0, 1, 0, 1, \omega, \bar{\omega})_2$	$3\cdot 2^4\cdot 15$	2	3
		Total	756		
3	$(3, 1^3, 0^2)$	$(0, \theta, 0, 1, \omega, \bar{\omega})_2$	$3\cdot 2^4\cdot 15\cdot 4$	2	18
	(1^6)	$(1, 1, \omega, \omega, \bar{\omega}, \bar{\omega})_2$	$3\cdot 2^6\cdot 6$	2	18
		Total	4032		
4	$(4^2, 0^4)$	$(2, 2\omega, 0^4)$	$3\cdot 2^2\cdot \binom{6}{2}\cdot 3$	12	$9b$
	$(4, 1^4, 0)$	$(2, 1, 0, 1, \omega, \bar{\omega})_2$	$3\cdot 2^5\cdot 15\cdot 6$	$8a$	$45c$
	$(3^2, 1^2, 0^2)$	$(0, \theta, 0, \theta, \omega, \bar{\omega})_2$	$3\cdot 2^4\cdot 15\cdot 6$	$4a$	$72b$
	$(3, 1^5)$	$(\theta, 1, \omega, \omega, \bar{\omega}, \bar{\omega})_2$	$3\cdot 2^6\cdot 6\cdot 6$	12	$36c$
		Total	20412		

obtained from any of the 15 projectively distinct hexacodewords of weight 4. There are $3\cdot 2^4\cdot 15$ vectors of this shape. The fifth line refers to the $3\cdot 2^6\cdot 6$ vectors such as

$$\omega^{\nu}(\pm 1, \pm 1, \pm\omega, \pm\omega, \pm\bar{\omega}, \pm\bar{\omega})_2 \quad (\nu = 0, 1, 2)$$

obtained from the 6 projectively distinct hexacodewords of weight 6.

The Mitchell group G_0 may now be defined as $\mathrm{Aut}(\Lambda^{(2)})$. It is not difficult to show that this group is generated by the reflections in the minimal vectors of $\Lambda^{(2)}$. For it is easily verified that the subgroup generated by these reflections acts transitively on the vectors of norms 2, 3, 4 and 5, and that $\mathrm{Aut}(\Lambda^{(2)})$ contains no other elements. Note also that $\mathrm{Aut}(\Lambda^{(2)})$ contains a monomial subgroup $(2^6 \times 3)\,A_6$. This is a split extension of the subgroup 2^6 (generated by all sign changes) by $\mathrm{Aut}(C_{\mathrm{hex}})$ (which is a nonsplit extension $3A_6$ of a cyclic group of order 3 by the alternating group of degree 6). We call this the *visible group* of $\Lambda^{(2)}$. The rows of Table 1 show the orbits of short vectors under the visible group.

2·3. *Definition of* $\Lambda^{(3)}$. Let $\theta = \omega - \bar{\omega}$, with $\theta^2 = -3$. Then (see (23))

$$\Lambda^{(3)} := \{(x_1, \ldots, x_6)_3 : x_i \in \mathbf{Z}[\omega], x_1 \equiv \ldots \equiv x_6 \,(\mathrm{mod}\,\theta), \Sigma x_i \equiv 0 \,(\mathrm{mod}\,3)\}.$$

In the notation of (53) this is an application of construction B to the ternary code $\{0^6, +1^6, -1^6\}$, using the fact that

$$\mathbf{Z}[\omega]/\theta\mathbf{Z}[\omega] \cong \mathrm{GF}(3). \tag{3}$$

The visible group of $\Lambda^{(3)}$ is a split extension $(2\times 3^5){:}\,S_6$. The short vectors are listed in Table 2, in orbits under the visible group. (They are also given in (23).) For example the third line of the table refers to the vectors $\pm(\omega^a, \omega^b, \ldots, \omega^f)_3$, where $a, b, \ldots, f \in \{0, 1, 2\}$ and $a+b+\ldots+f \equiv 0\,(\mathrm{mod}\,3)$. A more condensed notation is used for the vectors of norms 3 and 4.

Table 2. *The short vectors of* $\Lambda^{(3)}$, *which is* Λ_6^ω *written in the* 3-*base: the visible group is* (2×3^5): $\mathbf{S}_6$

Norm of vector	Norm of coordinates	Coordinates	Number	Class mod 2	Class mod θ
0	(0^6)	$(0^6)_3$	1	1	1
2	$(3^2, 0^4)$	$\pm(\omega^a\theta, -\omega^b\theta, 0^4)_3$	$2\cdot 3^2\cdot\binom{6}{2}$	2	3
	(1^6)	$\pm(\omega^a, \omega^b, \ldots, \omega^f)_3$	$2\cdot 3^5$	2	3
		Total	756		
3	$(9, 0^5)$	$(3, 0^5)_3$	$2\cdot 3\cdot 6$	2	18
	$(4, 1^5)$	$(-2, 1^5)_3$	$2\cdot 3^5\cdot 6$	2	18
	$(3^3, 0^3)$	$(\theta^3, 0^3)_3$	$2\cdot 3^3\cdot\binom{6}{3}$	2	18
		Total	4032		
4	$(7, 1^5)$	$(1+3\omega, 1^5)_3$	$2\cdot 3^5\cdot 6$	12	$18b$
	$(7, 1^5)$	$(1+3\bar\omega, 1^5)_3$	$2\cdot 3^5\cdot 6$	12	$18b$
	$(4^2, 1^4)$	$(-2^2, 1^4)_3$	$2\cdot 3^5\cdot\binom{6}{2}$	$6a$	$45b$
	$(3^4, 0^2)$	$(\theta^2, -\theta^2, 0^2)_3$	$2\cdot 3^4\cdot 45$	$6a$	81
		Total	20412		

Table 3. *The short vectors of* $\Lambda^{(4)}$, *which is* Λ_6^ω *written in the* 4-*base: the visible group is* (3×2^5): $\mathbf{S}_6$

