JOURNAL OF COMBINATORIAL THEORY, Series A **42**, 61–86 (1986)

A New Approach to the Covering Radius of Codes

N. J. A. SLOANE

Mathematical Sciences Research Center,
AT& T Bell Laboratories, Murray Hill, NJ 07974

Communicated by the Managing Editors

Received June 27, 1985

We introduce a new approach which facilitates the calculation of the covering radius of a binary linear code. It is based on determining the *normalized covering radius* ρ. For codes of fixed dimension we give upper and lower bounds on ρ that are reasonably close. As an application, an explicit formula is given for the covering radius of an arbitrary code of dimension $\leqslant 4$. This approach also sheds light on whether or not a code is normal. All codes of dimension $\leqslant 4$ are shown to be normal, and an upper bound is given for the norm of an arbitrary code. This approach also leads to an amusing generalization of the Berlekamp–Gale switching game.

I. INTRODUCTION

The new approach introduced here is aimed at finding the covering radius of a code with repeated coordinates, or in other words, finding the covering radius of codes with fixed dimension k and increasing block length n.

Suppose C is an $[n, k]$ code of known covering radius R, having a generator matrix in which every column is distinct and nonzero. By choosing suitable multiplicities $m_1,..., m_n$ and taking m_i copies of the ith column of C (for $i = 1,..., n$) we obtain a new code C^*, a "blown-up" version of C, of length $n^* = \sum m_i$. The multiplicities may be 0 (or 1), so any code can be obtained in this way.

It is easily shown (see Sect. II) that the covering radius R^* of C^* is at least $\sum [m_i/2]$, so it makes sense to define the *normalized covering radius* of C^* to be

$$\rho = \rho_C(m_1,..., m_n) = R^* - \sum_{i=1}^{n} \left[\frac{m_i}{2}\right].$$

The investigation of this function is the main subject of the paper.

0097-3165/86 $3.00

We show that determining ρ is an integer programming problem (Theorems 1, 3) and give an integer programming bound (Theorem 4). Sections V and VII give a lower bound on ρ (Theorem 5) and several other upper bounds (Theorems 7, 8, 11). In some cases it is possible to determine ρ exactly: if all the multiplicities m_i are 1 then of course $\rho = R$, and if they are all 0 then $\rho = 0$ (Theorem 6). If the dimension k is at most 4 then ρ is given explicitly in Theorems 13, 14, and 16. The Hamming and Golay codes are treated in Corollary 12. The monotonicity property of Theorem 2 is a useful general result.

In order to prove Theorems 14 and 16 we must classify the codes of length $n \leqslant 15$ and dimension $k \leqslant 4$. This is done in Section VIII (see Tables I–III).

The norm of a code and the concept of a normal code were introduced in [10] and are further studied in [5]. The definitions are given in Section VI. Theorem 9 gives a sufficient condition for a code to be normal, which is used in Sections IX and X to show that all codes of dimension $\leqslant 4$ are normal. Theorem 18 gives an upper bound on the norm of any code.

The integer programming approach of Section III leads to an amusing generalization of the Berlekamp–Gale switching game (see Sect. IV).

Section II gives the definition of covering radius and of several other terms, including the important notion of the height of a vector. Further information about the covering radius of codes may be found in [4, 5, 10].

We conclude this introduction with two examples illustrating how the normalized covering radius ρ may be used: (i) Suppose a code C^* is formed from the [23, 12] Golay code by taking m_i copies of the ith column (for $i = 1, \ldots, 23$). Then the covering radius of C^* is between

$$\sum_{i=1}^{23} \left[\frac{m_i}{2}\right] \quad \text{and} \quad \sum_{i=1}^{23} \left[\frac{m_i}{2}\right] + 3$$

TABLE I

$\Phi(n, k)$, the Number of $[n, k]$ Codes with Distinct Nonzero Coordinates (from [13])

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0
2	0	1	1	0	0	0	0	0	0	0	0	0	0	0	0
3	0	0	1	2	1	1	1	0	0	0	0	0	0	0	0
4	0	0	0	1	3	4	5	6	5	4	3	2	1	1	1
5	0	0	0	0	1	4	8	15	29	46	64	89	112	128	144
6	0	0	0	0	0	1	5	14	38	105	273	700	1794	4579	11635

TABLE II

Components Used in Table III

F_1^n	$= [n, n]$ 0 code $\mathbf{F}_2^n$
E_n	$= [n, n-1]$ 1 even weight code
O_n	= empty component of length n
C_5	$= E_2^2 O_1 = [5, 3]$ 1 code (Fig. 4)
C_{6a}	$= [6, 3]$ 2 code of (28)
C_{6b}	$= E_2^3 = [6, 4]$ 1 code (Fig. 4)
C_{6c}	$= E_2 E_3 O_1 = [6, 4]$ 1 code (Fig. 4)
C_{7a}	$= E_2^3 O_1 = [7, 4]$ 2 code (Fig. 4)
C_{7b}	$= E_2^2 O_3 = [7, 4]$ 2 code (Fig. 4)
C_{7c}	$= E_2 O_5 = [7, 4]$ 2 code (Fig. 4)
H_n	$= [n = 2^m - 1, n - m]$ 1 Hamming code
S_k	$= [2^k - 1, k]$ $2^{k-1} - 1$ simplex code

(by Corollary 12). If all the m_i are even the lower bound is attained, while if they are all odd the upper bound is attained. (ii) Consider the [8, 4] extended Hamming code with generator matrix

$$\begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \end{array}$$
$$\begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{bmatrix} \tag{1}$$

and let C^* be obtained by taking the ith column of C with multiplicity m_i, where

$$(m_1, \ldots, m_8) = (4, 0, 1, 3, 1, 3, 5, 0).$$

TABLE III

List of $[n, k]$ Codes

$n = 1$	$k = 1$	F_1
$n = 2$	$k = 2$	F_1^2
$n = 3$	$k = 2$	E_3
	$k = 3$	F_1^3
$n = 4$	$k = 3$	$E_3 F_1, E_4$
	$k = 4$	F_1^4
$n = 5$	$k = 3$	C_5
	$k \geqslant 4$	$E_3 F_1^2, E_4 F_1, E_5, F_1^5$
$n = 6$	$k = 3$	C_{6a}
	$k = 4$	$E_3^2, C_5 F_1, C_{6b}, C_{6c}$
	$k \geqslant 5$	$E_3 F_1^3, E_4 F_1^2, E_5 F_1, E_6, F_1^6$
$n = 7$	$k = 3$	S_3
	$k = 4$	$C_{6a} F_1, C_{7a}, C_{7b}, C_{7c}, H_7$

Thus C^* has generator matrix

$$\begin{bmatrix} 0&0&0&0&0&0&0&0&1&1&1&1&1&1&1&1&1 \\ 0&0&0&0&1&1&1&1&0&0&0&0&1&1&1&1&1 \\ 0&0&0&0&0&1&1&1&0&1&1&1&0&0&0&0&0 \\ 1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1&1 \end{bmatrix}.$$

