The number of bits required to represent defect information in Theorem 1 can be reduced with the following scheme. Let $M0$ be an $(m+1) \times n$ matrix whose last row is an all-ones vector and whose first m rows contain all possible distinct binary m-tuples as columns. Matrix $M0$ can be considered a generator matrix of the first-order RM code. Next, let $M1$ be the matrix that contains as rows all row vectors of $M0$ and the sums of all possible pairs of row vectors of the first m rows of $M0$. The number of rows in $M1$ is equal to $1 + m(m+1)/2$. Finally, let M be the matrix formed by placing matrix $M1$ on top of the complement of $M1$. We prove the following lemma.

Lemma 1: The row vectors of a set of any three columns of M contain all possible eight binary three-tuples.

Proof: Let U be the matrix of three columns of M. Matrix U contains $(0,0,0)$ as a row vector. Because the null space of the row space of $M0$ has a minimum distance of four, the columns of U are linearly independent. Without loss of generality, let $x_1 = (1,1,1)$, x_2, and x_3 be a set of independent rows of U. Then all eight possible linear combinations of x_1, x_2, and x_3 are distinct and are members of the rows of U. Q.E.D.

Lemma 1 says that matrix M taken as $G0$ satisfies condition 1 with $t = 3$. The next problem is to determine the number of bits required to represent the defect information. Let s be the minimum number of columns of M such that the rows of the matrix formed by the columns are all distinct. Then from condition 2 the number of bits required to represent the defect information is s. Since there are a total of $2 + m(m+1)$ rows in M, s is greater than or equal to $\log_2(2 + m(m+1))$. To obtain an upper bound on s, we first prove the following lemma.

Lemma 2. If there exists an $(m+1, m+1-s)$ binary linear code with minimum distance six, then there exits a set of s columns of M that contain as row vectors distinct s-tuples.

Proof: Let $H0$ be the generator matrix of the null space of an $(m+1, m+1-s)$ code with minimum distance of six. The dimension of the row space of the matrix is s.

Matrix $H0$ can be arranged so that the all-ones vector appears in the last column. Let W be the set of vectors that contains all the columns of $H0$, and the vectors of the sums of all possible pairs of the first m columns of $H0$. Because any five columns of $H0$ are linearly independent, all $2 + m(m+1)$ vectors of W and the complements of the vectors of W are distinct. Now, the transpose of $H0$, an $(m+1) \times s$ matrix with an all-ones row at the bottom, is a submatrix of $M0$ because the first m rows of $M0$ contain all possible m-tuples as columns. Thus, there exists a set of s columns in M that contain as row vectors the elements of W and their complements. The row vectors of the matrix formed by these s columns of M are distinct. Q.E.D.

Let $K(n)$ denote the dimension of the smallest binary linear code of lenth n whose dual code has a minimum distance of six. Combining Lemmas 1 and 2, we have the following results.

Theorem 2: Let C be an (n,k) binary linear code that contains the first-order Reed–Muller code as a subcode, where $n = 2^m$. Assume that the minimum distance of C is greater than $2T$. Then C can be used to mask defect patterns of weight less than or equal to three and correct random error patterns of weight less than or equal to T. The number of redundant bits required for the defect information is at most equal to $K(m+1)$.

The BCH bound on the minimum distance [5] implies that the number of check bits required for a linear code of length n with a minimum distance of six is less than or equal to $1 + 2\log_2 n$. From this bound we have the following corollary.

Corollary: The number of redundant bits required for the defect information in Theorem 2 is less than or equal to $1 + 2\log_2(m+1)$.

TABLE I
NUMBER OF BITS REQUIRED FOR MASKING 3-BIT DEFECTS

Number of bits	$m = \log_2 n$
5	5
6	6
7	7–8
8	9–11
9	12–17
10	18–22

Table I lists some of the number of required redundant bits for the defect information stated in Theorem 2. As an example, the table shows that a distance 4 (79, 71) code can be obtained for the encoding of 64 data bits. The code can be used to mask all three-bit stuck errors, correct all single bit random errors, and detect all double bit random errors.