Norm of vector	Norm of coordinates	Coordinates	Number	Class mod 2	Class mod θ
0	(0^6)	$(0^6)_4$	1	1	1
2	$(4^2, 0^4)$	$(2^2, 0^4)_4$	$3\cdot 2\cdot\binom{6}{2}$	2	3
	$(3, 1^5)$	$(\theta, 1^5)_4$	$3\cdot 2^5\cdot 6$	2	3
		Total	756		
3	$(7, 1^5)$	$(2-\theta, 1^5)_4$	$3\cdot 2^5\cdot 6$	2	18
	$(7, 1^5)$	$(-2-\theta, 1^5)_4$	$3\cdot 2^5\cdot 6$	2	18
	$(4^3, 0^3)$	$(2, 2\omega, 2\bar\omega, 0^3)_4$	$3\cdot 2^3\cdot\binom{6}{3}\cdot 2$	2	$6d$
	$(3^3, 1^3)$	$(\theta^3, 1^3)_4$	$3\cdot 2^5\cdot\binom{6}{3}$	2	$12d$
		Total	4032		
4	$(16, 0^5)$	$(4, 0^5)_4$	$3\cdot 2\cdot 6$	12	$3e$
	$(12, 4, 0^4)$	$(2\theta, 2, 0^4)_4$	$3\cdot 2^2\cdot 30$	$4a$	$30e$
	$(9, 3, 1^4)$	$(-3, \theta, 1^4)_4$	$3\cdot 2^5\cdot 30$	$10b$	$12f$
	$(7, 3^2, 1^3)$	$(2-\theta, \theta^2, 1^3)_4$	$3\cdot 2^5\cdot 60$	$6c$	$24f$
	$(7, 3^2, 1^3)$	$(-2-\theta, \theta^2, 1^3)_4$	$3\cdot 2^5\cdot 60$	$6c$	$24f$
	$(4^4, 0^2)$	$(2^4, 0^2)_4$	$3\cdot 2^4\cdot\binom{6}{2}$	$8a$	$3f$
	$(4^4, 0^2)$	$((2\omega)^2, (2\bar\omega)^2, 0^2)_4$	$3\cdot 2^4\cdot 90$	12	$18f$
	$(3^5, 1)$	$(\theta^5, 1)_4$	$3\cdot 2^5\cdot 6$	$2b$	$48e$
		Total	20412		

2·4. *Definition of* $\Lambda^{(4)}$.

$$\Lambda^{(4)} := \{(x_1, \ldots, x_6)_4 : x_i \in \mathbf{Z}[\omega],\ x_1 \equiv \ldots \equiv x_6 \equiv m \text{ (say) } (\bmod 2) \text{ and } x_1 + \ldots + x_6 \equiv 2\bar{\omega}m \ (\bmod 4)\}.$$

The visible group of $\Lambda^{(4)}$ is $(3 \times 2^5)\colon \mathbf{S}_6$, and the orbits of short vectors under the visible group are listed in Table 3.

2·5. *Definition of* $\Lambda^{(7)}$.

Let $\alpha = 2 + 3\omega$, with $\alpha^2 - \alpha + 7 = 0$, $N(\alpha) = 7$, $\mathbf{Z}[\omega]/\alpha\mathbf{Z}[\omega] \cong \mathrm{GF}(7)$. Then

$$\Lambda^{(7)} := \{(x_0, \ldots, x_6)_7 : x_i \in \mathbf{Z}[\omega],\ x_0 \equiv \ldots \equiv x_6 \ (\bmod \alpha),\ \Sigma x_i = 0\}.$$

The minimal vectors consist of all permutations of

$$\pm\,\omega^\nu(1, 1, 1, 1, -2-\omega, -2-\omega, 2\omega)_7,$$
$$\pm\,\omega^\nu(\alpha, -\alpha, 0, 0, 0, 0, 0)_7,$$

where $\nu = 0, 1, 2$, a total of

$$6 \cdot \frac{7!}{4!\,2!} + 6\binom{7}{2} = 756.$$

The visible group of $\Lambda^{(7)}$ is $6\colon \mathbf{S}_7$.

3. *Congruences mod 2 and mod θ*

It is useful (for example in proving the uniqueness of Λ_6^ω, as we shall see in the next section) to have a description of the congruence classes of $\Lambda_6^\omega/2\Lambda_6^\omega$ and $\Lambda_6^\omega/\theta\Lambda_6^\omega$ in the 2-, 3- and 4-bases. It follows from (1) and (3) that $\Lambda_6^\omega/2\Lambda_6^\omega$ and $\Lambda_6^\omega/\theta\Lambda_6^\omega$ are elementary abelian groups of orders 2^{12} and 3^6 respectively.

We first consider $\Lambda_6^\omega/2\Lambda_6^\omega$. The following facts are easy to establish by the argument used to prove theorem 2 of (10). Every congruence class contains short vectors. A vector v is always congruent to $-v$ modulo $2\Lambda_6^\omega$, and if v has norm 2 or 3 these are the only congruences. The vectors of norm 4 fall into congruence classes of size 12 modulo $2\Lambda_6^\omega$, each set forming a coordinate frame. For example in $\Lambda^{(2)}$ the 12 vectors

$$\begin{array}{lll} \pm(2, 2, 0, 0, 0, 0)_2, & \pm(0, 0, 2, 2, 0, 0)_2, & \pm(0, 0, 0, 0, 2, 2)_2, \\ \pm(2, -2, 0, 0, 0, 0)_2, & \pm(0, 0, 2, -2, 0, 0)_2, & \pm(0, 0, 0, 0, 2, -2)_2 \end{array} \tag{4}$$

are congruent modulo $2\Lambda^{(2)}$ and form six mutually orthogonal pairs. Another class of $\Lambda^{(2)}/2\Lambda^{(2)}$ contains the coordinate frame

$$\begin{array}{lll} \pm(2, 1, 0, 1, \omega, \bar{\omega})_2, & \pm(0, 1, 2, 1, -\omega, -\bar{\omega})_2, & \pm(0, \theta, 0, -\theta, \omega, -\bar{\omega})_2, \\ \pm(-2, 1, 0, 1, \omega, \bar{\omega})_2, & \pm(0, 1, -2, 1, -\omega, -\bar{\omega})_2, & \pm(0, 1, 0, -1, \omega\theta, -\bar{\omega}\theta)_2. \end{array} \tag{5}$$

Thus modulo $2\Lambda_6^\omega$

the vectors of norm 0 fall into 1 class of size 1,

the vectors of norm 2 fall into 378 classes of size 2,

the vectors of norm 3 fall into 2016 classes of size 2,

the vectors of norm 4 fall into 1701 classes of size 12,

and indeed

$$u_0 + \frac{u_2}{2} + \frac{u_3}{2} + \frac{u_4}{12} = 2^{12}. \tag{6}$$

The fifth column of Tables 1–3 shows how the classes are divided among the rows of each table. For example the first entry 12 in Table 1 indicates that all 12 vectors (4) are found in this row of the table. The following entries $8a$ and $4a$ indicate that 8 out of the 12 vectors (5) are found in one row and 4 in the next.