To find the covering radius of C^*, Theorem 16 instructs us to form the contracted code $\tilde{C}$, by taking one copy of each column of C^* that has odd multiplicity. Thus $\tilde{C}$ has generator matrix

$$\begin{bmatrix} 0&0&1&1&1 \\ 1&1&0&0&1 \\ 0&1&0&1&0 \\ 1&1&1&1&1 \end{bmatrix}$$

or equivalently

$$\begin{bmatrix} 1&1&0&0&0 \\ 1&0&1&0&0 \\ 1&0&0&1&0 \\ 0&0&0&0&1 \end{bmatrix}.$$

This is the [5, 4] code E_4F_1 of Table IV, hence $\rho = 1$. Therefore C^* has covering radius

$$R^* = \sum_{i=1}^{8} \left\lceil \frac{m_i}{2} \right\rceil + \rho = 6 + 1 = 7.$$

II. Covering Radius and Height

Throughout this paper we only consider codes that are binary, linear, and have no coordinate that is identically zero. An $[n, k]$ code C has *covering radius* R if

$$\begin{aligned} R &= \max_{x \in \mathbf{F}_2^n} \min_{c \in C} d(x, c) \\ &= \max_{x \in \mathbf{F}_2^n} \min_{c \in C} wt(x + c), \end{aligned} \tag{2}$$

where d is Hamming distance and wt is Hamming weight (see [4, 10, 12]

TABLE IV

Normalized Covering Radius $\rho = \rho^{(4)}(m_1, ..., m_{15})$. $\bar{C}$ in the Second Half of the Table Is the Complement of $\bar{C}$ in the First Half

s	$\bar{C}$	$\bar{R}$	ρ	s	$\bar{R}$	ρ
0	0	0	0	15	7	7
1	F_1	0	0	14	6	6
2	F_1^2	0	0	13	5	5
3	E_3	1	0	12	4	4 or 5
	F_1^3	0	0		4	4
4	E_3F_1	1	1	11	3	3 or 4
	E_4	1	1		4	4
	F_1^4	0	0		4	4
5	C_5	1	1	10	3	3
	$E_3F_1^2$	1	1		4	4
	E_4F_1	1	1		3	3
	E_5	1	1		4	4
6	C_{6a}	2	2	9	3	3
	E_3^2	2	2		3	3
	C_5F_1	1	1		3	3
	C_{6b}	1	1		3	3
	C_{6c}	1	1		3	3
7	S_3	3	3	8	3	3
	$C_{6a}F_1$	2	2		2	2
	C_{7a}	2	2		3	3
	C_{7b}	2	2		2	2
	C_{7c}	2	2		2	2
	H_4	1	1 or 2		2	2 or 3

for any undefined terms). Then C is said to be an $[n, k]$ R code. For example, the $[n, 1]$ *repetition code* $\{00\cdots 0, 11\cdots 1\}$ has covering radius

$$R = \left[\frac{n}{2}\right]. \tag{3}$$

This is as large as R can be: for any code

$$R \leqslant \left[\frac{n}{2}\right]. \tag{4}$$

(see [4, Theorem 6]).

If C has a generator matrix in which every column is distinct (and nonzero) then we say that C has *distinct coordinates*. This implies $n \leqslant 2^k - 1$. Our approach however is aimed at codes with repeated coordinates. Any such code may be obtained by starting with an $[n, k]$ R code C with distinct coordinates, assigning arbitrary nonnegative multiplicities $m_1, \ldots, m_n$, and taking m_i copies of the ith coordinate (for $i = 1, \ldots, n$). The resulting code C^*, a "blown-up" version of C, is an $[n^*, k^*]$ R^* code (say), where $n^* = \sum m_i$ and $k^* \leqslant k$. The n^* coordinates are divided naturally into n blocks, and we shall partition vectors $x \in \mathbf{F}_2^{n^*}$ as

$$x = (x^{(1)}, \ldots, x^{(n)}), \tag{5}$$

where length $(x^{(i)}) = m_i$. A code word $c \in C$ blows up to a code word $c^* = (c^{(1)}, \ldots, c^{(n)}) \in C^*$, where $c^{(i)} = c_i c_i \cdots c_i$ (m_i times).

There is an obvious lower bound on R^*. For let $x \in \mathbf{F}_2^{n^*}$ contain exactly $[m_i/2]$ 1's in the ith block, i.e., let $wt(x^{(i)}) = [m_i/2]$ for all i. In view of (3), $d(c, C^*) \geqslant \sum [m_i/2]$, and so

$$R^* \geqslant \sum_{i=1}^{n} \left[\frac{m_i}{2}\right]. \tag{6}$$

We now define the *normalized covering radius* of C^* to be

$$\rho = \rho_C(m_1, \ldots, m_n) = R^* - \sum_{i=1}^{n} \left[\frac{m_i}{2}\right]. \tag{7}$$

Then $\rho \geqslant 0$, and, since $C = C^*$ when all $m_i = 1$,

$$\rho_C(1, 1, \ldots, 1) = R. \tag{8}$$

From (3), if C has dimension 1, $\rho = 0$.

The $[n = 2^k - 1, k]$ $R = 2^{k-1} - 1$ *simplex code* S_k plays a particularly important role, since it has as generator matrix a $k \times (2^k - 1)$ matrix whose columns are all direct nonzero binary k-tuples. We arrange these k-tuples in increasing order, so for example the generator matrix for S_3 is

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix}$$
$$\begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{9}$$

(see also (60)). The normalized covering radius of S_k will be denoted by

$$\rho^{(k)}(m_1, \ldots, m_{2^k-1}),$$

where m_i is the number of times the column representing the integer i appears. For example, when $k=1$, Eq. (3) implies

$$\rho^{(1)}(m_1)=0. \tag{10}$$

Since any code C^* is a blown-up version of some S_k, it is possible to express the normalized covering radius of any code in terms of $\rho^{(k)}$. However, the notation introduced in (7) is often more convenient.

For $x=(x^{(1)},..., x^{(n)})$ as above, we define the *height* of $x^{(i)}$ to be

$$ht(x^{(i)})=wt(x^{(i)})-\left[\frac{m_i}{2}\right], \tag{11}$$

the *height vector* of x to be

$$(ht(x^{(1)}),..., ht(x^{(n)}),$$

and the *height* of x itself to be

$$ht(x)=\sum_{i=1}^{n} ht(x^{(i)}). \tag{12}$$

Since $0\leqslant wt(x^{(i)})\leqslant m_i$,

$$-\left[\frac{m_i}{2}\right]\leqslant ht(x^{(i)})\leqslant\left\lceil\frac{m_i}{2}\right\rceil. \tag{13}$$

In particular, if all the m_i are 1, $C^*=C$ has distinct coordinates, and

$$ht(x^{(i)})=0 \quad \text{or} \quad 1 \quad (i=1,..., n). \tag{14}$$

By analogy with the formula

$$d(x, C^*)=\min_{c\in C^*} d(x, c)=\min_{c\in C^*} wt(x+c)$$

for the distance of x from C^*, we call

$$ht(x, C^*)=\min_{c\in C^*} ht(x+c)$$

the *height of x above C^**. Then we have (from (2) and (7)),

$$\begin{aligned}\rho=\rho_C(m_1,..., m_n)&=\max_x \min_{c\in C^*} ht(x+c)\\ &=\max ht(x, C^*).\end{aligned} \tag{15}$$

A vector x such that

$$d(x, C^*) = R^*,$$

i.e.,

$$ht(x, C^*) = \rho$$

is called a *deep hole* in C^* (by analogy with [6]).

