ON THE VORONOI REGIONS OF CERTAIN LATTICES*

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Abstract. The Voronoi region of a lattice $L_n \subseteq \mathbb{R}^n$ is the convex polytope consisting of all points of $\mathbb{R}^n$ that are closer to the origin than to any other point of L_n . In this paper we calculate the second moments of the Voronoi regions of the lattices E_6^* , E_7^* , K_{12} , Λ_{16} and Λ_{24} . The results show that these lattices are the best quantizers presently known in dimensions 6, 7, 12, 16 and 24. The calculations are performed by Monte Carlo integration, and make use of fast algorithms for finding the closest lattice point to an arbitrary point of the space. We also establish two general theorems concerning the number of faces of the Voronoi region of a lattice.

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1. Introduction. The Voronoi region of an n -dimensional lattice $L_n \subseteq \mathbb{R}^n$ is the convex polytope

$$\mathcal{P} := \{x \in \mathbb{R}^n : N(x) \leq N(x-l) \text{ for all } l \in L_n\},$$

where $N(x) = x \cdot x$ denotes the norm of a vector (cf. [10] and the references given there). Figure 1 for example shows the Voronoi region of the body-centered cubic lattice A_3^* . If L_n is used as a quantizer (or analog-to-digital convertor), its average mean squared error per symbol is given by

$$(1) \quad G(L_n) := \frac{1}{n \det(L_n)} \frac{\int_{\mathcal{P}} x \cdot x \, dx}{(n+2)/(2n)},$$

where $\det(L_n)$, the determinant of L_n , is the square of the volume of $\mathcal{P}$ ([10], [22], [39]). $G(L_n)$ is a normalized second moment of $\mathcal{P}$ about the origin. (The formula (1) assumes that the input to the quantizer is uniformly distributed over a large region of $\mathbb{R}^n$, and the number of output levels is very large.)

FIG. 1. The Voronoi region of the body-centered cubic lattice A_3^* ($\cong D_3^*$) is a truncated octahedron. The open circles represent lattice points, the solid circles the vertices of the Voronoi region.

$G(L_n)$ measures the average error introduced when points of $\mathbb{R}^n$ are replaced by the closest lattice points. If we use the 1-dimensional integer lattice $\mathbb{Z}$ as a quantizer, the average error is $G(\mathbb{Z}) = 1/12 = 0.08333 \dots$. But by using higher-dimensional lattices this can be reduced (see Table 1 and Fig. 2). The expression for $G(L_n)$ in (1) has been scaled so as to provide a proper comparison between quantizers of different dimensions.

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TABLE 1
Smallest mean squared error $G(L_n)$ of any known n -dimensional
lattice L_n

dimension n	lattice L_n	mean squared error $G(L_n)$
1	$\mathbb{Z}$	0.0833333...
2	A_2	0.0801875...
3	A_3^*	0.0785433...
4	D_4	0.0766032...
5	D_5^*	0.0756254...
6	E_6^*	0.074239 ± 0.000018
7	E_7^*	0.073124 ± 0.000013
8	E_8	0.0716821...
12	K_{12}	0.070100 ± 0.000024
16	Λ_{16}	0.068299 ± 0.000027
24	Λ_{24}	0.065771 ± 0.000074

FIG. 2. Normalized second moment G for various lattices, and the Zador and sphere bounds. It is known that the best quantizers must lie between the two bounds.

In 1953 Fejes Tóth ([20], see also [30]) proved that the hexagonal lattice A_2 is the optimal lattice quantizer in two dimensions, i.e., has the smallest value of $G(L_2)$, namely $5/36\sqrt{3} = 0.0801875 \dots$. He also showed that no nonlattice quantizer can do better. Gershon [22] computed $G(A_3)$, $G(A_3^*)$ and $G(D_4)$, and conjectured that $A_3^* (\cong D_3^*)$ is the optimal lattice in three dimensions. This conjecture was established in [2]. In an earlier paper [10] we determined the Voronoi regions and evaluated G for all the root lattices $A_n (n \geq 1)$, $D_n (n \geq 3)$, E_6 , E_7 , $E_8 (=E_8^*)$ and the dual lattices $A_n^* (n \geq 1)$ and $D_n^* (n \geq 3)$. We observed that the optimal lattice quantizer was often

the dual of the densest lattice packing, and conjectured that this may be true in general. It is true in dimensions 1, 2 and 3, and is supported by the available data in dimensions 4, 5 and 8. In the present paper we evaluate G for the duals of the densest known packings in 6, 7, 12, 16 and 24 dimensions, namely for the lattices E_6^* , E_7^* , the Coxeter–Todd lattice K_{12} ($\cong K_{12}^*$), the Barnes–Wall lattice Λ_{16} ($\cong \Lambda_{16}^*$), and the Leech lattice Λ_{24} ($= \Lambda_{24}^*$). The results are summarized in Table 1 and Fig. 2 and support our conjecture.