Similarly, every congruence class of $\Lambda_6^\omega/\theta\Lambda_6^\omega$ also contains short vectors. The vectors v, ωv and $\bar{\omega}v$ are all congruent modulo $\theta\Lambda_6^\omega$, and if v has norm 2 these are the only congruences. The vectors of norm 3 are divided into congruence classes of size 18, consisting of scalar multiples of a set of six mutually orthogonal vectors $\{\omega^\nu v_i : \nu = 0, 1, 2, 1 \leqslant i \leqslant 6$, with $v_i \cdot v_j = 0$ if $i \neq j\}$. The vectors of norm 4 are divided into congruence classes of size 81. For example in $\Lambda^{(3)}$ the following vectors:

$$a_i := (\bar{\omega}+3, \bar{\omega}^5)_3 \quad (1 \leqslant i \leqslant 6) \tag{7}$$

with the $\bar{\omega}+3$ in position i,

$$b_j := (\omega+3, \omega^5)_3 \quad (1 \leqslant j \leqslant 6) \tag{8}$$

with the $\omega+3$ in position j,

$$c_{ij} := (-2^2, 1^4)_3 \quad (1 \leqslant i < j \leqslant 6) \tag{9}$$

with the -2's in positions i and j, together with their multiples by ω and $\bar{\omega}$, are all congruent modulo $\theta\Lambda^{(3)}$. Thus modulo $\theta\Lambda_6^\omega$

the vectors of norm 0 fall into 1 class of size 1,
the vectors of norm 2 fall into 252 classes of size 3, (10)
the vectors of norm 3 fall into 224 classes of size 18,
the vectors of norm 4 fall into 252 classes of size 81,

and indeed

$$u_0 + \frac{u_2}{3} + \frac{u_3}{18} + \frac{u_4}{81} = 3^6. \tag{11}$$

The right-hand columns in Tables 1–3 indicate how the classes are divided among the rows of each table. For example the entries $18b$, $18b$, $45b$ in the last column of Table 2 refer to the 81 congruent vectors described in (7)–(9).

4. *Uniqueness of* Λ_6^ω

In this section we give two different characterizations of Λ_6^ω (a third characterization is given in (18) – see the Introduction). The first (Theorem 1) will imply that the lattices $\Lambda^{(2)}$, $\Lambda^{(3)}$, $\Lambda^{(4)}$ and $\Lambda^{(7)}$ constructed in Section 2 are isomorphic. We *define* Λ_6^ω to be the lattice $\Lambda^{(4)}$ constructed in Section 2·4.

THEOREM 1. *Let L be a 6-dimensional integral $\mathbf{Z}[\omega]$-lattice in which $u_0 = 1$, $u_1 = 0$, $u_2 = 756$, $u_3 = 4032$ and $u_4 = 20412$ (where u_i is the number of vectors in L of norm i). Then L is isomorphic to Λ_6^ω.*

Proof. The method of proof is essentially the same as that used to characterize the Leech lattice in (10). First, the argument used to prove theorem 2 of (10) shows, using equation (6), that the division of the short vectors of L into congruence classes modulo $2L$ is the same as the division of short vectors of Λ_6^ω into classes modulo $2\Lambda_6^\omega$ (see Section 3). We may now suppose that L contains all vectors of the shapes $(\pm 4, 0^5)_4$ and $(\pm 2^2, 0^4)_4$, these being 6 mutually orthogonal pairs of vectors of norm 4 and their halved differences. Let $x = (x_1, x_2, \ldots, x_6)_4$ be an arbitrary vector of L. By considering the inner product of x with the vectors $(\pm 4, 0^5)_4$ and $(\pm 2^2, 0^4)_4$ we see that all $x_i \in \mathbf{Z}[\omega]$ and $x_1 \equiv \ldots \equiv x_6 \equiv m$ (say) (mod 2). So far we have identified 180 vectors of norm 2, namely $\pm\omega^\nu(2^2, 0^4)_4$, corresponding to $m = 0$. Let x be one of the remaining 576 norm 2 vectors, so that $\Sigma N(x_i) = 8$ and $m = 1$. The elements of $\mathbf{Z}[\omega]$ have norms $0, 1, 3, 4, \ldots$, so the only possibility is that one x_i has norm 3 and the other five have norm 1. Therefore without loss of generality we may assume that $(\theta, 1^5)_4 \in L$. Then all the vectors $\omega^\nu(\pm\theta, \pm 1^5)_4$ are in L. Since these vectors span $\Lambda^{(4)}$, $L \supseteq \Lambda^{(4)}$. On the other hand, by considering the inner product of an arbitrary vector x with $(\theta, 1^5)_4$, we find that $\Sigma x_i \equiv 2\bar{\omega}m$ (mod 4). Thus $L \subseteq \Lambda^{(4)}$, and so $L = \Lambda^{(4)} = \Lambda_6^\omega$ as required.

COROLLARY 2.

$$\Lambda^{(2)} \cong \Lambda^{(3)} \cong \Lambda^{(4)} \cong \Lambda^{(7)} \cong \Lambda_6^\omega.$$

Proof. We have seen in Tables 1–3 that $\Lambda^{(2)}$, $\Lambda^{(3)}$ and $\Lambda^{(4)}$ have the same values of u_0, u_1, u_2, u_3, u_4, and it is not hard to show that $\Lambda^{(7)}$ does also. The result then follows from Theorem 1.

For example

$$\sigma: (x_1, x_2, x_3, x_4, x_5, x_6)_2 \to (y_1, y_2, y_3, y_4, y_5, y_6)_3, \tag{12}$$

where $(x_1, x_2, \ldots, x_6) \cdot P = (y_1, y_2, \ldots, y_6)$ and

$$P = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \bar{\omega} & \omega & \bar{\omega} & \omega & 1 & \\ 1 & \omega & \bar{\omega} & 1 & \bar{\omega} & \omega \\ \bar{\omega} & 1 & 1 & \omega & \bar{\omega} & \omega \\ 1 & 1 & \bar{\omega} & \omega & \omega & \bar{\omega} \\ \bar{\omega} & \omega & 1 & 1 & \omega & \bar{\omega} \end{bmatrix}$$

is an isomorphism from $\Lambda^{(2)}$ to $\Lambda^{(3)}$, and

$$\tau: (x_1, x_2, x_3, x_4, x_5, x_6)_2 \to (x_1 + x_2, x_1 - x_2, \ldots, x_5 + x_6, x_5 - x_6)_4 \tag{13}$$

is an isomorphism from $\Lambda^{(2)}$ to $\Lambda^{(4)}$.

The following alternative characterization of Λ_6^ω was first obtained by Feit (29) via direct enumeration.