III. The Exact Value of ρ as an Integer Programming Problem

The problem of finding ρ (and equivalently R^*) can be phrased as an integer programming problem. To see this, let us examine what happens when a code word $c^* = (c^{(1)},..., c^{(n)}) \in C^*$ is added to a vector $x = (x^{(1)},..., x^{(n)})$ having height vector $(h_1,..., h_n)$ and height $h_1 + \cdots + h_n$. If $c^{(i)} = \mathbf{0}$, $x^{(i)}$ and h_i are unchanged. If $c^{(i)} = \mathbf{1}$, $x^{(i)}$ is complemented, $wt(x^{(i)})$ is changed to $m_i - wt(x^{(i)})$, and so h_i is changed to $-h_i$ if m_i is even, or to $1 - h_i$ if m_i is odd.

Let $\pi_i = \text{parity}\,(m_i) = 0$ if m_i is even, $= 1$ if m_i is odd. Then, by what we have just said, the effect of adding c^* on the height of x is expressed by the formula

$$ht(x + c^*) = ht(x) + \sum_{i=1}^{n} \pi_i c_i - 2 \sum_{i=1}^{n} c_i h_i, \tag{16}$$

where the c_i ($= 0$ or 1) in (16) are regarded as real numbers.

Suppose now that x is a deep hole in C^*. Since C^* is linear, we may assume that $\mathbf{0}$ is a closest code word to x. Therefore adding $c^* \in C^*$ to x must not increase the height of x, i.e.,

$$ht(x + c^*) \geqslant ht(x), \qquad \text{all } c^* \in C^*,$$

or in other words we have, from (16),

$$\sum_{i=1}^{n} c_i h_i \leqslant \tfrac{1}{2} \sum_{i=1}^{n} \pi_i c_i, \qquad \text{all } c \in C. \tag{17}$$

In view of Eq. (15) we have established the following result.

THEOREM 1. *The normalized covering radius $\rho = \rho_C(m_1,..., m_n)$ is given by the solution to the following integer programming problem*:

$$\textit{maximize } h_1 + \cdots + h_n \tag{18}$$

subject to

$$h_i \in Z, \tag{19}$$

$$-\left[\frac{m_i}{2}\right] \leqslant h_i \leqslant \left[\frac{m_i}{2}\right], \tag{20}$$

$$\sum_{i=1}^{n} c_i h_i \leqslant \tfrac{1}{2} \sum_{i=1}^{n} \pi_i c_i \tag{21}$$

for all $c \in C$. *The maximum value of* $h_1 + \cdots + h_n$ *is equal to* ρ.

COROLLARY 2. *The monotonicity theorem. If* $m_i \leqslant m_i'$ *and* $m_i \equiv m_i'$ (mod 2) *for all* i,

$$\rho_C(m_1,..., m_n) \leqslant \rho_C(m_1',..., m_n').$$

Proof. For the conditions (20) on the h_i are weakened, while (21) is unchanged.

For C itself all the m_i are 1, and we have (using (14)):

THEOREM 3. *The covering radius* $R = \rho_C(1, 1,..., 1)$ *of* C *is given by the solution to the above integer programming problem, but with* (20) *replaced by*

$$h_i = 0 \text{ or } 1. \tag{22}$$

The maximum value of $h_i + \cdots + h_n$ *is equal to* R.

Finally, if we drop (20) altogether, we get an upper bound on ρ.

THEOREM 4. *Let* ρ_∞ *be the solution to the problem* (18), (19), *and* (21). *Then* $\rho_C(m_1,..., m_n) \leqslant \rho_\infty$.

As an illustration of Theorem 1 we consider one of the four "wild" codes that arise in Theorem 16. This is a [12, 4] $R = 4$ code C with generator matrix

$$\begin{bmatrix} 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}. \tag{23}$$
$$\begin{matrix} a & b & c & d & e & f & g & h & i & j & k & l \end{matrix}$$

We sketch a proof that if these twelve columns have odd multiplicities m_i, then

$$\rho_C(m_1,..., m_{12}) = 4. \tag{24}$$

Suppose on the contrary that there is a deep hole of height 5, i.e., a height vector $(h_1, \ldots, h_{12}) = (a, b, \ldots, l)$, say, satisfying

$$a+b+\cdots+l=5 \tag{25}$$

and also satisfying the 16 inequalities (21), one for each code word in C, remembering that all $\pi_i = 1$. The inequalities corresponding to the first row of (23) and the sum of the first and fourth rows are

$$b+d+f+h+j+l \leqslant 3,$$

$$a+c+f+h+i+k \leqslant 3,$$

which together with (25) imply

$$-e+f-g+h \leqslant 1.$$

Similarly we obtain

$$\begin{aligned} -1 &\leqslant e-f-g+h \leqslant 1, \\ -1 &\leqslant e-f+g-h \leqslant 1, \\ -1 &\leqslant e+f-g-h \leqslant 1. \end{aligned} \tag{26}$$

From the third and fourth rows of (23) we obtain in the same way

$$1 \leqslant e+f+g+h \leqslant 3 \tag{27}$$

and (26), (27) imply $0 \leqslant e \leqslant 1$. By symmetry, since the group of C is transitive on the coordinates, we have

$$0 \leqslant a, b, \ldots, l \leqslant 1.$$

But since R is known to be 4 for the original code C (by computer), we know that there is no (0, 1)-vector of height 5. Therefore $\rho \leqslant 4$.

In the other direction, one can either verify that $(a, b, \ldots, l) =$ (000001000111) is a feasible solution of height 4, or else use the monotonicity theorem to obtain $\rho \geqslant R = 4$. This completes the proof of (24).

IV. A Generalization of the Berlekamp-Gale Switching Problem

The above integer programming problems give rise to a nice generalization of the Berlekamp–Gale switching game. In the original version of this game (cf. [1, 3, 9]) there is an $l \times m$ rectangular array of lightbulbs, controlled by $l+m$ switches, one for each row and column of the array. When a switch is thrown, all lights in that row or column which are off turn on, and those which are on turn off. For each initial pattern x of lights, let $f(x)$ be the minimal number of lights that are on after throwing the switches in any way. The problem is to determine $\max_x f(x)$, which is precisely the covering radius of a certain $[n = lm, k = l + m - 1]$ product code (see [10, Sect. V]).

We generalize this as follows. For simplicity we suppose that all the multiplicities m_i in C^* are odd. First, instead of n lightbulbs that are on or off, we use n cells labeled $1,..., n$, the ith cell containing an integer h_i. When that cell is switched, h_i changes to $1 - h_i$. Second, instead of switching on the rows and columns of a rectangular array, we may now switch on any subset $\{i_1,..., i_w\}$ of cells such that $i_1,..., i_w$ are the positions of the 1's in some nonzero code word of C.