It is worth pointing out that this conjecture would imply the somewhat surprising result that the best quantizer is in general different from the most efficient lattice covering of space by spheres. Indeed, the two problems already have different answers in dimensions 4 and 5. There the best lattice coverings are known to be A_4^* and A_5^* [33], yet D_4^* and D_5^* are better quantizers (see Fig. 2). In higher dimensions, the best coverings known are A_n^* , if $n \leq 23$, and then various lattices constructed from Λ_{24} [1]. So it seems likely that, in all dimensions between 4 and 23, the best lattice quantizers and coverings are distinct.

The values of G for various lattices are compared in Fig. 2, the values for A_n , A_n^* , D_n , D_n^* , E_6 , E_7 and E_8 being taken from [10]. In 1964 Zador proved by a nonconstructive argument that good quantizers exist in sufficiently high dimensions, and observed that the second moment of a sphere gives a bound in the other direction (see [39], and also [22], [10, Eq. (3)]). These two bounds are also plotted in the figure.

We use Monte Carlo integration to compute G . The technique is briefly described in § 2. It requires that we have a fast algorithm for performing the quantizing, that is, given an arbitrary point of $\mathbb{R}^n$, for finding the closest lattice point. (If the lattice is used as a code for a band-limited channel, this algorithm performs the decoding [9], [11].) The best quantizing algorithms we have found for these lattices, and the values of G that were obtained, are given in §§ 3–7.

If a more complete description of the polytopes $\mathcal{P}$ were available, we could determine G exactly by decomposing each polytope into simplices and using the methods of [10]. Unfortunately little is known about the polytopes of these lattices. For E_6^* and E_7^* , for example, even the covering radius (the distance of the furthest vertex of $\mathcal{P}$ from the origin) is unknown, although it is known for all the other lattices mentioned [8], [10], [13], [15], [17]. In § 8 we establish two theorems which help determine the number of $(n-1)$ -dimensional faces of $\mathcal{P}$, and in the last section we use them to study the Voronoi region of K_{12} .

Notation. If L_n is a lattice in $\mathbb{R}^n$ (the subscript indicates the dimension), the dual lattice $L_n^* = \{x \in \mathbb{R}^n : x \cdot y \in \mathbb{Z} \text{ for all } y \in L_n\}$. Two lattices L_n and M_n are equivalent, written $L_n \cong M_n$, if they differ only by a rotation and possibly a change of scale. The direct sum of k copies of L_n is written L_n^k . For further information about these lattices see [4], [10], [13]–[17], [27], [31], [36], [37].

2. Monte Carlo integration over a Voronoi region. The Monte Carlo technique that we use to calculate G is slightly unusual. We wish to find $I := \int_{\mathcal{P}} x \cdot x \, dx$. Conventional Monte Carlo methods ([24], [25], [28], [34]) would begin by replacing I by $\int_{\mathcal{Q}} \chi_{\mathcal{P}}(x) x \cdot x \, dx$, where $\mathcal{Q}$ is a region enclosing $\mathcal{P}$, usually a sphere or a cube, and $\chi_{\mathcal{P}}(x)$ is 1 if $x \in \mathcal{P}$, 0 otherwise. This is wasteful, since points in $\mathcal{Q} \setminus \mathcal{P}$ do not contribute to the estimate. The following approach avoids this difficulty, by exploiting our fast quantizing algorithms.

Let $v^{(1)}, \dots, v^{(n)}$ be linearly independent vectors spanning the lattice, and let $u_1, \dots, u_n$ be independent random numbers, uniformly distributed between 0 and 1. Then $y = \sum u_i v^{(i)}$ is uniformly distributed over the fundamental parallelepiped generated

by the $v^{(i)}$. Let l be the closest lattice point to y (found by the quantizing algorithm for this lattice); then $w(y) := y - l$ is uniformly distributed over the Voronoi region $\mathcal{P}$. Figure 3 illustrates this for the hexagonal lattice A_2 .