III. Conclusions

We have constructed a class of binary linear codes that are capable of masking three-bit defect patterns and correcting multiple random errors. The number of additional redundant bits required for masking the defects grows linearly with $\log(\log n)$.

In the testing of logic circuits, an approach that applies n-bit patterns that exercise all possible combinations of t-tuples at any t positions has been suggested [6], [7]. The design of a matrix $G0$ that satisfies condition 1 is a solution to the problem of the generation of n-bit test patterns. Thus, the techniques used in the construction of codes for masking defects can be applied to exhaustive pattern generation for logic testing.

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A Lower Bound on the Average Error of Vector Quantizers

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***Abstract*—A lower bound is proposed for the mean-squared error of an n-dimensional vector quantizer with a large number of output points.**

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Although no formal proof has been found, a plausible geometrical argument is given for believing that the bound is correct. The new bound is analogous to the Rogers bound for packing spheres, the Coxeter–Few–Rogers bound for covering space with spheres, and the Coxeter–Böröczky bound for packing spherical caps. It is significantly stronger than Zador's sphere bound for quantizers.

I. Introduction

An n-dimensional vector quantizer [3], [11] with output points $y_1, \cdots, y_M \in \mathbb{R}^n$ maps an input point $x \in \mathbb{R}^n$ into a closest output point y_i. If x has probability density function $p(x)$, the mean-squared error per symbol of this quantizer is

$$E = \frac{1}{n} \int_{\mathbb{R}^n} \|x - y_i\|^2 p(x)\, dx,$$

where $\|x\| = (x \cdot x)^{1/2}$. We wish to choose $y_1, \cdots, y_M$ to minimize E. Zador [25] showed under quite general assumptions about $p(x)$ that the mean-squared error of an optimal quantizer satisfies

$$\lim_{M \to \infty} M^{2/n} E = G_n \left(\int_{\mathbb{R}^n} p(x)^{n/(n+2)}\, dx \right)^{(n+2)/n}, \qquad (1)$$

where G_n depends only on n. He also showed that

$$G_n \geq \frac{1}{(n+2)\pi} \Gamma\left(\frac{n}{2} + 1\right)^{2/n} \qquad (2)$$

(the "sphere bound"), and

$$G_n \leq \frac{1}{n\pi} \Gamma\left(\frac{n}{2} + 1\right)^{2/n} \Gamma\left(1 + \frac{2}{n}\right). \qquad (3)$$

Since the probability density function $p(x)$ only appears in the last term of (1), we may choose any convenient $p(x)$ when attempting to find G_n. In particular, this implies that G_n is the mean-squared error of an optimal quantizer in the case when the input x is uniformly distributed over a large sphere in $\mathbb{R}^n$, and $M \to \infty$, which is an important question in its own right.

In this correspondence we propose a stronger lower bound than (2). We give a plausible geometrical argument for believing that G_n cannot be less than the quantity

$$\frac{n + 3 - 2H_{n+2}}{4n(n+1)} (n+1)^{1/n} (n!)^{4/n} f_n(n)^{2/n}, \qquad (4)$$

where

$$H_m = \sum_{i=1}^{m} \frac{1}{i} \qquad (m = 1, 2, \cdots) \qquad (5)$$

is a harmonic sum, and $f_n(x)$ is Schläfli's function (see Section II).

Some numerical values of (4) are given in Fig. 1 and Table I, together with Zador's bounds (2) and (3). We also show some upper bounds on G_n that were obtained in [3], [5] by evaluating the mean-squared error of certain lattices when used to quantize uniformly distributed inputs. The following lattices were used: the so-called Voronoi lattices of the first type, A_n^* (the subscript indicates the dimension, and the asterisk that these are the duals of the root lattices A_n—see [3]); the lattices D_n^*; and, in the "Other" column of Table I, the lattices E_6^*, E_7^*, the Gosset lattice E_8, the Coxeter-Todd lattice K_{12}, the Barnes-Wall lattice Λ_{16}, and the Leech lattice Λ_{24} (see [3]–[5]).

