Unsolved Problems Related to the Covering Radius of Codes*

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0. Introduction

Codes with low covering radius have applications in source coding and data compression (see [6]). Although there has been considerable activity in recent years in studying these codes ([2]-[4], [6], [7], [9], [10], [12], [13]), many open questions remain. The following are some of the most important. Other problems may be found in [2], [6].

1. What is the solution to Berlekamp's light-bulb game?

In the Math. Dept. Commons Room at Bell Labs in Murray Hill there is a light-bulb game built by Elwyn Berlekamp nearly twenty years ago. There are 100 light-bulbs, arranged in a 10×10 array. At the back of the box there are 100 individual switches, one for each bulb. On the front there are 20 switches, one for each row and column. Throwing one of the rear switches changes the state of a single bulb, while throwing one of the front switches changes the state of a whole row or column.

Suppose some subset S of the 100 bulbs are turned on using the rear switches. Let $f(S)$ be the minimum number of illuminated bulbs that can now be attained by throwing any sequence of row and column switches. **The problem** is to determine

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$$R = \max_S f(S)$$

It is known [1] that $32 \le R \le 37$.

The preceding problem is in fact equivalent to finding the covering radius of a certain code. Let C be an $[n, k]$ binary, linear code. The *covering radius* R of C is the maximal distance of any vector $x \in \mathbf{F}_2^n$ from C, i.e.

$$R = \max_{x \in \mathbf{F}_2^n} \min_{c \in C} \operatorname{dist}(x, c) \ . \tag{1}$$

Let us define a *light-bulb* code $L_{a,b}$ to be the $[n = ab, k = a + b - 1]$ linear code spanned by the rows and columns of an $a \times b$ rectangular array. Figure 1 shows some typical codewords of $L_{3,3}$ (which might also be called the tic-tac-toe code). Berlekamp's game asks for the covering radius of $L_{10,10}$. Since there are potentially 2^{100} choices for x in (1), a brute force attack will not succeed!

More generally one may ask for the covering radius of $L_{a,b}$. The Table gives the known bounds on $L_{a,a}$. For large a it is known [1], [6] that

$$\frac{a^2}{2} - \frac{a^{3/2}}{2} + o\,(a^{3/2}) \le R \le \frac{a^2}{2} - \frac{a^{3/2}}{\sqrt{2\pi}} + o\,(a^{3/2}) \ .$$

See also [5], [9].

My reason for giving Berlekamp's game as the first problem is that it appears that light-bulb codes, and codes closely related to them, such as those in Equations (46), (47) of [6], often have unusually low covering radii. It would therefore be valuable to have a better understanding of these codes.

A related question: determine the exact covering radius of the codes obtained by the extended direct sum construction given in (79) and (81) of [6].

2. Is there a code of length 15, dimension 6 and covering radius 3?

Two general questions in this subject are: (i) find the smallest possible covering radius $t[n, k]$ of any $[n, k]$ linear code, and (ii) exhibit explicit codes that attain or come reasonably close to this bound (see [6]). The value of $t[n\ k]$ is known exactly if $k \leq 5$, or if $n \leq 14$, and a table of bounds on $t[n, k]$ for $n \leq 64$ is given in [6]. The first gap occurs when $n = 15$ and $k = 6$. A $[15, 6]$ codes exists with $R = 4$, but the best bound only guarantees that $R \geq 3$. **Problem**: is $t[15, 6] = 3$ or 4?

3. Find an abnormal linear code

The ''amalgamated direct sum'' construction for constructing codes with low covering radius given in [6] works best when applied to *normal* codes (the definition is given below). It seems likely that almost all linear codes are abnormal, although at the time of writing (August 1985) not a single example of an abnormal linear code is known. Every code that has been studied so far has turned out to be normal! **Problem**: find an abnormal linear code, or prove that all linear codes are normal. Abnormal *nonlinear* codes are known to exist (see [7]).

The definition. Let C be an $[n, k]$ code with covering radius R, and let $C_a^{(i)}$ denote the set of codewords $(c_1, \ldots, c_n)\ \varepsilon\ C$ with $c_i = a$ (for $i = 1, \cdots, n$ and $a = 0$ or 1). Then C is *normal* if, for some i,

$$\mathrm{dist}(x, C_0^{(i)}) + \mathrm{dist}(x, C_1^{(i)}) \leq 2R + 1$$

holds for all $x \in \mathbf{F}_2^n$. Many classes of codes are known to be normal, including all codes of minimal distance $d \le 5$, or with dimension $k \le 5$, or with covering radius $R \le 2$, or with length $n \le 14$ (see [3], [7], [13]).