FIG. 3. Hexagonal lattice A_2 with spanning vectors $v^{(1)}, v^{(2)}$; y is a random point in the parallelogram $0, v^{(1)}, v^{(2)}, v^{(1)} + v^{(2)}$; l is the closest lattice point to y ; and $w = y - l$ is a random point in the Voronoi region $\mathcal{P}$.

Then if $m = gh$ random points $y^{(0)}, \dots, y^{(m-1)}$ are selected in the manner just described,

$$(2) \quad \hat{I} = \frac{1}{m} \sum_{i=0}^{m-1} N(w(y^{(i)}))$$

is an estimate of I , and when suitably scaled (see (1)) produces our estimate $\hat{G}$ of G . To estimate the variance of $\hat{I}$, we group the measurements into g sets of h , and use the jackknife estimator (see [29], [32], [38]):

$$(3) \quad \widehat{\text{var}}(\hat{I}) = \frac{1}{g(g-1)} \sum_{j=0}^{g-1} \left(\frac{1}{h} A_j - \hat{I} \right)^2,$$

where

$$(4) \quad A_j = \sum_{i=jh}^{(j+1)h-1} N(w(y^{(i)})), \quad j = 0, 1, \dots, h-1.$$

By taking the square root of (3) and scaling, we obtain an estimate $\hat{\sigma}$ for the standard deviation of $\hat{G}$, and then $\hat{G} \pm 2\hat{\sigma}$ is our final estimate for G . The portable random number generator on the PORT library [21], which combines a congruential generator and a Tausworthe generator, was used to produce the u_i 's.

To test this procedure we applied it first to the lattices E_6, E_7 and E_8 , for which the exact value of G is known [10]. The estimates agreed closely with the exact values. For example, for E_8 we found using $m = 10^7$ points (g was always taken to be 100) that

$$\hat{G}(E_8) = 0.071689 \pm 0.000008,$$

while the true value is

$$G(E_8) = \frac{929}{12960} = 0.0716821 \dots$$

3. The lattices E_6 and E_6^* . E_6 is most easily obtained from the complex 3-dimensional $\mathbb{Z}[\omega]$ -lattice with generator matrix

$$(5) \quad \begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 1 & 1 & 1 \end{bmatrix}, \quad \omega = e^{2\pi i/3}, \quad \theta = \omega - \bar{\omega} = i\sqrt{3},$$

([16, p. 421], [35, Example 5], [36, § 5.8.2], [37, § 17]), and the dual lattice E_6^* from

$$(6) \quad \begin{bmatrix} \theta & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 0 & -1 \end{bmatrix}.$$

Thus the real 6-dimensional lattices E_6 and E_6^* have generator matrices

$$(7a) \quad \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \sqrt{3} & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ \frac{-3}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{-3}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & \frac{-1}{2} & \frac{\sqrt{3}}{2} & \frac{-1}{2} & \frac{\sqrt{3}}{2} \end{bmatrix},$$

$$(7b) \quad \begin{bmatrix} 0 & \sqrt{3} & 0 & 0 & 0 & 0 \\ 1 & 0 & -1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ \frac{-3}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & \frac{1}{2} & \frac{-\sqrt{3}}{2} & 0 & 0 \\ \frac{-1}{2} & \frac{\sqrt{3}}{2} & 0 & 0 & \frac{1}{2} & \frac{-\sqrt{3}}{2} \end{bmatrix}$$

respectively. (For example (7a) is obtained from the real and imaginary parts of (5) and ω times (5).)

To find a quantizing algorithm for any of these lattices, we proceed as follows, following [11]. Inside our lattice L_n we look for a sublattice S_n of small index t (say), for which quantizing is easy. Suppose

$$(8) \quad L_n = (a^{(0)} + S_n) \cup \cdots \cup (a^{(t-1)} + S_n),$$

where $a^{(0)}, \dots, a^{(t-1)}$ are coset representatives for S_n in L_n . Let $\phi: \mathbb{R}^n \rightarrow S_n$ be a quantizer for S_n , so that $\phi(x)$ is the closest point of S_n to a given vector x . Then a quantizer for L_n is obtained by taking the given vector w , forming the t candidates

$$c^{(i)} = \phi(w - a^{(i)}) + a^{(i)}, \quad i = 0, \dots, t-1,$$

and choosing the closest candidate to w .