It is known that A_1^* (the integers) and A_2^* (the familiar two-dimensional hexagonal lattice) are optimal for quantizing uniformly distributed inputs in one and two dimensions [8], [10], [18], and that the body-centered cubic lattice A_3^* is similarly optimal among *lattice* quantizers in three dimensions [1]. It can be seen from Table I that (4) is stronger than Zador's sphere bound (2) and gives considerably better estimates of G_n for $n \geq 2$. In dimensions 8 and 24, in particular, the bounds are

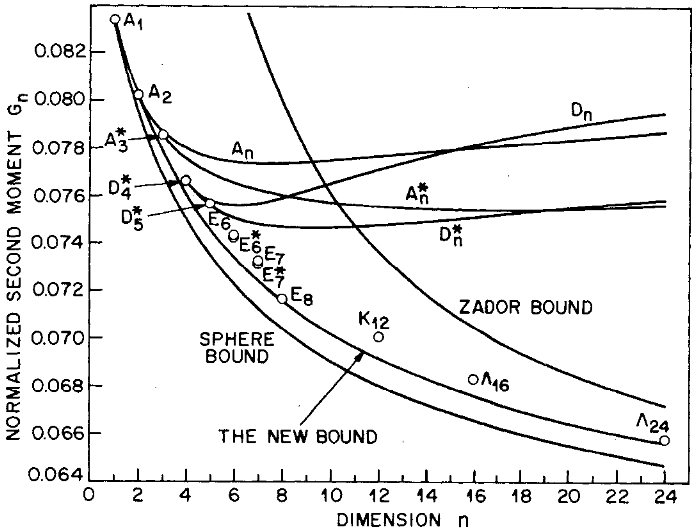

Fig. 1. Comparison of the new bound with the sphere bound and Zador's bound.

TABLE I
BOUNDS ON G_n (MULTIPLIED BY 100)

Dimension n	Lower bounds: Sphere bound (2)	Lower bounds: New bound (4)	Upper bounds: A_n^*	Upper bounds: D_n^*	Upper bounds: Other	Upper bounds: Zador (3)
1	8.333	8.333	8.333			50.000
2	7.958	8.019	8.019			15.915
3	7.697	7.787	7.854	7.854		11.580
4	7.503	7.609	7.756	7.660		9.974
5	7.352	7.465	7.692	7.563		9.132
6	7.230	7.347	7.649	7.512	7.424	8.608
7	7.130	7.248	7.619	7.486	7.312	8.248
8	7.045	7.163	7.597	7.474	7.168	7.982
9	6.973	7.090	7.582	7.469		7.778
10	6.910	7.026	7.570	7.470		7.614
12	6.807	6.918	7.557	7.480	7.010	7.367
14	6.724	6.831	7.551	7.495		7.189
16	6.657	6.759	7.549	7.513	6.830	7.053
18	6.600	6.698	7.550	7.530		6.945
20	6.552	6.646	7.553	7.548		6.857
22	6.511	6.600	7.558	7.565		6.784
24	6.475	6.561	7.563	7.581	6.577	6.722

remarkably close together. When Coxeter [6] published his conjectured bound for spherical caps—which, as we shall see in Section II, is analogous to (4)—he observed a similar phenomenon in eight dimensions and remarked that this was "a manifestation of the extraordinary near-regularity" of the Gosset lattice E_8. The Leech lattice Λ_{24} is just as extraordinary as Gosset's (see [4], [5], [15], [16], [19], [24]), and the existence of these two lattices explains the small gaps in Table I at 8 and 24 dimensions. Asymptotically, as $n \to \infty$, all three bounds (2), (3), (4) agree, and $G_n \to 1/(2\pi e)$. (The asymptotic form of $f_n(n)$ may be found in [14, p. 676]; see also [21].)