4. What is the covering radius of a first-order Reed-Muller code?

First-order Reed-Muller codes are among the simplest, most elegant, and most important of all codes [8, Chapter 14]. These codes have length $n = 2^m$, dimension $k = m + 1$ and minimal distance 2^{m-1}. For m even Rothaus [12] showed that

$$R = \frac{n}{2} - \frac{\sqrt{n}}{2} .$$

But for m odd it is only known in general that

$$\frac{n}{2} - \sqrt{\frac{n}{2}} \le R < \frac{n}{2} - \frac{\sqrt{n}}{2}$$

(see [2] for references), and for odd $m \ge 15$ that

$$\frac{n}{2} - \frac{27}{32}\sqrt{\frac{n}{2}} \le R < \frac{n}{2} - \frac{\sqrt{n}}{2}$$

(Patterson and Wiedemann [10]). **Problem**. Determine R when m is odd.

This problem can be stated another way: which boolean functions of m arguments are most difficult to approximate by linear functions?

For m even these codes are known to be normal [6]. **Problem**. Show that first-order Reed-Muller codes of length 2^m, m odd, are normal. (This would improve certain asymptotic estimates in [6].)

5. Find the covering radius of cyclic codes of length 63

In searching for codes with low covering radius it was found that one of the cyclic codes of length 31, the [31, 11] five-error-correcting BCH code, has an exceptionally low covering radius, namely $R = 7$ (see the tables in [4], [6]). It is likely that some cyclic codes of greater length will also have low R. **Problem**. Determine the covering radius of cyclic codes of length 33 - 63. (Tables of these codes may be found in [11].)

References

[1] T. A. Brown and J. H. Spencer, *Minimization of* ± 1 *matrices under line shifts*, Colloq. Math. **23** (1971), 165-171.

[2] G. D. Cohen, M. G. Karpovsky, H. F. Mattson, Jr. and J. R. Schatz, *Covering radius - survey and recent results*, IEEE Trans. Information Theory, **IT-31** (1985), 328-343.

[3] G. D. Cohen, A. C. Lobstein and N. J. A. Sloane, *Further results on the covering radius of codes*, IEEE Trans. Information Theory, submitted.

[4] D. E. Downie and N. J. A. Sloane, *The covering radius of cyclic codes of length up to 31*, IEEE Trans. Information Theory, **IT-31** (1985), 446-447.

[5] Y. Gordon and H. S. Witsenhausen, *On extensions of the Gale-Berlekamp switching problem and constants of* l_p*-spaces*, Israel J. Math., **11** (1972), 216-229.

[6] R. L. Graham and N. J. A. Sloane, *On the covering radius of codes*, IEEE Trans. Information Theory, **IT-31** (1985), 385-401.

[7] K. E. Kilby and N. J. A. Sloane, *On the covering radius problem*, in preparation.

[8] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*, North-Holland, Amsterdam, 1977.

[9] H. F. Mattson, Jr., *An improved upper bound on covering radius*, preprint.

[10] N. J. Patterson and D. H. Wiedemann, *The covering radius of the* $(2^{15}, 16)$ *Reed-*

Muller code is at least 16276, IEEE Trans. Information Theory, **IT-29** (1983), 354-356.

[11] W. W. Peterson and E. J. Weldon, Jr., *Error-Correcting Codes*, M.I.T. Press, Cambridge, Mass., 2nd edition, 1972.

[12] O. S. Rothaus, *On ''bent'' functions*, J. Combinatorial Theory, **20A** (1976), 300-305.

[13] N. J. A. Sloane, *A new approach to the covering radius of codes*, J. Combinatorial Theory, to appear.

List of Figure Captions

Figure 1. Some codewords in the light-bulb code $L_{3,3}$.

Table. Covering radius of light-bulb code $L_{a,a}$, from [1], [6] (n = length, k = dimension, R = covering radius, $t[n, k]$ = world record).

a	n	k	R	$t[n, k]$
1	1	1	0	0
2	4	3	1	1
3	9	5	2	2
4	16	7	4	3 or 4
5	25	9	7	5 or 6
6	36	11	?	8-10
7	49	13	$\leq$ 16	12-15
8	64	15	22-23	
9	81	17	$\leq$ 29	
10	100	19	32-37	

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ABSTRACT

Some of the principal unsolved problems related to the covering radius of codes are described. For example, although it is almost twenty years since it was built, Elwyn Berlekamp's light-bulb game is still unsolved. [Added later: this problem has since been solved — see P. C. Fishburn and N. J. A. Sloane, ''The Solution to Berlekamp's Switching Game,'' *Discrete Math*., Vol. 74, 1989, pp. 263–290.]

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$$\boxed{\begin{matrix}1&1&1\\0&0&0\\0&0&0\end{matrix}} \longleftrightarrow (111\quad 000\quad 000)$$

$$\boxed{\begin{matrix}1&0&0\\1&0&0\\1&0&0\end{matrix}} \longleftrightarrow (100\quad 100\quad 100)$$

$$\boxed{\begin{matrix}0&1&1\\1&0&0\\1&0&0\end{matrix}} \longleftrightarrow (011\quad 100\quad 100)$$