Both E_6 and E_6^* contain a sublattice S_6 isomorphic to A_2^3 , namely the real version of the lattice $\theta\mathbb{Z}[\omega]^3$ with generator matrix

$$\begin{bmatrix} \theta & 0 & 0 \\ 0 & \theta & 0 \\ 0 & 0 & \theta \end{bmatrix}.$$

S_6 has index 3 in E_6 , with coset representatives

$$a^{(0)} = (0, 0, 0, 0, 0, 0), \quad a^{(1)} = (1, 0, 1, 0, 1, 0), \quad a^{(2)} = -a^{(1)},$$

and has index 9 in E_6^* , with coset representatives given in Table 2.

TABLE 2
Coset representatives $a^{(0)}, \dots, a^{(8)}$ for S_6 in E_6^*

(0, 0, 0, 0, 0, 0)
$\pm(1, 0, -1, 0, 0, 0)$
$\pm(0, 0, 1, 0, -1, 0)$
$\pm(-1, 0, 0, 0, 1, 0)$
$\pm(1, 0, 1, 0, 1, 0)$

It is easy to design a quantizer for the hexagonal lattice A_2 (and hence A_2^3), either using the fact that A_2 is the union of a rectangular lattice and a translate, as suggested by Gersho [23, p. 165], or via 3-dimensional coordinates as we suggest in [10, § VII]. Gersho's method seems slightly simpler and is the one we adopted. From the previous discussion we have the following quantizing algorithms for E_6 and E_6^* .

QUANTIZING ALGORITHMS FOR E_6 ($t=3$) AND E_6^* ($t=9$)

Given $w = (w_1, \dots, w_6)$, to find the closest lattice point of E_6 (or E_6^*).

Subtract one of t coset representatives $a^{(i)}$, obtaining

$$z = w - a^{(i)} = (z_1, \dots, z_6).$$

Divide the corresponding complex vector by θ , i.e., form

$$z' = \left(\frac{z_2}{\sqrt{3}}, \frac{-z_1}{\sqrt{3}}, \frac{z_4}{\sqrt{3}}, \frac{-z_3}{\sqrt{3}}, \frac{z_6}{\sqrt{3}}, \frac{-z_5}{\sqrt{3}} \right).$$

Apply the quantizer for A_2 to the three pairs

$$\left(\frac{z_2}{\sqrt{3}}, \frac{-z_1}{\sqrt{3}} \right), \left(\frac{z_4}{\sqrt{3}}, \frac{-z_3}{\sqrt{3}} \right), \left(\frac{z_6}{\sqrt{3}}, \frac{-z_5}{\sqrt{3}} \right)$$

obtaining say

$$(m_1, m_2), (m_3, m_4), (m_5, m_6).$$

Multiply by “ θ ”, to get

$$m' = (-\sqrt{3}m_2, \sqrt{3}m_1, -\sqrt{3}m_4, \sqrt{3}m_3, -\sqrt{3}m_6, \sqrt{3}m_5).$$

Then $c^{(i)} = m' + a^{(i)}$ is the i th candidate.

The final answer is the candidate which minimizes $N(w - c^{(i)})$.

We used this algorithm in the Monte Carlo procedure described in § 2, in order

to estimate $G(E_6^*)$. With 5×10^6 random points we obtained

$$(9) \quad \hat{G}(E_6^*) = 0.074239 \pm 0.000018.$$

(As a check, 10^7 points for E_6 gave $\hat{G}(E_6) = 0.074342 \pm 0.000013$, while the exact value from [10] is $G(E_6) = 3^{-1/6} 5/56 = 0.07434671 \dots$, well within the range of the estimate.)