II. The Proposed New Bound

In this section we derive (4) and give our reasons for believing that it is a lower bound for G_n. The following argument has been applied to a number of geometrical problems. In three dimensions it is possible to arrange four equal spheres so that they all touch one another, the centers then being at the vertices of a regular tetrahedron. Since the four spheres cannot move closer together, this is the tightest possible packing of four spheres. Suppose that new spheres are added to this configuration one at a time, so as to form a new tetrahedron at each step. If this process could be continued indefinitely, it is plausible that the resulting configuration would have the highest packing density of any arrangement of spheres, since every tetrahedral configuration would be packed as densely as possible [24]. Unfortunately, however, regular tetrahedra do not fit together perfectly to fill space, and after a while this construction cannot be extended. Nevertheless, Rogers showed in [20] (see also [22]) that the density of this hypothetical arrangement is indeed an upper bound on the density of any three-dimensional packing, and that a similar result holds in any dimension.

In order to give a precise statement of Rogers' bound, let T_n denote the n-dimensional regular simplex (T_2 is a triangle, T_3 is a tetrahedron) having the $n+1$ vertices $(1,0,\cdots,0)$, $(0,1,0,\cdots,0),\cdots,$ $(0,0,\cdots,0,1)$, where there are $n+1$ coordinates. The edge length of this simplex is $\sqrt{2}$. If spheres of radius $1/\sqrt{2}$ are placed at the vertices, they form a set of $n+1$ mutually touching (but nonoverlapping) spheres. Let σ_n denote the ratio of the volume of the part of T_n covered by the spheres to the total volume of T_n. Then Rogers proved that the density Δ of any packing of spheres in $\mathbb{R}^n$ cannot exceed σ_n.

A formula for σ_n can be given in terms of Schläfli's function $f_n(x)$. This function has been studied by several authors (see [6], [14], and their references), and we just mention that it is defined recursively by

$$f_1(x) = 1,$$
$$f_2(x) = \frac{\operatorname{arcsec} x}{\pi},$$
$$f_n(x) = \frac{1}{\pi}\int_{n-1}^{x} \frac{f_{n-2}(x-2)\,dx}{x\sqrt{x^2-1}}.$$

Rogers' bound can then be written

$$\frac{\Delta}{V_n} \le 2^{-3n/2}\sqrt{n+1}\,(n!)^2 f_n(n) = U(n) \quad \text{(say)}, \tag{6}$$

where V_n is the volume of a unit sphere in $\mathbb{R}^n$ (see Leech [14, p. 675]).

Similar reasoning led Coxeter, Few, and Rogers [7] (see also [22, chap. 8]) to give a lower bound on the density of any *covering* of $\mathbb{R}^n$ by overlapping spheres. For this problem, however, instead of spheres of radius $1/\sqrt{2}$, they placed spheres of radius $\{n/(n+1)\}^{1/2}$ at the vertices of T_n, this radius being chosen so that the spheres just cover T_n. Let τ_n denote the ratio of the sum of the volumes of the "sectors" of these spheres lying in T_n to the total volume of T_n. Then Coxeter, Rogers, and Few proved that the density Δ of any covering of $\mathbb{R}^n$ by spheres cannot be less than τ_n. Since

$$\tau_n = \left(\frac{2n}{n+1}\right)^{n/2} \sigma_n, \tag{7}$$

this bound can also be expressed in terms of Schläfli's function.

Coxeter [6] (see also [9]) used essentially the same argument (but replacing T_n by a regular spherical simplex) to propose an upper bound on the density of nonoverlapping spherical caps placed on an n-dimensional sphere. Coxeter was unable to prove this bound, and it was only established 15 years later by Böröczky [2]. Both Rogers' and Coxeter's bounds have since been improved in many cases [12], [17], [19], [23].