THEOREM 3 (Feit (29)). *Let L be a unimodular $\mathbf{Z}[\omega]$-lattice of dimension $n < 12$ containing no vectors of norm 1. Then $n = 6$ and $L \cong \Lambda_6^\omega$.*

Proof. It follows from theorem 9 of (52) that the theta-series of L,

$$\Theta_L(q) = \sum_{x \in L} q^{N(x)},$$

is an element of the graded polynomial ring $\mathbf{C}[\phi_0, \phi_1]$, where

$$\begin{aligned} \phi_0 = \Theta_{\mathbf{Z}[\omega]}(q) &= \sum_{a=-\infty}^{\infty} \sum_{b=-\infty}^{\infty} q^{a^2-ab+b^2} \\ &= 1 + 6q + 6q^3 + 6q^4 + 12q^7 + \ldots, \\ \phi_1 &= q \prod_{a=1}^{\infty} (1-q^a)^6 (1-q^{3a})^6 \\ &= q - 6q^2 + 9q^3 + 4q^4 + 6q^5 - 54q^6 - 40q^7 + \ldots . \end{aligned}$$

The only element of this ring in dimension less than 12 that contains no q term is the 6-dimensional series

$$\phi_0^6 - 36\phi_1 = 1 + 756q^3 + 4032q^3 + 20\,412q^4 + 60\,480q^5 + 139\,860q^6 + 326\,592q^7 + \dots \quad (14)$$

Therefore $n = 6$ and (14) is the theta-series of L. From Theorem 1, $L \cong \Lambda_6^\omega$.

5. *The group order; congruence bases*

The calculation of the order of the automorphism group of a lattice is sometimes simplified by the following definition. Let L_n be an n-dimensional J-lattice for some ring J (i.e. a free J-module), and let π be a prime ideal in J. It may happen that certain congruence classes of $L_n/\pi L_n$ have the property that the minimal representatives in the class consist of scalar multiples of a set of n mutually orthogonal vectors. If so we call these minimal representatives a *congruence base* for the lattice. Expressing the lattice in terms of a congruence base often leads to simple coordinates for the vectors.

For example, for the Leech lattice Λ_{24}, any congruence class of $\Lambda_{24}/2\Lambda_{24}$ containing vectors of norm 8 contains 48 vectors $\pm e_1, \dots, \pm e_{24}$ of norm 8 forming a coordinate frame (see (**10**)). By expressing the vectors of Λ_{24} in terms of $e_1, \dots, e_{24}$ we obtain the standard coordinates for this lattice. Other examples are the complex Leech lattice (see (**60**)) and the lattice E_8 (see (**17**)).

Let the vectors of a congruence base have norm r, and suppose that Aut(L_n) acts transitively on the vectors of L_n of norm r. If H is the subgroup of Aut(L_n) fixing a congruence base, and s is the number of vectors in the base, then there are u_r/s ways of choosing a congruence base, and so

$$|\mathrm{Aut}\,(L_n)| = \frac{u_r}{s}|H|. \quad (15)$$

Usually H consists of monomial matrices and is closely related to the visible group mentioned in Section 2.

Continuing the example, for the Leech lattice $H = 2^{12}\cdot M_{24}$ (where M_{24} is a Mathieu group), and

$$\begin{aligned}|\mathrm{Aut}\,(\Lambda_{24})| = |\cdot 0| &= \frac{u_8}{48}|2^{12}\cdot M_{24}| \\ &= \frac{398\,034\,000}{48}\cdot 2^{12}\cdot 244\,823\,040 \\ &= 831\,555\,361\,308\,672\,000\,0\end{aligned}$$

(see (**9**)–(**11**)).

For Λ_6^ω we have two choices for the prime ideal π. In Section 3 we saw that a congruence class of $\Lambda_6^\omega/2\Lambda_6^\omega$ containing norm 4 vectors contains 12 norm 4 vectors forming a congruence base (e.g. (4) or (5)). The stabilizer of such a congruence base is $2^5\colon \mathrm{S}_6$ (one third of the visible group of $\Lambda^{(4)}$), and so

$$|\mathrm{Aut}\,(\Lambda_6^\omega)| = \frac{20\,412}{12}\cdot 2^5\cdot 6\,! = 2^9\cdot 3^7\cdot 5\cdot 7 = 108\cdot 9\,! = 39\,191\,040. \quad (16)$$

If we adjoin the antilinear symmetries (those which involve complex conjugation) the group order increases to $2^{10}\cdot 3^7\cdot 5\cdot 8$, which is the order of the automorphism group of the real lattice K_{12}.

Alternatively we could argue from congruences modulo $\theta\Lambda_6^\omega$. The vectors of norm 3 are divided into congruence bases of size 18, the stabilizer of a base being $3^5\colon \mathbf{S}_6$ (half the visible group of $\Lambda^{(3)}$). Then

$$|\mathrm{Aut}\,(\Lambda_6^\omega)| = \frac{4032}{18}\cdot 3^5\cdot 6\,! = 39\,191\,040,$$

is agreement with (16).

6. *Connections with the lattice E_6*

In the next two sections we examine the close connections between the lattices Λ_6^ω, E_6 and Λ_{24}. First, we recall the standard definition of the lattice E_8:

$$E_8 := \{(x_1,\dots,x_8)\colon \text{all } x_i\in\mathbf{Z} \text{ or all } x_i\in\mathbf{Z}+\tfrac{1}{2}, \text{ and } \Sigma x_i \text{ is even}\}$$

(see (4), (15), (41), (47)). E_8 is a six-dimensional sublattice of E_8 and (like Λ_6^ω) has many isomorphic definitions. One definition is

$$E_6 := \{(x_1,\dots,x_8)\in E_8\colon x_6 = x_7 = x_8\}. \tag{17}$$

In this version the 72 minimal vectors of E_6 consist of

$$\begin{array}{lll} 40 & \text{of shape} & (\pm 1^2, 0^6) \\ 32 & \text{of shape} & \pm(\pm\tfrac{1}{2}^5, \tfrac{1}{2}^3). \end{array}$$

Comparison of these vectors with the minimal vectors of $\Lambda^{(4)}$ listed in Table 3 suggests defining the map

$$x = (x_1, x_2, x_3, x_4, x_5, y, y, y)\in E_6 \mapsto \hat{x} = (2x_1, 2x_2, 2x_3, 2x_4, 2x_5, 2\theta y)_4\in\Lambda^{(4)}, \tag{18}$$

which preserves inner products and embeds E_6 in $\Lambda^{(4)}$. It is not difficult to verify that the lattice

$$\mathbf{Z}[\omega]\langle\hat{x}\colon x\in E_6\rangle$$

(the $\mathbf{Z}[\omega]$-span of the vectors $\hat{x}$, cf. (1)) has index 3 in $\Lambda^{(4)} \cong \Lambda_6^\omega$.