4. The lattices E_7 and E_7^* . E_7 may be obtained by applying Construction A of [27] to the little $[7, 3, 4]$ Hamming code $\mathcal{H}_1$, and a generator matrix for E_7 may be found on [37, p. 335] (see also [4], [16], [13], [27]). Similarly E_7^* may be obtained from the $[7, 4, 3]$ Hamming code $\mathcal{H}_2$. Both E_7 and E_7^* contain a sublattice $S_7 = 2\mathbb{Z}^7$, of index 8 and 16 respectively; the coset representatives are the codewords of either $\mathcal{H}_1$ or $\mathcal{H}_2$. Since there is a trivial quantizer for $\mathbb{Z}^7$ (see [11, § III]), this leads to fast quantizing algorithms for E_7 and E_7^* ; we omit the details. (Alternative algorithms, based on the sublattice A_7 , were proposed in [11, § VII].) Our Monte Carlo estimate for $G(E_7^*)$, based on 10^7 points, is

$$(10) \quad \hat{G}(E_7^*) = 0.073124 \pm 0.000013.$$

This is slightly better than the value for E_7 , which is $0.07323063 \dots$.

5. The Coxeter–Todd lattice K_{12} . The real 12-dimensional lattice K_{12} was first described in [18]. It is the subject of our earlier paper [15], and further properties will be found in § 9 below and in [13], [14], [27]. Since it is the densest sphere packing known in 12 dimensions [13], [27], and is also equivalent to its dual, according to the conjecture mentioned in § 1 it is a good candidate for a quantizer. The same remark applies to the lattices Λ_{16} and Λ_{24} studied in the following sections.

Regarded as a complex 6-dimensional $\mathbb{Z}[\omega]$ -lattice, K_{12} has the generator matrix

$$(11) \quad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & \omega & \bar{\omega} \\ 1 & 0 & 1 & 0 & \omega & \bar{\omega} \\ \omega & \bar{\omega} & 0 & 1 & 0 & 1 \end{bmatrix},$$

and so as a real 12-dimensional lattice it has the generator matrix shown in Fig. 4. K_{12} has a sublattice isomorphic to A_2^6 , of index 64, namely the real version of the lattice $2\mathbb{Z}[\omega]^6$. The coset representatives correspond to the codewords of the hexacode (the code over $GF(4) = \{0, 1, \omega, \bar{\omega}\}$ spanned by the last three rows of (11)). This leads to a quantizing algorithm similar to those for E_6 and E_6^* described in § 3.

There are alternative definitions of K_{12} (see [15]), which make four other sublattices visible, namely the real lattices corresponding to $\mathbb{Z}[\omega] \otimes A_6$, $\mathbb{Z}[\omega] \otimes D_6$, $\mathbb{Z}[\omega] \otimes E_6$, and the lattice $\{(x_1, \dots, x_6) : \text{all } x_i \in \mathbb{Z}[\omega], \sum x_i \equiv 0 \pmod{\theta}\}$. Each of these leads to a decoding algorithm. However the one described above seems to be the simplest.

Our Monte Carlo estimate, based on 10^6 points, is

$$(12) \quad \hat{G}(K_{12}) = 0.070100 \pm 0.000024.$$

6. The Barnes–Wall lattice Λ_{16} . The lattice Λ_{16} was first described in [3]. Other references are [13] and [27], and a generator matrix is given on page 336 of [37]. Λ_{16} has a sublattice $2D_{16}$ of index 32, with coset representatives which are the codewords of the $[16, 5, 8]$ first-order Reed–Muller code. Then the quantizing algorithm for D_n

2 0	0 0	0 0	0 0	0 0	0 0	0 0
0 0	2 0	0 0	0 0	0 0	0 0	0 0
0 0	0 0	2 0	0 0	0 0	0 0	0 0
0 0	1 0	0 0	1 0	-1/2 $\sqrt{3}/2$	-1/2 $-\sqrt{3}/2$	
1 0	0 0	1 0	0 0	-1/2 $\sqrt{3}/2$	-1/2 $-\sqrt{3}/2$	
-1/2 $\sqrt{3}/2$	-1/2 $-\sqrt{3}/2$	0 0	1 0	0 0	1 0	
-1 $\sqrt{3}$	0 0	0 0	0 0	0 0	0 0	
0 0	-1 $\sqrt{3}$	0 0	0 0	0 0	0 0	
0 0	0 0	-1 $\sqrt{3}$	0 0	0 0	0 0	
0 0	-1/2 $\sqrt{3}/2$	0 0	-1/2 $\sqrt{3}/2$	-1/2 $-\sqrt{3}/2$	1 0	
-1/2 $\sqrt{3}/2$	0 0	-1/2 $\sqrt{3}/2$	0 0	-1/2 $-\sqrt{3}/2$	1 0	
-1/2 $-\sqrt{3}/2$	1 0	0 0	-1/2 $\sqrt{3}/2$	0 0	-1/2 $\sqrt{3}/2$	