For any initial state $h^0 = (h_1^0, h_2^0,..., h_n^0)$, let f be the minimal value of $\sum h_i$ after any sequence of switches. The problem is to determine $\max_{h^0} f$, the *value* of the game. If the h_i are restricted by (20), this is clearly equivalent to the integer programming problem of Theorem 1, and the value of the game is $\rho_C(m_1,..., m_n)$. If the h_i are restricted to be 0's and 1's, as in the original Berlekamp–Gale game, the value of the game is $f_C(1, 1,..., 1) = R$ (see Theorem 3). Finally, if the h_i may be any integers, which is the most appealing version, the value of the game is ρ_∞ (see Theorem 4).

For small codes this game provides a convenient and amusing way to calculate ρ or ρ_∞. We illustrate with two examples.

The first example is shown in Fig. 1. Arbitrary integers $h_1^0,..., h_6^0$ are written in the six cells. Any three numbers along a line may be switched (e.g., h_1, h_5, h_6 may be changed to $1 - h_1, 1 - h_5, 1 - h_6$, respectively). The corresponding code C is the [6, 3] $R = 2$ code C_{6a} with generator matrix

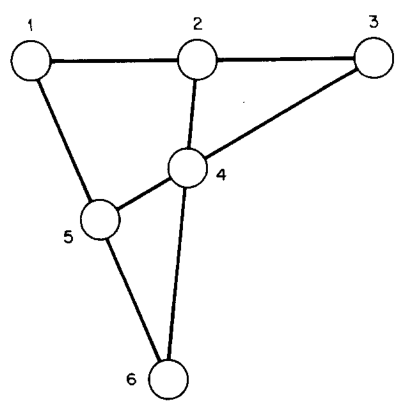

FIG. 1. Swiching game corresponding to [6, 3] code defined in (28).

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}. \tag{28}$$

The four lines in the figure correspond to the three rows of the generator matrix and to their sum. A series of switches along lines are made so as to minimize $h_1 + \cdots + h_6$. The value of the game is

$$\rho_\infty = \max_{h_1^0,\ldots,h_6^0} \min_{\text{switches}} h_1 + \cdots + h_6.$$

At first glance it is not at all obvious that the value is finite, since $h_1^0,\ldots, h_6^0$ may be arbitrarily large. In fact we shall see that $\rho_\infty = 2$. Setting

$$h_1 = h_4 = 1, \qquad h_2 = h_3 = h_5 = h_6 = 0 \tag{29}$$

shows that $\rho_\infty \geqslant 2$.

On the other hand, by switching on 123 (if necessary) we can make $h_3 \leqslant 0$, and by switching on 156 we can make $h_6 \leqslant 0$. Switching on 1245 (the modulo-2 sum of lines 123 and 345) changes $\theta = h_1 + h_2 + h_4 + h_5$ to $4 - \theta$. Since $\min\{0, 4-\theta\} \leqslant 2$ for all integers θ, we can make $h_1 + \cdots + h_6 \leqslant 2$, thus $\rho_\infty \leqslant 2$, and so $\rho_\infty = 2$. Because the deep hole (29) uses only 0's and 1's, we have

$$R = \rho_C(1, 1,\ldots, 1) = \rho_C(m_1,\ldots, m_6) = \rho_\infty = 2,$$

provided the m_i are odd. In other words restricting the h_i to be 0's and 1's does not reduce the value of the game. This is not true in general, as the second example shows.

Figure 2 is the switching game corresponding to the [10, 5] $R = 2$ code with generator matrix

$$\begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \end{matrix}$$
$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \end{bmatrix} \tag{30}$$

given in [10, Eq. (4)]. The six lines in the figure correspond to the rows of (30) and their sum. We invite the reader to verify that

$$\rho_C(1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) = 2,$$

$$\rho_C(3 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1 \quad 1) = \rho_\infty = 3.$$

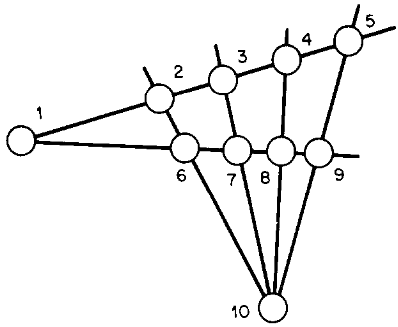

FIG. 2. Switching game corresponding to [10, 5] code defined in (30).

V. A LOWER BOUND ON ρ

Let C^* be obtainted by blowing up C with multiplicities m_i, as above. Let s, the *oddness* of C^*, be the number of odd m_i. Form a $k \times s$ matrix by taking one copy of each column of the generator matrix for C that occurs with odd multiplicity. The *contracted code* $\tilde{C}$ is the code spanned by the rows of this matrix. Let $\tilde{C}$ have parameters $[s, \tilde{k}]$ $\tilde{R}$, where $\tilde{k} \leqslant k$.

THEOREM 5.

$$\rho_C(m_1, ..., m_n) \geqslant \tilde{R}. \tag{31}$$

Proof. The generator matrix for C^* can put in the form $[AB]$, where each column of A occurs an even number of times and B is the $k \times s$ matrix mentioned earlier. Then $R^* \geqslant R_A + \tilde{R}$, where R_A is the covering radius of the code spanned by the rows of A. From (6), $R_A \geqslant \sum [m_i/2]$, and the desired result follows from (7).

THEOREM 6. *If all m_i are even, $\rho_C(m_1, ..., m_n) = 0$.*

Proof. $\tilde{R} = 0$, so $\rho \geqslant 0$ from (31). On the other hand $R^* \leqslant n/2$ from (4), so $\rho \leqslant 0$.

VI. NORMAL CODES

In this section C may or may not have repeated coordinates. The *norm* of C was introduced in [10]. We give here the definition from [5], which is slightly preferable. For $i = 1, ..., n$ and $a = 0, 1$ let $C_a^{(i)}$ denote the subset of code words of C with $c_i = a$, and for an arbitrary $x \in \mathbf{F}_2^n$ let

$$f_a^{(i)}(x) = d(x, C_a^{(i)}).$$

Then

$$N^{(i)} = \max_{x} \{f_0^{(i)}(x) + f_1^{(i)}(x)\} \tag{32}$$

is called the *norm of C with respect to the ith coordinate.* If

$$N^{(i)} \leqslant N \tag{33}$$

for at least one coordinate i, we say that C has *norm* N, and coordinates i for which (33) holds are called *acceptable.* Finally, C is *normal* if it has norm N satisfying

$$N \leqslant 2R + 1. \tag{34}$$

It follows from the definition that if C has norm N, it also has norm $N+1, N+2,...$ (just as a t-design is also an s-design for $s = t-1, t-2,...$). We take N as small as possible.

The importance of normal codes stems from the fact that they can be combined very efficiently using the amalgamated direct sum construction [5, 10]. At the end time of writing it is not known if an abnormal code exists. Various conditions which imply that normal codes are given in [5, 10], and some further conditions are given in Theorem 9, 15, 17, and Corollary 10.

It follows immediately from (32), (33) that, for any code,

$$R \leqslant \left[\frac{N}{2}\right] \quad \text{and} \quad 2R \leqslant N. \tag{35}$$

Many other properties of the norm may be found in [10].