The alternative definition

$$E_6 := \{(x_1,\dots,x_8)\in E_8\colon x_1+\dots+x_6 = x_7+x_8 = 0\} \tag{19}$$

will enable us to see the whole of Λ_6^ω. The minimal vectors in this version of E_6 consist of

$$\begin{array}{lll} 30 & \text{of shape} & (1, -1, 0^4 \mid 0^2), \\ 2 & \text{of shape} & (0^6 \mid 1, -1), \\ 40 & \text{of shape} & \left(\frac{1}{2}^3, \frac{-1}{2}^3 \,\middle|\, \frac{1}{2}, \frac{-1}{2}\right). \end{array}$$

The dual lattice E_6^* is the union of E_6 and two cosets $\pm\alpha_1+E_6$, the 27 minimal vectors of the coset α_1+E_6 in this definition being as follows:

$$\alpha_i := \left(\frac{5}{6}, \frac{-1}{6}^5 \,\middle|\, \frac{1}{2}, \frac{-1}{2}\right) \quad (1\leqslant i\leqslant 6), \tag{20}$$

$$\beta_i := \left(\frac{5}{6}, \frac{-1}{6}^5 \,\middle|\, \frac{-1}{2}, \frac{1}{2}\right) \quad (1\leqslant i\leqslant 6), \tag{21}$$

with the $\frac{5}{6}$ in position i, and

$$\gamma_{ij} := \left(\frac{-2}{3}^2, \frac{1}{3}^4 \,\middle|\, 0^2\right) \quad (1\leqslant i<j\leqslant 6), \tag{22}$$

with the $-\frac{2}{3}$'s in positions i and j.

THEOREM 4. *Regarding E_6 and E_6^* (with either definition) as embedded in complex space, we have*

$$\Lambda_6^{\omega} \cong \mathbf{Z}[\omega]\langle E_6, \theta E_6^* \rangle.$$

Proof. The 27 vectors α_i, β_i, γ_{ij} of E_6^* have the same mutual inner products, except for a factor of 3, as the norm 4 vectors a_i, b_j, c_{ij} of $\Lambda^{(3)}$ (see (7)–(9)). We have already seen that $\mathbf{Z}[\omega]E_6$ has index 3 in $\Lambda^{(3)} \cong \Lambda_6^{\omega}$, and by adjoining $\theta\alpha_i$, $\theta\beta_i$, $\theta\gamma_{ij}$ we obtain all of $\Lambda^{(3)}$.

7. *Connections with the Leech lattice* Λ_{24}

Vectors in $\mathbf{R}^{24}$ will be specified by MOG coordinates

w_1	w_2	w_3	w_4	w_5	w_6
x_1	x_2	x_3	x_4	x_5	x_6
y_1	y_2	y_3	y_4	y_5	y_6
z_1	z_2	z_3	z_4	z_5	z_6

(23)

as in (12), (13), (17), (24), and may be written as quaternionic vectors (in $\mathbf{H}^6$) by identifying (23) with

$$(w_1 + x_1 i + y_1 j + z_1 k, \ldots, w_6 + x_6 i + y_6 j + z_6 k). \tag{24}$$

We regard $\mathbf{Z}[\omega]$ as embedded in $\mathbf{H}$ via the identifications

$$\omega = \frac{-1+i+j+k}{2}, \quad \theta = i+j+k. \tag{25}$$

Let Re denote the subspace of $\mathbf{R}^{24}$ consisting of the vectors (23) for which $x_i = y_i = z_i$ for $i = 1, \ldots, 6$, and let $Im = Re^{\perp}$, the subspace of $\mathbf{R}^{24}$ with $w_i = 0$, $x_i + y_i + z_i = 0$ for $i = 1, \ldots, 6$.

We already know from ((17), figure 5) that the lattices

$$\Lambda^{(R)} := \Lambda_{24} \cap Re \quad \text{and} \quad \Lambda^{(I)} := \Lambda_{24} \cap Im$$

are isomorphic to K_{12}. More precisely, if we write vectors in Re with a subscript 4, and vectors in Im with a subscript 2, we find that

$$\Lambda^{(R)} = 2\Lambda^{(4)} \quad \text{and} \quad \Lambda^{(I)} = 2\Lambda^{(2)} \cdot (j-k). \tag{26}$$

For example the Leech vector

2	2	0	0	0	0
2	2	0	0	0	0
2	2	0	0	0	0
2	2	0	0	0	0

is in $\Lambda^{(R)}$, and by (24) and (25) is written as

$$(2+2\theta, 2+2\theta, 0, 0, 0, 0)_4 = 2(-2\bar{\omega}, -2\bar{\omega}, 0, 0, 0, 0)_4 \in 2\Lambda^{(4)}.$$

Similarly

$$\begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline 4 & 0 & 0 & 0 & 0 & 0 \\ -4 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}$$

is in $\Lambda^{(I)}$, and by (24) is written as $2(2,0,0,0,0,0)_2\cdot(j-k)\in 2\Lambda^{(2)}\cdot(j-k)$.

The Leech lattice Λ_{24} is therefore obtained by *gluing* (cf. (15), (17)) $\Lambda^{(R)}$ to $\Lambda^{(I)}$. Let $Re(\Lambda_{24})$ and $Im(\Lambda_{24})$ denote the projections of Λ_{24} onto the spaces Re and Im respectively. Then it can easily be seen that

$$Re(\Lambda_{24}) = 2\Lambda^{(4)}\cdot\frac{1}{\theta}, \quad Im(\Lambda_{24}) = 2\Lambda^{(2)}\cdot(j-k)\frac{1}{\theta}. \tag{27}$$

Consider for example the Leech vector

$$x = \begin{array}{|cc|cc|cc|} \hline 4 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

Its projections onto Re and Im are respectively

$$u = Re(x) = \begin{array}{|cc|cc|cc|} \hline 4 & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ \hline \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ \frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array},$$

$$v = Im(x) = \begin{array}{|cc|cc|cc|} \hline 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{8}{3} & 0 & 0 & 0 & 0 & 0 \\ \hline -\frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ -\frac{4}{3} & 0 & 0 & 0 & 0 & 0 \\ \hline \end{array}.$$

From (24),

$$u = \left(4+\frac{4\theta}{3}, 0^5\right)_4 = 2(4\omega, 0^5)_4\cdot\frac{1}{\theta}\in 2\Lambda^{(4)}\cdot\frac{1}{\theta}, \tag{28}$$

and similarly

$$v = 2(-2, 0^5)_2\cdot(j-k)\cdot\frac{1}{\theta}\in 2\Lambda^{(2)}\cdot(j-k)\cdot\frac{1}{\theta}, \tag{29}$$

in agreement with (27).