FIG. 4. Generator matrix for Coxeter-Todd lattice K_{12} .

described in [11, § IV] leads to an efficient quantizer for Λ_{16} . Using $2 \cdot 10^6$ points we found

$$(13) \quad \hat{G}(\Lambda_{16}) = 0.068299 \pm 0.000027.$$

7. The Leech lattice Λ_{24} . The Leech lattice Λ_{24} ([5]–[8], [12]–[14], [26], [27], [36], [37]) may be constructed in many ways. The standard MOG (or miracle octad generator [7], [8], [19]) basis is shown in Fig. 5. Λ_{24} has a sublattice $4D_{24}$ of index 8,192, with coset representatives

$$2c \text{ and } 2c + u,$$

where $u = (-3, 1, 1, \dots, 1)$, and c runs through the vectors of the [24, 12, 8] Golay code with generator matrix shown in Fig. 6. Because of the large index of the sublattice, this is by far the slowest of our quantizing algorithms. Using 25,000 points we found

$$(14) \quad \hat{G}(\Lambda_{24}) = 0.065771 \pm 0.000074.$$

8. The number of faces of the Voronoi region. The Voronoi region $\mathcal{P}$ of a lattice $L_n \subseteq \mathbb{R}^n$ may be expressed as

$$(15) \quad \mathcal{P} = \bigcap_{v \in L_n, v \neq 0} \mathcal{S}(v),$$

where $\mathcal{S}(v)$ is the half-space $\{x \in \mathbb{R}^n : x \cdot v \leq \frac{1}{2}v \cdot v\}$, bounded by the hyperplane

$$\Pi(v) = \{x \in \mathbb{R}^n : x \cdot v = \frac{1}{2}v \cdot v\}.$$

Of course only finitely many of the $\mathcal{S}(v)$ are really needed to define $\mathcal{P}$. Let

$$\mathcal{P} = \bigcap_{v \in \mathcal{R}} \mathcal{S}(v),$$

where $\mathcal{R}$ is a minimal subset of $L_n \setminus \{0\}$ that will define $\mathcal{P}$. We call the lattice vectors in $\mathcal{R}$ *relevant*, and the remaining vectors of L_n *irrelevant*. The number of relevant vectors is the number of $(n-1)$ -dimensional faces of $\mathcal{P}$. For example, in the body-centred cubic lattice (Fig. 1) there are 14 relevant vectors, namely the six minimal vectors and the eight vectors of the next smallest norm.

8 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 4 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 4 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 4	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	4 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 4 0 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 0 4 0	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
2 2 2 2	2 2 2 2	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 0 0 0	4 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 0 0 0	0 4 0 0	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 0 0 0	0 0 4 0	0 0 0 0	0 0 0 0	0 0 0 0
2 2 2 2	0 0 0 0	2 2 2 2	0 0 0 0	0 0 0 0	0 0 0 0
4 0 0 0	0 0 0 0	0 0 0 0	4 0 0 0	0 0 0 0	0 0 0 0
2 2 0 0	2 2 0 0	2 2 0 0	2 2 0 0	0 0 0 0	0 0 0 0
2 0 2 0	2 0 2 0	2 0 2 0	2 0 2 0	0 0 0 0	0 0 0 0
2 0 0 2	2 0 0 2	2 0 0 2	2 0 0 2	0 0 0 0	0 0 0 0
4 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0	4 0 0 0	0 0 0 0
2 0 2 0	2 0 0 2	2 2 0 0	0 0 0 0	2 2 0 0	0 0 0 0
2 0 0 2	2 2 0 0	2 0 2 0	0 0 0 0	2 0 2 0	0 0 0 0
2 2 0 0	2 0 2 0	2 0 0 2	0 0 0 0	2 0 0 2	0 0 0 0
0 2 2 2	2 0 0 0	2 0 0 0	2 0 0 0	2 0 0 0	2 0 0 0
0 0 0 0	0 0 0 0	2 2 0 0	2 2 0 0	2 2 0 0	2 2 0 0
0 0 0 0	0 0 0 0	2 0 2 0	2 0 2 0	2 0 2 0	2 0 2 0
-3 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1

FIG. 5. Generator matrix for Leech lattice Λ_{24} in standard MOG form.