VII. Upper Bounds on ρ

We return to the notation of Sections II–V, and give several upper bounds on $\rho = \rho_C(m_1,..., m_n)$. Even though the exact value of ρ can in principal be found from the integer program in Theorem 1, these upper bounds turn out to be useful, both for theoretical reasons and for calculating ρ in particular cases.

Our main upper bound on ρ (Theorem 7) is obtained by a geometrical approach, and for this it is convenient to take C to be the simplex code S_k. As mentioned in Section II, there is no loss of generality in doing so, since any code C^* of dimension k is a blown-up version of S_k for suitable multiplicities $m_1,..., m_{2^k-1}$. We take the generator matrix for S_k in the canonical form illustrated in (9), and denote the corresponding $\rho = \rho_{S_k}(m_1, \ldots)$ by

$$\rho^{(k)}(m_1, \ldots, m_{2^k-1}).$$

In view of Theorem 6 we may assume that at least one m_i is odd.

An upper bound on the covering radius R^* of C^* may be obtained as follows. We take an arbitrary vector $x \in \mathbf{F}_2^{n^*}$, where $n^* = \sum m_i$, and show that by adding suitable code words of C^* the distance of x from C^* can always be reduced to at most (38). Then (38) is an upper bound on R^*.

We first choose a Q $(1 \leqslant Q \leqslant 2^k - 1)$ called the *pivot*, such that $m_Q \neq 0$, and make $ht(x^{(Q)}) \leqslant 0$ by (if necessary) adding a code word of C^* for which $C^{(Q)} \neq 0$. We now further reduce the height of x by using the subcode of C^* consisting of all code words $(c^{(1)}, \ldots, c^{(n)})$ for which $c^{(Q)} = 0$. Before doing this it is useful to define two further codes.

Let $C_a^{[Q]}$ denote the set of all code words of C^* for which $c^{(Q)} = a, \ldots, a$, with the Qth block of coordinates deleted (for $a = 0, 1$). $C_0^{[Q]}$ is a code of length $n^* - m_Q$ and dimension $k - 1$, and has covering radius $R^{[Q]}$ (say). $C_1^{[Q]}$ is a translate of $C_0^{[Q]}$ and has the same covering radius.

In particular, $C_0^{[Q]}$ is a blown-up version of S_{k-1}, with multiplicities m'_i (say). The m'_i are related to the original multiplicities m_i as follows. The columns of the generator matrix of S_k are all the distinct nonzero k-tuples, and therefore may be identified with the $2^k - 1$ points of the projective geometry $PG(2, k-1)$ of geometric dimension $k - 1$ over $\mathbf{F}_2$. We remind the reader that each line in $PG(2, k-1)$ contains three points. Three points are collinear if and only if they sum to zero.

The original multiplicities $m_P (1 \leqslant P \leqslant 2^k - 1)$ are nonnegative integers assigned to the points $P \in PG(2, k-1)$. When we form the subcode $C_0^{[Q]}$, the m_P are combined in pairs to give the new multiplicities m'_P. Figure 3 illustrates this process for S_3, taking $Q = 4$ to be the pivot. The multiplicities m_P and m_R are combined if and only if QPR is a line. Thus

$$m'_P = m_P + m_R, \qquad \text{for } QPR \text{ a line in } PG(2, k-1). \tag{36}$$

In particular, the oddness of $C_0^{[Q]}$, s' say, is equal to the number of lines QPR for which either m_P is odd and m_R is even, or m_P is is even and m_R is odd.

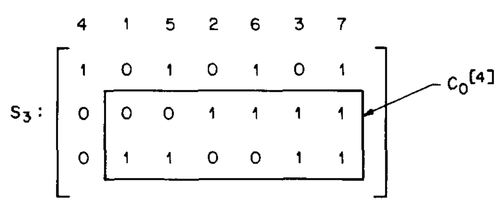

FIG. 3. Illustrating the definition of $C_0^{[Q]}$.

We now return to the problem of reducing the distance from x to C^*. We already have $ht(x^{(Q)}) \leqslant 0$, and now by adding a suitable code word of $C_0^{[Q]}$ we can make

$$ht(x) \leqslant \rho^{(k-1)}(m'_1, \ldots, m'_{2^{k-1}-1}).$$

i.e.,

$$d(x, C^*) \leqslant \left[\frac{m_Q}{2}\right] + \sum_{(P,R)} \left[\frac{m_P + m_R}{2}\right] + \rho^{(k-1)}(m'_1, \ldots, m'_{2^{k-1}-1}),$$

where the sum is over all lines QPR. If u, v are integers,

$$\left[\frac{u+v}{2}\right] = \left[\frac{u}{2}\right] + \left[\frac{v}{2}\right] + \begin{cases} 1 & \text{if } u, v \text{ odd}, \\ 0 & \text{otherwise}. \end{cases} \tag{37}$$

Therefore we can make $d(x, C^*)$ less than or equal to

$$\sum_{i=1}^{2^k-1} \left[\frac{m_i}{2}\right] + \eta + \rho^{(k-1)}(m'_1, \ldots, m'_{2^{k-1}-1}), \tag{38}$$

where

η is the number of lines QPR for which m_P and m_R are odd. (39)

In view of our earlier remark, (38) is an upper bound on R^*, and we have proved

THEOREM 7. *If* $m_Q \neq 0$,

$$\rho^{(k)}(m_1, \ldots, m_{2^k-1}) \leqslant \eta + \rho^{(k-1)}(m'_1, \ldots, m'_{2^{k-1}-1}) \tag{40}$$

where the m'_P *are given by* (36) *and* η *by* (39).

Remarks. Different choices for the pivot Q may give different bounds, so we may replace the right side of (40) by

$$\min_Q \{\eta + \rho^{(k-1)}(m'_1, \ldots, m'_{2^{k-1}-1})\}. \tag{41}$$

It appears best to choose Q so that m_Q is odd. Even so, (40) and (41) may not be tight: there may be no Q for which equality holds in (40). If there is such a Q we call C^* *tame*, otherwise *wild*. Numerous examples will be given in Sections IX and X. As a corollary we give an upper bound on $\rho_C(m_1, \ldots, m_n)$ for any code C.

THEOREM 8. *Suppose exactly s of the numbers m_i are odd. If either $s = 0$ or if the s odd columns of the generator matrix for C form a $PG(2, l-1)$ in $PG(2, k-1)$ for some $l \geqslant 1$ (this requires $s = 2^l - 1$) then*

$$\rho_C(m_1, \ldots, m_n) = \left[\frac{s}{2}\right]; \tag{42}$$

and otherwise

$$\rho_C(m_1,\ldots, m_n) \leqslant \left[\frac{s}{2}\right] - 1. \tag{43}$$

In particular,

$$\rho^{(k)}(m_1,\ldots, m_{2^k-1}) = 2^{k-1} - 1 \tag{44}$$

if all the m_i are odd.

Note that the right-hand sides of (42) and (43) are independent of n and k. The first few values of this bound are

$s=0$	1	2	3	4	5	6	7
$\rho\leqslant 0$	0	0	1	1	1	2	3
$s=8$	9	10	11	12	13	14	15
$\rho\leqslant 3$	3	4	4	5	5	6	7.