This $x = u+v$ is a *glue vector* connecting $\Lambda^{(R)}$ and $\Lambda^{(I)}$ in the Leech lattice, and the pairing $u \leftrightarrow v$ is the *gluing map*. Abstractly this gluing map is from congruence classes of $(1/\theta)\Lambda_6^{\omega}/\Lambda_6^{\omega} \cong (1/\theta)\Lambda^{(4)}/\Lambda^{(4)}$ to congruence classes of $(1/\theta)\Lambda_6^{\omega}/\Lambda_6^{\omega} \cong (1/\theta)\Lambda^{(2)}/\Lambda^{(2)}$.

From Section 3 we know that the norms of minimal representatives of these classes are 0, $\frac{2}{3}$, 1 or $\frac{4}{3}$, and the gluing map must pair classes of norm $\frac{2}{3}$ with classes of norm $\frac{4}{3}$, and classes of norm 1 with classes of norm 1. The gluing map has a simple description in terms of our quaternionic coordinates. It is essentially given by the map

$$\Delta_w : (a,b,c,d,e,f)_4 \mapsto (\omega a, \bar{\omega} b, \omega c, \bar{\omega} d, \omega e, \bar{\omega} f)_2, \tag{30}$$

which is a norm-doubling homomorphism from $\Lambda^{(4)}$ to $\Lambda^{(2)}$. (Δ_w uses the word

$$(\omega, \bar{\omega}, \omega, \bar{\omega}, \omega, \bar{\omega}) \in \mathbf{C}_{\text{hex}},$$

but in fact any hexacodeword of weight 6 would do.)

THEOREM 5 (the gluing map). *A vector $x = u + v$, $u \in Re$, $v \in Im$, is in Λ_{24} if and only if when u and v are written as vectors in* $\mathbf{H}^6$ *using the above conventions then*

$$u = 2U \cdot \frac{1}{\theta}, \quad v = 2V \cdot (j-k) \cdot \frac{1}{\theta} \tag{31}$$

with $U \in \Lambda^{(4)}$, $V \in \Lambda^{(2)}$ and

$$\Delta_w(U) \equiv V \pmod{\Lambda^{(2)} \cdot \theta}. \tag{32}$$

Thus the gluing map is given by

$$u \mapsto 2\Delta_w(U) \cdot (j-k) \cdot \frac{1}{\theta}, \tag{33}$$

for if (31) *and* (32) *hold then the right-hand side of* (33) *is congruent to v modulo* $\Lambda^{(I)}$.

We omit the straightforward proof. Example: According to the theorem, (28) is glued to

$$2(4\bar{\omega}, 0^5)_4 \cdot (j-k) \cdot \frac{1}{\theta},$$

and indeed this is congruent to (29) modulo $\Lambda^{(I)}$.

8. *Covering radius and deep holes*

In this section we use Simon Norton's method (cf. (48), (17), theorem 12) to find the covering radii of Λ_6^ω and K_{12}, and the deep holes in these lattices.

THEOREM 6. *The covering radius of Λ_6^ω is $\sqrt{\frac{8}{3}}$ times the packing radius. Any deep hole is congruent modulo Λ_6^ω to a vector $(1/\theta)v$, where $v \in \Lambda_6^\omega$, $N(v) = 4$. All deep holes are equivalent under the action of* $\text{Aut}(\Lambda_6^\omega)$.

Proof. The proof resembles that of theorem 12 of (17), and the reader is referred to that paper for the justification of certain steps. Let the covering radius of Λ_6^ω be $\sqrt{d}$; then the covering radius of $(1/\theta)\Lambda_6^\omega$ is $\sqrt{(d/3)}$. Since the minimal representatives of some congruence classes of $(1/\theta)\Lambda_6^\omega/\Lambda_6^\omega$ have norm $\frac{4}{3}$ (by Section 3), we know $d \geq \frac{4}{3}$. Suppose x is a deep hole in Λ_6^ω, and let z be the closest point of $(1/\theta)\Lambda_6^\omega$ to x. Again using our knowledge of the congruence classes of $(1/\theta)\Lambda_6^\omega/\Lambda_6^\omega$, z can be written as $z = -(1/\theta)v_r + l$, where $v_r, l \in \Lambda_6^\omega$ and $r := N(v_r)$ is 2, 3 or 4. Then $x' = x - l$ is also a deep hole, and the closest point of $(1/\theta)\Lambda_6^\omega$ to x' is $-(1/\theta)v_r$. Let $x' = -(1/\theta)v_r + x''$, where $N(x'') \leq d/3$.

Now $-(1/\theta)v_r = (\omega v_r - \omega^2 v_r)/3$ is the centre of the triangle 0, ωv_r, $-\omega^2 v_r$ (see Fig.

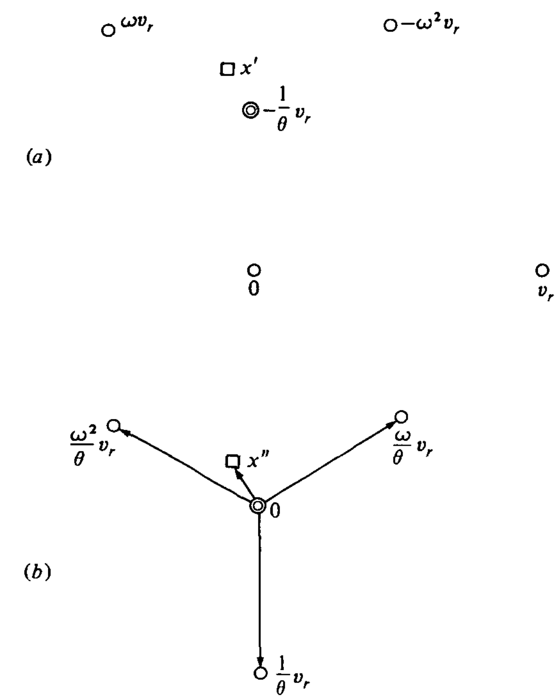

Figure 1. (a) A deep hole x' (small square), the closest point of $(1/\theta)\Lambda_6^\omega$ (double circle), and some nearby points of Λ_6^ω (small circles). (b) The same points referred to the new origin.