1 1 1 1	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0	0 0 0 0
1 1 1 1	0 0 0 0	1 1 1 1	0 0 0 0	0 0 0 0	0 0 0 0
1 1 0 0	1 1 0 0	1 1 0 0	1 1 0 0	0 0 0 0	0 0 0 0
1 0 1 0	1 0 1 0	1 0 1 0	1 0 1 0	0 0 0 0	0 0 0 0
1 0 0 1	1 0 0 1	1 0 0 1	1 0 0 1	0 0 0 0	0 0 0 0
1 0 1 0	1 0 0 1	1 1 0 0	0 0 0 0	1 1 0 0	0 0 0 0
1 0 0 1	1 1 0 0	1 0 1 0	0 0 0 0	1 0 1 0	0 0 0 0
1 1 0 0	1 0 1 0	1 0 0 1	0 0 0 0	1 0 0 1	0 0 0 0
0 1 1 1	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0	1 0 0 0
0 0 0 0	0 0 0 0	1 1 0 0	1 1 0 0	1 1 0 0	1 1 0 0
0 0 0 0	0 0 0 0	1 0 1 0	1 0 1 0	1 0 1 0	1 0 1 0
1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1	1 1 1 1

FIG. 6. Generator matrix for Golay code in standard MOG form.

[10, § III] contains two theorems which give sufficient conditions for a lattice to have the property that the minimal vectors are the only relevant vectors. The lattices A_n , D_n , E_6 , E_7 and E_8 have this property [10, Corollary to Theorem 5]. The following theorems can also be used to show that certain vectors are irrelevant.

THEOREM 1. *If L_n has covering radius R_c (cf. [8]), then any vector $v \in L_n$ of norm $\geq 4R_c^2$ is irrelevant.*

Proof. Suppose $v \in L_n$ is relevant and $N(v) \geq 4R_c^2$. Then $\Pi(v)$ meets $\mathcal{P}$ in a set of positive measure. On the other hand the closest point of $\Pi(v)$ to the origin, $\frac{1}{2}v$, has norm $\geq R_c^2$, so $\Pi(v)$ can contain at most a single vertex of $\mathcal{P}$, which is a contradiction. Q.E.D.

For example, Theorem 1 and the main result of [8] imply that for the Leech lattice only the minimal vectors and those of the next smallest norm are relevant. Thus the Voronoi region has 16,969,680 faces.

When attempting to show that a vector $v \in L_n$ is irrelevant, it is sometimes possible to prove that the point $\frac{1}{2}v$ is not in the Voronoi region. In general this is not enough to prove that v is irrelevant, since it is certainly possible for the hyperplane $\Pi(v)$ to intersect $\mathcal{P}$ in an asymmetric region not containing $\frac{1}{2}v$. The next theorem establishes a condition under which this does not happen.

THEOREM 2. *Let $\mathcal{G}_v$ be the subgroup of the automorphism group $\text{Aut}(L_n)$ fixing a lattice vector v . Suppose $\mathcal{G}_v$ fixes exactly one 1-dimensional subspace, namely the 1-dimensional subspace containing v . Then if the hyperplane $\Pi(v)$ contains a point x of the Voronoi region $\mathcal{P}$ different from $\frac{1}{2}v$, it follows that $\frac{1}{2}v \in \mathcal{P}$.*

Proof. Consider the point

$$y = \frac{1}{|\mathcal{G}_v|} \sum_{g \in \mathcal{G}_v} x^g.$$

Since $x \in \mathcal{P} \cap \Pi(v)$, $x^g \in \mathcal{P} \cap \Pi(v)$. Because $\mathcal{P}$ is convex, $y \in \mathcal{P} \cap \Pi(v)$. Clearly y is fixed by $\mathcal{G}_v$. But the only point of $\Pi(v)$ fixed by $\mathcal{G}_v$ is $\frac{1}{2}v$. Therefore $y = \frac{1}{2}v$ and $\frac{1}{2}v \in \mathcal{P}$. Q.E.D.

An application of these theorems will be found in the following section.

9. The Voronoi region of K_{12} . We shall determine the number of faces of the Voronoi region $\mathcal{P}$ of K_{12} . We assume K_{12} is scaled so that the minimal norm is 2. It was shown in [15] that the covering radius of K_{12} is $\sqrt{4/3}$, and that there are 20,412 vertices of $\mathcal{P}$ at this distance from the origin, all equivalent under $\text{Aut}(K_{12})$. Unfortunately nothing is known about other vertices of $\mathcal{P}$.