Proof. The proof is by induction on $k = \dim C^*$. The result is true for $k=1$ by (10), since then $s=0$ or 1 and in both cases $\rho=0$, in agreement with (42), (43). We now fix k, and temporarily write

$$\rho^{(k)}(s) = \max \rho^{(k)}(m_1,\ldots, m_{2^k-1}),$$

where the maximum is taken over all choices for the m_i with s of them odd. (a) If $s=0$ we have $\rho^{(k)}(0)=0$, by Theorem 6. (b) If $s>0$ and even, say $s=2\sigma$, we take any odd point Q (i.e., a point such that m_Q is odd) as the pivot. If s' is the oddness of $C_0^{[Q]}$, then from (39)

$$1 + s' + 2\eta = 2\sigma. \tag{45}$$

From (40), (45), and the induction hypothesis,

$$\begin{aligned}\rho^{(k)}(s) &\leqslant \eta + \rho^{(k-1)}(s') \\ &\leqslant \sigma - \frac{s'}{2} - \frac{1}{2} + \frac{s'}{2} \\ &= \left[\frac{s}{2}\right] - 1,\end{aligned}$$

as required. (c) Suppose s is odd, $s = 2\sigma + 1$, and the s odd points do not form a $PG(2, l-1)$ for any l. It is easy to show that there is an odd point Q and a line through Q containing just one further odd point. Taking Q as the pivot we have $s' \geqslant 1$ and

$$1 + 2\eta + s' = 2\sigma + 1, \tag{46}$$

so s' is even and ≥ 2. Then (40), (46), and the induction hypothesis imply

$$\rho^{(k)}(s) \leq \left[\frac{s}{2}\right] - 1,$$

as required. (d) Finally, suppose the $s = 2^l - 1$ odd points form a $PG(2, l-1)$ for some $l \geq 1$. Now all the lines through any odd point contain either 0 or 2 further odd points. Thus $\eta = (s-1)/2$, $s' = 0$ and we obtain

$$\rho^{(k)}(s) \leq \left[\frac{s}{2}\right].$$

On the other hand,

$$\rho^{(l)}(1, 1, ..., 1) = 2^{l-1} - 1,$$

since this is the covering radius of the simplex code S_l. By the monotonicity theorem, if all m_i are odd,

$$\rho^{(l)}(m_1, ..., m_{2^l - 1}) \geq 2^{l-1} - 1,$$

and (again by the monotonicity theorem)

$$\rho^{(k)}(2^l - 1) \geq 2^{l-1} - 1 = \left[\frac{s}{2}\right]$$

if the $s = 2^l - 1$ odd points form a $PG(2, l-1)$. This completes the proof of the theorem.

Remark. The weaker result

$$\rho_C(m_1, ..., m_n) \leq [s/2] \tag{47}$$

is an immediate consequence of (4) and (7).

Using the notation of Theorem 7 we can also give a useful sufficient condition for a code to be normal.

THEOREM 9. *If, for some choice of the pivot Q, m_Q is odd, and the covering radius $R^{[Q]}$ of $C_0^{[Q]}$ satisfies*

$$R^{[Q]} - \sum_{P \neq Q} \left[\frac{m_P}{2}\right] \leq \rho^{(k)}(m_1, ..., m_{2^k - 1}) + 1 \tag{48}$$

or equivalently if

$$R^{[Q]} \leq R^* - \left[\frac{m_Q}{2}\right] + 1 \tag{49}$$

then C^* is normal.

Proof. (a) Suppose

$$R^{[Q]} \leqslant R^* - \left[\frac{m_Q}{2}\right]. \tag{50}$$

Consider an arbitrary vector $x = (x^{(1)}, ..., x^{(n)})$, and let $wt(x^{(Q)}) = a$ $(0 \leqslant a \leqslant m_Q)$. Then from (50),

$$f_0^{(Q)}(x) \leqslant a + R^* - \left[\frac{m_Q}{2}\right] \tag{51}$$

and similarly

$$f_1^{(Q)}(x) \leqslant m_Q - a + R^* - \left[\frac{m_Q}{2}\right]. \tag{52}$$

Therefore

$$\begin{aligned} f_0^{(Q)}(x) + f_1^{(Q)}(x) &\leqslant m_Q - 2\left[\frac{m_Q}{2}\right] + 2R^* \\ &\leqslant 2R^* + 1, \end{aligned} \tag{53}$$

so C^* has norm $2R^* + 1$, as required.

(b) Suppose

$$R^{[Q]} = R^* - \left[\frac{m_Q}{2}\right] + 1. \tag{54}$$

Then instead of (51)–(53) we have

$$f_0^{(Q)}(x) \leqslant a + R^* - \left[\frac{m_Q}{2}\right] + 1, \tag{55}$$

$$f_1^{(Q)}(x) \leqslant m_Q - a + R^* - \left[\frac{m_Q}{2}\right] + 1, \tag{56}$$

$$f_0^{(Q)}(x) + f_1^{(Q)}(x) \leqslant 2R^* + 3.$$

Let y be obtained from x by deleting the $x^{(Q)}$ block. Case (i). If both $d(y, C_0^{[Q]}) \leqslant R^{[Q]} - 1$ and $d(y, C_1^{[Q]}) \leqslant R^{[Q]} - 1$ then we can reduce the right-hand sides of (55) and (56) by 1, establishing (53). Case (ii). Suppose $d(y, C_0^{[Q]}) = R^{[Q]}$. Let z be obtained by changing the $x^{(Q)}$ block of x to a vector of weight $[m_Q/2]$. Then

$$\begin{aligned} f_0^{(Q)}(z) &= d(y, C_0^{[Q]}) + \left[\frac{m_Q}{2}\right] \\ &= R^* + 1 \end{aligned}$$

by (54). Therefore $f_1^{(Q)}(z) \leqslant R^*$, i.e.,

$$\left[\frac{m_Q}{2}\right] + 1 + d(y, C_1^{[Q]}) \leqslant R^*,$$

$$d(y, C_1^{[Q]}) \leqslant R^* - \left[\frac{m_Q}{2}\right] - 1,$$

$$f_1^{(Q)}(x) \leqslant m_Q - a + R^* - \left[\frac{m_Q}{2}\right] - 1,$$

which saves 2 over (56), and again proves (53). This completes the proof.

COROLLARY 10. *All tame codes are normal.*

THEOREM 11. *Suppose C has the property that the code words of minimal weight $d_{\min}$ form a t-design with $t \geqslant R$. Then*

$$\rho_C(m_1, ..., m_n) \leqslant \left[\frac{d_{\min}}{2}\right]. \tag{57}$$

Proof. Consider an arbitrary vector $x = (x^{(1)}, ..., x^{(n)})$ with height vector $(h_1, ..., h_n)$. Define $e = (e_1, ..., e_n)$ by $e_i = 1$ if $h_i > 0$, $e_i = 0$ if $h_i \leqslant 0$. There is a code word $c \in C$ with $d(e, c) \leqslant R$. Then it is easy to check that the height vector of $y = x + c^*$ has at most R positive components. Choose a code word b of weight $d_{\min}$ which is 1 at these coordinates, and let $z = y + b^*$. It is now straightforward to check that either $ht(y) \leqslant [d_{\min}/2]$ or $ht(z) \leqslant [d_{\min}/2]$.