Our justification of (4) uses the same reasoning, and its status is that of Coxeter's bound in 1963: it seems very plausible, but we cannot give a proof. (We do not expect that (4) will be easy to prove. Certainly the methods used by Rogers [22] do not seem to be applicable.) We consider quantizers in $\mathbb{R}^n$ with a uniformly distributed input and a large number of output points. We argue as above that an optimal quantizer would be obtained if we could fill space with copies of T_n. This cannot be done, except in one and two dimensions, but nevertheless the mean-squared error of this hypothetical quantizer should be a lower bound on the error of any quantizer, and therefore a lower bound on G_n. We now evaluate the mean-squared error of this hypothetical quantizer and show that it is given by (4).

Each output point y_i of a quantizer belongs to a Voronoi region

$$V(y_i) = \{x \in \mathbb{R}^n : \|x - y_i\| \le \|x - y_j\| \text{ for all } j\}.$$

If the Voronoi regions are all congruent, to some polytope P say, then the mean-squared error per symbol of the quantizer is given by

$$\frac{1}{n}\frac{\int_P \|x - \hat{x}\|^2\,dx}{\left(\int_P dx\right)^{(n+2)/n}}, \tag{8}$$

where $\hat{x}$ is the centroid of P (see [3], [11], [25]). To find the Voronoi region of our quantizer, we first decompose T_n into $(n+1)!$ congruent smaller simplices L by joining the centroid of T_n to the midpoints of all i-dimensional faces, for all $i = 0,1,\cdots,n$. Then $n!$ copies of L meet at each vertex of T_n. Fig. 2 illustrates the case $n = 2$.

As shown in Fig. 2, six copies of T_2 fit together around any vertex. In n dimensions, imagine a small sphere drawn around

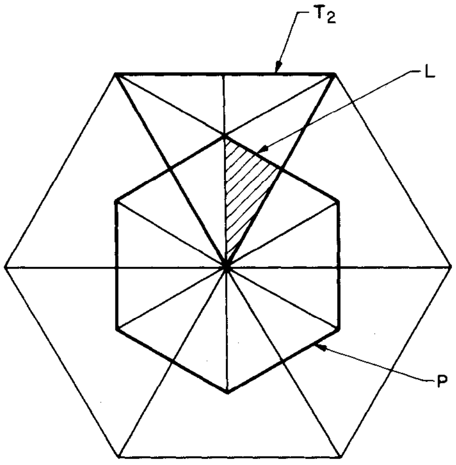

Fig. 2. The two-dimensional case, when six copies of T_2 fit together at a vertex, and the Voronoi region P of that vertex is made up of 12 copies of the small simplex L.

one vertex of T_n, and let κ be the ratio of the area, or $(n-1)$-dimensional content, of this sphere to the area of the intersection with T_n. If our hypothetical quantizer existed, κ would be an integer, and κ copies of T_n would fit together perfectly around a vertex. In two dimensions $\kappa = 6$. Fig. 3 shows a sketch of the case $n = 3$, when $\kappa = 4\pi/(3 \operatorname{arcsec} 3 - \pi) = 22.79\cdots$.

The Voronoi region of the hypothetical quantizer is then the disjoint union of $\kappa \cdot n!$ copies of L arranged around a vertex of T_n (see Fig. 2). The mean-squared error can be evaluated using (8) and the fact that there is an explicit formula for the second moment of an arbitrary simplex L with vertices $Q_0, Q_1, \cdots, Q_n$ about any point [3, (15)]. The second moment about Q_0 (say) is given by

$$\frac{\int_L x \cdot x\,dx}{\int_L dx} = \frac{n+1}{n+2}\|\hat{Q} - Q_0\|^2 + \frac{1}{(n+1)(n+2)}\sum_{i=0}^{n}\|Q_i - Q_0\|^2, \tag{9}$$