K_{12} is best studied via the corresponding 6-dimensional lattice Λ_6^ω (see [15]). The latter has several equivalent definitions, one of which may be seen in (11). In this section, however, another construction is more convenient, the so-called 3-base. In this form Λ_6^ω is defined as the set of vectors

$$\{(x_1, \dots, x_6)_3 : \text{all } x_i \in \mathbb{Z}[\omega], x_1 \equiv \dots \equiv x_6 \pmod{\theta} \text{ and } \Sigma x_i \equiv 0 \pmod{3}\},$$

where $(x_1, \dots, x_6)_3$ is an abbreviation for $\theta^{-1}(x_1, \dots, x_6)$.

THEOREM 3. *The Voronoi region $\mathcal{P}$ of K_{12} has 4,788 11-dimensional faces, and is bounded by the hyperplanes determined by the 756 vectors of norm 2 and the 4,032 vectors of norm 3.*

Proof. We shall show that the vectors of norm ≥ 4 in K_{12} are irrelevant. Theorem 1 already implies that the vectors of norm ≥ 6 are irrelevant. Also it is easy to verify that $\text{Aut}(K_{12})$ is transitive on the vectors of norms 2, 3, 4 and 5. The vectors of norms 2, 3 and 4, expressed in the 3-base, are listed in Table II of [15].

(i) *The vectors of norm 5 are irrelevant.* In view of the transitivity just mentioned, it is enough to show that a single norm 5 vector, say $v_5 = (2\theta, \theta, 0, 0, 0, 0)_3$, is irrelevant. We first show that $\frac{1}{2}v_5 \notin \mathcal{P}$. In fact it is easily checked that $\frac{1}{2}v_5$ is outside the hyperplane $\Pi(v_2)$ determined by $v_2 = (\omega, \omega, \omega, \omega, \bar{\omega}, 1)_3$, and therefore by (15) is not in $\mathcal{P}$.

Next, let $\mathcal{F}$ be the stabilizer of v_5 in $\text{Aut}(\Lambda_6^\omega)$. We show that $\mathcal{F}$ only fixes a (complex) 1-dimensional subspace. Since $\mathcal{F}$ contains the diagonal matrices

$$\text{diag}\{1, 1, 1, 1, \omega, \bar{\omega}\},$$

where the ω and $\bar{\omega}$ may be in any of the last four positions, any vector fixed by $\mathcal{F}$ must have the form $f = (\alpha, \beta, 0, 0, 0, 0)_3$. Also $\mathcal{F}$ contains the reflection

$$R_w : x \mapsto x - x \cdot \bar{w} \cdot w,$$

where $w = (1, -2, 1, 1, 1, 1)_3$, and so $f = (2\beta, \beta, 0, 0, 0, 0)_3$; in other words f is a multiple of v_5 .

Let $\mathcal{G}_{v_5}$ be the stabilizer in $\text{Aut}(K_{12})$ of the real vector corresponding to v_5 . Since $\text{Aut}(K_{12})$ contains a transformation which corresponds to complex conjugation in Λ_6^ω , only real multiples of v_5 are fixed by $\mathcal{G}_{v_5}$. Thus $\mathcal{G}_{v_5}$ only fixes a single 1-dimensional space. We can conclude from Theorem 2 that v_5 is irrelevant.

(ii) *The vectors of norm 4 are irrelevant.* It is enough to consider one vector of norm 4, say $v_4 = (1 + 3\omega, 1, 1, 1, 1, 1)_3$. One can now show that the equation to the hyperplane $\Pi(v_4)$ is already implied by the equations to the ten hyperplanes $\Pi(v_2)$ passing through $\frac{1}{2}v_4$, and defined by the following vectors v_2 :

$$\begin{aligned} &(\omega, \bar{\omega}, 1, 1, 1, 1)_3, \\ &\quad \dots \\ &(\omega, 1, 1, 1, 1, \bar{\omega})_3, \\ &(\theta, -\omega\theta, 0, 0, 0, 0)_3, \\ &\quad \dots \\ &(\theta, 0, 0, 0, 0, -\omega\theta)_3. \end{aligned}$$

We leave this verification to the reader, as well as the easy justification that the vectors of norms 2 and 3 are relevant. Q.E.D.

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