COROLLARY 12. *Hamming codes of length $\geqslant 7$, extended Hamming codes of length $\geqslant 8$, the Golay code of length 23, and the extended Golay code of length 24 share the following property. They are $[n, k]$ R codes C such that*

$$0 \leqslant \rho_C(m_1, ..., m_n) \leqslant R, \tag{58}$$

and if all m_i are odd then $\rho_C(m_1, ..., m_n) = R$. The appropriate values of R are 1, 2, 3, *and* 4, *respectively.*

Proof. For a Hamming code we have $R = 1$, $t = 2$, $d_{\min} = 3$, and $\rho \leqslant 1$ from (57). If all m_i are odd then $\rho \geqslant \tilde{R} = R = 1$. Similarly in the other cases.

VIII. CLASSIFICATION OF SMALL CODES

In the following sections we need to know what possible contracted codes $\tilde{C}$ can occur, and so we give here a classification of all codes of

$$\begin{bmatrix} 1&1&0&0&0 \\ 0&0&1&1&0 \\ 0&1&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&0&0&1&1 \\ 0&1&0&1&0&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0 \\ 0&0&1&1&0&0 \\ 0&0&1&0&1&0 \\ 0&1&0&0&1&1 \end{bmatrix}$$

$$C_5 \qquad\qquad C_{6b} \qquad\qquad C_{6c}$$

$$\begin{bmatrix} 1&1&0&0&0&0&0 \\ 0&0&1&1&0&0&0 \\ 0&0&0&0&1&1&0 \\ 0&1&0&1&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 0&0&1&1&0&0&0 \\ 0&1&0&1&1&0&1 \\ 0&0&0&1&0&1&1 \end{bmatrix} \quad \begin{bmatrix} 1&1&0&0&0&0&0 \\ 0&1&0&0&0&1&1 \\ 0&0&1&0&1&0&1 \\ 0&0&0&1&1&1&0 \end{bmatrix}$$

$$C_{7a} \qquad\qquad C_{7b} \qquad\qquad C_{7c}$$

FIG. 4. Generator matrices for certain codes in Table II.

length $n \leqslant 15$ and dimension $k \leqslant 4$. These codes were enumerated by Slepian [13] (see also [2]), but the codes themselves do not seem to have been published before.

Let $\Phi(n, k)$ be the numbers of inequivalent binary linear $[n, k]$ codes with distinct nonzero coordinates, where two codes are equivalent if one can be obtained from the other by a permutation of coordinates. The values of $\Phi(n, k)$ (obtained from [13]) are given in Table I and the corresponding codes in Table III. Just as in the enumeration of self-dual codes given in [7, 8, 11], the codes are described in terms of *components* held together by *glue vectors*. The components used are listed in Table II and Fig. 4.

The codes of dimension 4 and lengths $n = 8, 9, \ldots, 15$ are obtained by deleting from S_4 the columns corresponding to a $[15-n, k']$ code with $k' \leqslant 4$. Similarly in general, for $2^{k-1} \leqslant n \leqslant 2^k - 1$,

$$\Phi(n, k) = \sum_{k'=1}^{k} \Phi(2^k - 1 - n, k). \tag{59}$$

IX. THE COVERING RADIUS OF CODES OF DIMENSION 1, 2, AND 3

For codes of dimension at most 4, the upper bound on ρ given in Theorem 7 agrees with the lower bound of Theorem 5 exepct for four codes of dimension 4.

THEOREM 13. (a) $\rho^{(1)}(m_1) = 0$,

(b) $\rho^{(2)}(m_1, m_2, m_3) = 1$ *if* m_1, m_2, m_3 *odd*, $= 0$ *otherwise*.

Proof. (a) Equation (10). (b) Eqs. (43), (44).

THEOREM 14. *Suppose s of $m_1,\dots, m_7$ are odd. Then $\rho^{(3)}(m_1,\dots, m_7)$ is equal to*

- 0 *if $s = 0, 1, 2$, or $s = 3$ and the three odd points are independent,*
- 1 *if $s = 3$ and the three odd points are dependent, or $s = 4, 5$,*
- 2 *if $s = 6$,*
- 3 *if $s = 7$.*

Proof. In every case the upper bound of Theorem 7 agrees with the lower bound of Theorem 5. Consider, for example, the case of three odd m_i's, when the corresponding points of $PG(2, 2)$ are independent. There is essentially only one way to choose these points, for example, columns 1, 2, and 4 of (9). With 1 as the pivot, the three lines through 1 are

123	(containing two odd points),
145	(containing two odd points),
167	(containing one odd point).

Therefore $\eta = 0$ and $s' = 2$, so from Theorems 7 and 13

$$\rho \leqslant \eta + \rho^{(2)}(m'_1, m'_2, m'_3) = 0.$$

Therefore $\rho = 0$. On the other hand, suppose the three odd points are collinear, say 1, 2, and 3. With 1 as the pivot we find $\eta = 1$, $s' = 0$ and so $\rho \leqslant 1$. To apply Theorem 5 we must find the contracted code $\tilde{C}$, which is spanned by the rows of

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{pmatrix}.$$

Thus $\tilde{C} = E_3$ and $\tilde{R} = 1$. Therefore $\rho = 1$. The other cases are handled in exactly the same way. The contracted codes $\tilde{C}$ will be found in Tables II and III.

Since all these codes are tame, we have

THEOREM 15. *All codes of dimension $\leqslant 3$ are normal.*

X. THE COVERING RADIUS OF CODES OF DIMENSION 4

Consider an arbitrary code C^* of dimension 4, obtained by assigning multiplicities $m_1,\dots, m_{15}$ to the columns of the generator matrix

$$\begin{array}{ccccccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 & 12 & 13 & 14 & 15 \end{array}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix} \tag{60}$$

of the simplex code S_4. If s of the m_i are odd, let $\tilde{C}$ be the corresponding $[s, \tilde{k}]\ \tilde{R}$ contracted code. For $s \geqslant 8$ it is simpler to specify which columns are not in $\tilde{C}$, i.e., to give the *complement* $\bar{\tilde{C}}$. In all except four cases, $\rho = \rho^{(4)}(m_1, ..., m_{15})$ is determined solely by $\tilde{C}$, and is equal to $\tilde{R}$; these codes are tame. The four wild codes are H_4, $\bar{S}_4$, $\overline{E_3 F_1}$, and $\bar{E}_3$. Incidentally $\bar{S}_4$ is the $[8, 4]\ 2$ extended Hamming code with generator matrix (1), and Eq. (23) shows a generator matrix for $\overline{E_3 F_1}$.

THEOREM 16. *$\rho = \rho^{(4)}(m_1, ..., m_{15})$ is as shown in Table IV. The precise value of ρ in the four ambigous cases is determined as follows.*

For $\tilde{C} = H_4$, there is a unique further column R which is the sum of any three distinct columns of H_4. If $m_i = 0$ except for $i \in H_4 \cup \{R\}$ then $\rho = 1$; otherwise $\rho = 2$.