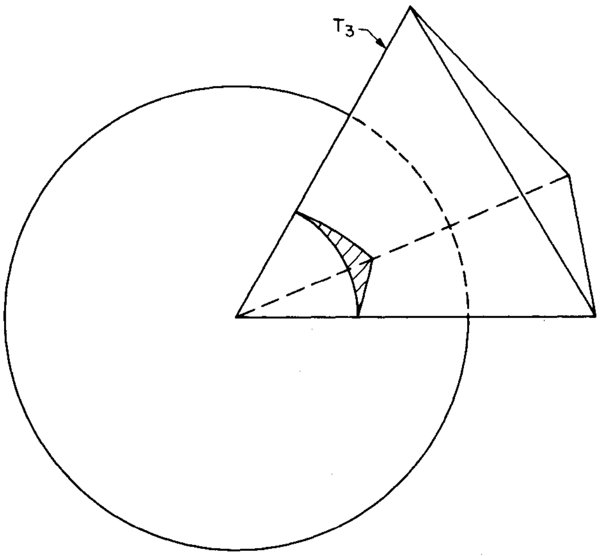

Fig. 3. A sketch of the three-dimensional case, showing the intersection of a regular tetrahedron with a small sphere around one vertex. The shaded area occupies $(1/\kappa)$-th of the surface of the sphere, where $\kappa = 22.79\cdots$.

where $\hat{Q} = (n+1)^{-1}\Sigma Q_i$ is the centroid of L. In our case

$$Q_0 = P_0 = (1,0,0,0,\cdots)$$
$$Q_1 = 2^{-1}(1,1,0,0,\cdots)$$
$$Q_2 = 3^{-1}(1,1,1,0,\cdots)$$
$$\vdots$$
$$Q_n = (n+1)^{-1}(1,1,\cdots,1)$$

so

$$\hat{Q} = (n+1)^{-1}(H_{n+1} - H_0, H_{n+1} - H_1, \cdots, H_{n+1} - H_n),$$

where H_m is defined in (5). From this it can be shown that

$$\|\hat{Q} - Q_0\|^2 = \frac{n+3}{n+1} - \frac{2n+3}{(n+1)^2}H_{n+1}.$$

We omit the details of the algebra. The right-hand side of (9) then simplifies to

$$\frac{n+3-2H_{n+2}}{n+1}. \tag{10}$$

The volumes of T_n, L, and P are, respectively,

$$\frac{\sqrt{n+1}}{n!}, \frac{1}{\sqrt{n+1}(n!)^2}, \frac{\kappa}{\sqrt{n+1}\, n!}, \tag{11}$$

and with the aid of (10) and (11), (8) becomes

$$\frac{n+3-2H_{n+2}}{n(n+1)}(n+1)^{1/n}(n!)^{2/n}\kappa^{-2/n}. \tag{12}$$

It remains to find κ. But this is exactly what the Schläfli functions are designed for, and from [6, (4.1)] we find

$$\kappa = \frac{2^n}{n!f_n(n)}. \tag{13}$$

Equation (4) now follows from (12) and (13).

Leech (see [6, Sec. 10]) has calculated $f_n(n)$ for $n \le 8$, and a more extensive but unpublished table has been obtained by Lang [13]. Fortunately, Rogers' bound $U(n)$ is given in [15] and [16] for $n \le 24$, and our bound (4) can be written in terms of Rogers' as

$$\frac{2(n+3-2H_{n+2})}{n(n+1)}U(n)^{2/n}, \tag{14}$$

using (6). The numerical values of our bound that are given in Table I were computed with the aid of (14).

Finally, we remark that it is clear that the proposed bound can be generalized to any kth power distortion measure. We have restricted attention to the mean-squared error criterion because it is the most popular, and because in this case the integrals can be evaluated explicitly.

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