1(a)). Let us move the origin of coordinates to the point $-(1/\theta)v_r$. Then the points of Λ_6^ω nearest to the new origin include

$$\frac{1}{\theta}v_r,\quad \frac{\omega}{\theta}v_r,\quad \frac{\omega^2}{\theta}v_r \tag{34}$$

(see Fig. 1(b)). Since x' is a deep hole, $N(x'-l) \geqslant d$ for all $l \in \Lambda_6^\omega$. Now

$$N(x'-l) = N(x') - 2\operatorname{Re}(x' \cdot l) + N(l). \tag{35}$$

If we take l to be one of the points (34) we obtain

$$N(x'') + 2\operatorname{Re}\left(x'' \cdot \frac{\omega^\nu \bar{v}_r}{\theta}\right) + \frac{r}{3} \geqslant d \quad (\nu = 0, 1, 2), \tag{36}$$

and therefore

$$N(x'') + \frac{r}{3} \geqslant d.$$

But $N(x'') \leqslant d/3$, so $r \geqslant 2d \geqslant \frac{8}{3}$, i.e. $r = 3$ or 4.

The case $r = 3$. Using the 3-base we may assume $v_r = v_3 = \pm(0, \ldots, 0, 3\omega^j, 0, \ldots)_3$, $x'' = (x_1, \ldots, x_6)_3$. Then (36) implies

$$N(x'') \pm \frac{2}{\sqrt{3}}\operatorname{Im}(\omega^\nu x_k) + 1 \geqslant d \tag{37}$$

for $\nu = 0, 1, 2, k = 1, \ldots, 6$, hence

$$|\operatorname{Im}(\omega^\nu x_k)| \leqslant \frac{\sqrt{3}}{18} \quad (\nu = 0, 1, 2). \tag{38}$$

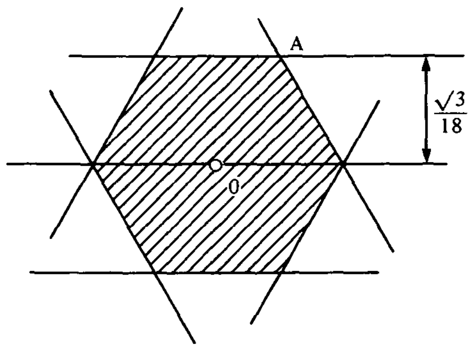

FIGURE 2. The region defined by equation (38), with $|0A| = \frac{1}{9}$.

Therefore x_k is in the shaded region shown in Fig. 2, and $|x_k| \leqslant \frac{1}{9}$, $N(x'') \leqslant \frac{2}{27}$. Substituting this into (37) we find that all the numbers $\pm 2 \cdot 3^{-\frac{1}{2}} \operatorname{Im}(\omega^\nu x_k)$ (for $\nu = 0, 1, 2$) are $\leqslant -\frac{7}{27} < 0$, which is impossible.

The case $r = 4$. Now the closest point of $(1/\theta)\Lambda_6^\omega$ to x' is $-(1/\theta)v_4$, where $N(v_4) = 4$ (referred to the old origin). From Section 3 there are 81 points of norm $\frac{4}{3}$ in $(1/\theta)\Lambda_6^\omega$ that are congruent to $-(1/\theta)v_4$ modulo Λ_6^ω (for example, the points (7)–(9), divided by θ). Changing to the new origin, let

$$\phi_i := \psi_i + \frac{1}{\theta} v_4 \quad (i = 1, \ldots, 81),$$

which are points of Λ_6^ω, now having norm $\frac{4}{3}$. Let us define a map from $\mathbf{C}^6$ to $\mathbf{C}^6$ by

$$T(y) := \sum_{i=1}^{81} \operatorname{Re}(\phi_i \cdot \bar{y})\, \phi_i.$$

Then, by an argument similar to that in (17), using the fact that the 12-dimensional real vectors corresponding to $\phi_1, \ldots, \phi_{81}$ form a eutactic star (cf. (23), §4), it follows that

$$\operatorname{Re}(T(y) \cdot \bar{y}) = cy \cdot \bar{y}$$

for some constant c. Taking y to have a single nonzero coordinate we find $c = 81$. Thus

$$\operatorname{Re}(T(y) \cdot \bar{y}) = \sum_{i=1}^{81} \{\operatorname{Re}(\phi_i \cdot \bar{y})\}^2 = 81 N(y),$$

and so

$$\operatorname{Re}(\phi_i \cdot \bar{y}) \geqslant \sqrt{N(y)} \tag{39}$$

for some $i \in \{1, \ldots, 81\}$. Set $y = x''$, $N(x'') = a^2$, $a \geqslant 0$, and $d = \frac{4}{3} + \delta$, $\delta \geqslant 0$. Since x' is a deep hole, $N(\phi_i - x'') \geqslant d$, which using (35), (39) becomes

$$\tfrac{4}{3} - 2a + a^2 \geqslant \tfrac{4}{3} + \delta, \quad (a-1)^2 \geqslant \delta + 1,$$

and so either

$$a - 1 \geqslant \sqrt{(\delta + 1)}, \tag{40}$$

or

$$a - 1 \leqslant -\sqrt{(\delta + 1)}. \tag{41}$$

(40) and $a^2 \leqslant d/3$ lead immediately to a contradiction. On the other hand (41) and $a \geqslant 0$ imply $\delta \leqslant 0$, hence $\delta = 0$ and $d = \frac{4}{3}$.

Therefore the deep holes are elements of $(1/\theta)\Lambda_6^\omega$ that are congruent to $-(1/\theta)v_4$ modulo Λ_6^ω. Since $\mathrm{Aut}(\Lambda_6^\omega)$ is transitive on the 20412 vectors of norm 4, there is unique type of deep hole. This completes the proof of Theorem 6.

COROLLARY 7. *The covering radius of* K_{12} *is* $\surd(8/3)$ *times the packing radius, and there is a unique type of deep hole.*

The argument used to prove theorem 3 of (17) now leads to the following result.

THEOREM 8. *Let* L_{12+r} $(0 \leqslant r \leqslant 8)$ *be any* $(12+r)$*-dimensional lattice of minimal norm* M *and containing a copy of* K_{12} *with the same minimal norm. Then*

$$\det L_{12+r} \geqslant 3^{6-r}\left(\frac{M}{4}\right)^{12+r}\lambda_r, \tag{42}$$

where $\lambda_0, \ldots, \lambda_8$ *are* 1, 4, 12, 32, 64, 128, 192, 256, 256 *respectively.*

By inspection of Table II of (17) we see that equality holds in (42) for the lattices $K_{12}, \ldots, K_{18}$.