For $\tilde{C} = \bar{S}_4$, if $m_i = 0$ for $i \notin \tilde{C}$ then $\rho = 2$; otherwise $\rho = 3$.

For $\tilde{C} = \overline{E_3 F_1}$, let R be the column corresponding to the omitted F_1. If $m_i = 0$ except for $i \in \tilde{C} \cup \{R\}$ then $\rho = 3$; otherwise $\rho = 4$.

For $\tilde{C} = \bar{E}_3$, if $m_i = 0$ for $i \notin \tilde{C}$ then $\rho = 4$; otherwise $\rho = 5$.

Proof. (i) $\rho \geqslant \tilde{R}$ from Theorem 5. (ii) The upper bound on ρ obtained from Theorem 7 agrees with the lower bound except for the four wild codes $\tilde{C}$, where there is a gap of 1. (iii) Suppose the $s = 7$ odd columns form a Hamming code H_4. The bounds give $1 \leqslant \rho \leqslant 2$. We verify by computer that if the eight remaining m_i are 0 except for a single $m_i = 2$ at a column that is neither in H_4 nor is the special column R (the sum of three distinct columns of H_4) then $\rho = 2$. By the monotonicity theorem $\rho = 2$ if any $m_i > 0$ for $i \notin H_4 \cup \{R\}$. We now use the integer programming method of Theorem 1 to show that if the only nonzero m_i are in $H_4 \cup \{R\}$ then $\rho = 1$. This may be done by hand, as illustrated (for one of the other wild codes) at the end of Section III. This proves the stated result for $\tilde{C} = H_4$. The other wild codes are handled in the same way.

All the codes in Table IV satisfy the hypothesis of Theorem 9, so we have

THEOREM 17. *All codes of dimension 4 are normal.*

XI. Upper Bound on the Norm

In [10] is was shown that any code C^* has norm N^* satisfying

$$N^* \leqslant 4R^* + 2. \tag{61}$$

We now give some bounds which improve on (61) at least in the case of k fixed and n large.

THEOREM 18. *With the notation as above, C^* has norm N^* satisfying*

$$N^* \leqslant 2R^* + s - 2\tilde{R}, \tag{62}$$

and therefore

$$N^* \leqslant 2R^* + s - 2t[s, \tilde{k}] \tag{63}$$

where $\tilde{k} = \dim \tilde{C} \leqslant k$, and $t[s, \tilde{k}]$ is the minimal covering radius of any $[s, \tilde{k}]$ code, with the convention that $t[s, \tilde{k}] = 0$ if $s < \tilde{k}$.

Remarks. The function t is extensively studied in [10]. Of course $s \leqslant 2^k - 1$.

Proof. (63) follows immediately from (62). If all m_i are even then $N^* = 2R^* = n^*$ and (62) is true. So we may assume at least one m_i is odd, say m_Q, and take Q as the pivot. As in part (a) of the proof of Theorem 9 we have, using (38) and (47),

$$f_0^{(Q)}(x) \leqslant a + \sum_{P \neq Q} \left[\frac{m_P}{2}\right] + \eta + \rho^{(k-1)}(m'_1, \dots),$$

$$\leqslant a + \sum_{P \neq Q} \left[\frac{m_P}{2}\right] + \eta + \left[\frac{s'}{2}\right],$$

$$f_1^{(Q)}(x) \leqslant m_Q - a + \sum_{P \neq Q} \left[\frac{m_P}{2}\right] + \eta + \left[\frac{s'}{2}\right],$$

$$f_0^{(Q)}(x) + f_1^{(Q)}(x) \leqslant 2 \sum_{i=1}^{2^k - 1} \left[\frac{m_i}{2}\right] + 2\eta + 1 + 2\left[\frac{s'}{2}\right], \tag{64}$$

and so N^* does not exceed the right-hand side of (64). On the other hand, from Theorem 5,

$$R^* \geqslant \sum_{i=1}^{2^k - 1} \left[\frac{m_i}{2}\right] + \tilde{R},$$

so

$$N^* \leqslant 2R^* - 2\tilde{R} + 2\eta + 1 + 2\left[\frac{s'}{2}\right].$$

But $s = 1 + 2\eta + s'$, so

$$N^* \leqslant 2R^* - 2\tilde{R} + s,$$

as claimed.

By combining Theorem 18 with the known bounds on $t[n, k]$ from [10] we can deduce, for example, that

$$\text{for } k = 5, \qquad N^* \leqslant 2R^* + 6, \tag{65}$$

$$\text{for } k = 6, \qquad N^* \leqslant 2R^* + 9, \tag{66}$$

and for large k,

$$N^* \leqslant 2R^* + \alpha\sqrt{k}\, 2^{k/2}(1 + o(1)) \tag{67}$$

where α is a constant.

Acknowledgments

The author would like to thank Ron Graham, Karen Kilby, and Hans Witsenhausen for numerous helpful discussions.

References

1. J. Beck and J. H. Spencer, Balancing matrices with line shifts, *Combinatorica* **3** (1983), 299–304.
2. J. E. Blackburn, H. H. Crapo, and D. A. Higgs, A catalogue of combinatorial geometries, *Math. Comp.* **27** (1973), 155–166 and microfiche supplement.
3. T. A. Brown and J. H. Spencer, Minimization of ± 1 matrices under line shifts, *Colloq. Math.* **23** (1971), 165–171.
4. G. D. Cohen, M. R. Karpovsky, H. F. Mattson, Jr., and J. R. Schatz, Covering radius: Survey and recent results, *IEEE Trans. Inform. Theory* **IT-31** (1985), 328–343.
5. G. D. Cohen, A. C. Lobstein, and N. J. A. Sloane, Further results on the covering radius of codes, *IEEE Trans. Inform. Theory* **IT-32** (1986), in press.
6. J. H. Conway, R. A. Parker, and N. J. A. Sloane, The covering radius of the Leech lattice, *Proc. Royal Soc. London Ser. A* **380** (1982), 261–290.
7. J. H. Conway and V. Pless, On the enumeration of self-dual codes, *J. Combin. Theory Ser. A* **28** (1980), 26–53.
8. J. H. Conway, V. Pless, and N. J. A. Sloane, Self-dual codes over GF (3) and GF (4) of length not exceeding 16, *IEEE Trans. Inform. Theory* **IT-25** (1979), 312–322.

9. Y. GORDON AND H. S. WITSENHAUSEN, On extensions of the Gale–Berlekamp switching problem and constants of l_p-spaces, *Israel J. Math.* **11** (1972), 216–229.
10. R. L. GRAHAM AND N. J. A. SLOANE, On the covering radius of codes, *IEEE Trans. Inform. Theory* **IT-31** (1985), 385–401.
11. J.S. LEON, V. PLESS, AND N. J. A. SLOANE, Self-dual codes over GF (5), *J. Combin. Theory Ser. A* **32** (1982), 178–194.
12. F. J. MACWILLIAMS AND N. J. A. SLOANE, "The Theory of Error-Correcting Codes," North-Holland, Amsterdam, 1977.
13. D. SLEPIAN, Some further theory of group codes, *Bell Syst. Tech. J.* **39** (1960), 1219-1252.