COROLLARY 9. *The lattices* $K_{12}, \ldots, K_{18}$ *are densest possible lattices containing* K_{12} *and having the same minimal norm.*

9. *New sphere packings obtained from* Λ_6^ω *and* Λ_{24}

Λ_6^ω, K_{12} and Λ_{24} all have both norm-doubling and norm-trebling endomorphisms, which can be used to obtain new lattices in higher dimensions via the constructions in (1), (3). A norm-doubling map for Λ_6^ω is the endomorphism of $\Lambda^{(2)}$ given by

$$\lambda := \Delta_w \circ \tau\colon (a,b,c,d,e,f)_2 \mapsto (\omega a + \omega b, \bar{\omega} a - \bar{\omega} b, \ldots, \omega e + \omega f, \bar{\omega} e - \bar{\omega} f)_2, \tag{43}$$

which satisfies $\lambda^2 + \lambda + 2 = 0$. A norm-trebling map for Λ_6^ω is

$$\Theta\colon (a,b,c,d,e,f)_2 \mapsto (\theta a, \theta b, \ldots, \theta f)_2, \tag{44}$$

and satisfies $\Theta^2 + 3 = 0$.

We mention in passing that the dual lattice K_{12}^* is the 12-dimensional real lattice corresponding to $\Theta^{-1}\Lambda_6^\omega$.

Both λ and Θ lead to new lattices, but those obtained from Θ are not very dense and we shall only describe those obtained from λ.

We use the construction given in (3), applying it to λ (which we now regard as acting on K_{12}). Then, as described in section 3 of (3), especially equation (15), we obtain lattice packings in $\mathbf{R}^{12n}$ for $1 \leqslant n \leqslant 2^6 + 1$ having centre density δ given by

$$\log_2 \delta = n \log_2 (3^{-3}) + 6(an - 2^{a+1} + a + 2), \tag{45}$$

where $a = [\log_2 n]$. In dimensions 228, 240, 252, …, 780, these appear to be denser than any lattice previously constructed.

Lattices obtained from a norm-doubling map for the Leech lattice Λ_{24} have already been analysed in (1) and (3), and produce record packings in dimensions up to $24(2^{12}+1)$. If vectors of Λ_{24} are represented by quaternions, as in (23)–(24), left multiplication by $\theta = i + j + k$ is a norm-trebling endomorphism Θ of Λ_{24}. Again applying the construction in (3), but now using codes over $\mathrm{GF}(3^{12})$, we obtain lattice packings in $\mathbf{R}^{24n}$ for $1 \leqslant n \leqslant 3^{12} + 1$ having centre density δ given by

$$\log_2 \delta = 12 \log_2 3 \left\{an - \frac{3^{a+1}-1}{2} + a + 1\right\}, \tag{46}$$

where $a = [\log_3 n]$. These set new records in dimensions $24(2^{12}+1) < 24n \leqslant 24(3^{12}+1)$. For example in dimension 1048584 the new record is $\delta = 2^{6917505 \cdot 44 \ldots}$, a considerable improvement over the old record (compare (3)).

10. *The invariants of the Mitchell group*

Let R^{G_0} denote the ring of invariants of the Mitchell group G_0, i.e. the set of polynomials $f \in \mathbf{C}[x_1, \ldots, x_6]$ such that

$$A \circ f = f \quad \text{for all} \quad A \in G_0,$$

where

$$A \circ f(x_1, \ldots, x_6) := f\left(\sum_{i=1}^{6} a_{1i} x_i, \ldots, \sum_{i=1}^{6} a_{6i} x_i\right).$$

It appears that no basis for R^{G_0} has ever been published, and we therefore provide one in this section.

In Todd's 1950 paper 'The invariants of a finite collineation group in five dimensions' (58), he computes the Molien series (cf. (51), (56)) $\sum_0^\infty a_d \lambda^d$, where a_d is the number of linear independent homogeneous invariants of degree d in R^{G_0}, and shows that

$$\sum_{d=0}^{\infty} a_d \lambda^d = \frac{1}{(1-\lambda^6)(1-\lambda^{12})(1-\lambda^{18})(1-\lambda^{24})(1-\lambda^{30})(1-\lambda^{42})}. \tag{47}$$

He remarks that it is 'tempting to infer' that there exist six algebraically independent invariants $\theta_6, \ldots, \theta_{42}$ such that

$$R^G = \mathbf{C}[\theta_6, \theta_{12}, \theta_{18}, \theta_{24}, \theta_{30}, \theta_{42}], \tag{48}$$

and shows that the lowest degree invariant θ_6 is given by

$$\theta_6 = \sum_{v \in S} (v_1 x_1 + \ldots + v_6 x_6)^6, \tag{49}$$

where S is (in effect) the set of 756 minimal vectors of Λ_6^ω. Todd computes θ_6 explicitly in the 2-, 3-, 4- and 7-bases. In the 4-base, for example, apart from a constant factor,

$$\theta_6 = \sum_{i}^{(6)} x_i^6 + 15 \sum_{i,j}^{(30)} x_i^4 x_j^2 - 30 \sum_{i<j<k}^{(20)} x_i^2 x_j^2 x_k^2 + 240\theta x_1 \ldots x_6. \tag{50}$$

In a companion paper, Hartley (38) investigates the properties of the surface $\theta_6 = 0$.

In their 1954 paper (50), Shephard and Todd state that G_0 'possesses a system of invariants of degrees 6, 12, 18, 24, 30, 42 whose Jacobian is of degree 126. The existence of these forms was indicated by Todd (58) and the slight reservation expressed there about their possible interdependence can be settled by a calculation like that made by Coxeter ((20), p. 777) showing that for a certain special set of values of the variables the Jacobian of these forms does not vanish.' The assertion (48) was established beyond any doubt the following year by Chevalley (7) (see also ((22), § 13·5), ((32), theorem 2·1), ((55), theorem 4·2·5), ((56), theorem 4·1)) as a special case of a general property of reflection groups. Since then there appears to have been no further work on the invariants of this group.

For any *real* reflection group which is the automorphism group of a regular polytope, Flatto and Wiener (see (30)–(34), ((22), p. 179)), have shown that a set of basic invariants is given by the polynomials $\sum_{v \in S} (v_1 x_1 + \ldots + v_n x_n)^k$ of the appropriate degrees, where

S is the set of vertices of the polytope. We now prove, by the type of calculation suggested above, that the analogous result holds for G_0.

THEOREM 10. *Let S denote the set of 756 minimal vectors of Λ_6^ω, and let*

$$\mu_k := \sum_{v \in S} (v_1 x_1 + \ldots + v_6 x_6)^k$$

for $k = 0, 1, \ldots$. Then the ring of invariants of the Mitchell group is

$$R^{G_0} = \mathbf{C}[\mu_6, \mu_{12}, \mu_{18}, \mu_{24}, \mu_{30}, \mu_{42}].$$

Proof. The algebraic symbol manipulation program Macsyma (45) was used to compute the Jacobian of $\mu_6, \ldots, \mu_{42}$ at the point $x = (1, 2, 4, 5, 6, 8)$, modulo the prime 99991. The result was $-400073\theta - 43754 \neq 0$. The theorem now follows from Chevalley's result quoted above.

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