ON THE COVERING RADIUS PROBLEM FOR CODES I. BOUNDS ON NORMALIZED COVERING RADIUS*

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Abstract. In this two-part paper we introduce the notion of a stable code and give a new upper bound on the normalized covering radius of a code. The main results are that, for fixed k and large n , the minimal covering radius $t[n, k]$ is realized by a normal code in which all but one of the columns have multiplicity 1; hence $t[n + 2, k] = t[n, k] + 1$ for sufficiently large n . We also show that codes with $n \leq 14$, $k \leq 5$ or $d_{\min} \leq 5$ are normal, and we determine the covering radius of all proper codes of dimension $k \leq 5$. Examples of abnormal nonlinear codes are given. In Part I we investigate the general theory of normalized covering radius, while in Part II [this Journal, 8 (1987), pp. 619–627] we study codes of dimension $k \leq 5$, and normal and abnormal codes.

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1. Introduction. Let C be an $[n, k]$ binary linear code. The covering radius R (also denoted by $CR(C)$) is given by

$$R = \max_{x \in \mathbb{F}^n} \min_{c \in C} d(x, c),$$

where $\mathbb{F} = \{0, 1\}$ and $d(\cdot, \cdot)$ is Hamming distance. Let $t[n, k]$ denote the smallest R for any $[n, k]$ code. Two central problems in this subject are to determine $t[n, k]$ and to construct codes with $R = t[n, k]$ (see [1]–[3], [9], [10] for further background information).

Before describing the new results, we define the normalized covering radius, which as we shall see is easier to work with than the covering radius itself. Let C have generator matrix G . In general, G may contain repeated columns. We assume throughout, however, that no column of G is zero. Let a be the number of distinct columns occurring in G , and let $m_1, \dots, m_a$ be their multiplicities, with $m_1 + \dots + m_a = n$. Then

$$R \geq \sum_{i=1}^a \left\lceil \frac{m_i}{2} \right\rceil,$$

and, following [10], we define the *normalized covering radius* ρ of C to be

$$(1) \quad \rho = R - \sum_{i=1}^a \left\lfloor \frac{m_i}{2} \right\rfloor,$$

a nonnegative integer. Then

$$(2) \quad R = \sum_{i=1}^a \left\lceil \frac{m_i}{2} \right\rceil + \rho = \frac{n}{2} - \frac{\text{no. of odd } m_i}{2} + \rho.$$

Summary of results. A *stable code* (§ 3) has the property that ρ does not increase when any number of pairs of identical columns of any length are adjoined to it. Many small codes are stable (§ 6 of Part II), so this often provides a quick method for determining the covering radius. The *contracted code* $\tilde{C}$ (§ 3) is spanned by the rows of the matrix formed by taking one copy of each column of G that has odd multiplicity, where G is

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a generator matrix for C . If $\tilde{C}$ is stable, $\rho(C) = \rho(\tilde{C})$. As an illustration, consider the code C with generator matrix

$$\begin{bmatrix} 111 & 000 & 00 & 00 & 11 \\ 000 & 111 & 00 & 00 & 11 \\ 000 & 000 & 11 & 00 & 11 \\ 000 & 000 & 00 & 11 & 11 \end{bmatrix}$$

(with multiplicities 3, 3, 2, 2, 2), encountered in the proof of Theorem 27 of [2]. The contracted code $\tilde{C}$ has generator matrix $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, and is stable with $\rho = 0$, so C has covering radius $0 + \sum [m_i/2] = 5$. Stable codes are normal (see Theorem 4).

Theorems 6 and 7 give improved upper bounds on ρ . Section 5 considers how ρ increases when the multiplicities of the columns are increased, subject to the constraint that the *parities* of the multiplicities are unchanged. More precisely, fix an $[n_B, k]$ projective code B (i.e., one with distinct columns), and consider all $[n, k]$ codes C with $\tilde{C} = B$. For sufficiently large n , $\rho_\infty(B) = \max_C \rho(C)$ and $\rho_*(B) = \min_C \rho(C)$ are independent of n . Theorem 8 investigates how rapidly $\rho_\infty(B)$ can be reached. Theorem 9 shows that $\rho_*(B)$ can be realized by a normal code having a very special structure, in which all columns have multiplicity 1 except for one column that has large multiplicity. Furthermore a normal code C has $\rho_*(C) = \rho(C)$ (see Theorem 11).

For fixed k and large n , the minimal covering radius of any $[n, k]$ code is given by

$$t[n, k] = \frac{n}{2} + \min_B \left\{ \rho_*(B) - \frac{n_B}{2} \right\},$$

where B ranges over the projective codes of dimension k or $k - 1$ (equation (52)). It follows (see Theorem 12) that, for large n , $t[n, k]$ can be attained by a normal code having the above-mentioned special structure. This establishes Conjectures A and D of [2] for sufficiently large n . A heuristic justification for the special structure of these codes is given at the end of § 6 of Part II.

Codes of dimension $k \leq 4$ were studied in [10]. We have now determined the covering radius of every projective code of dimension 5. If C is any $[n, 5]$ code, then

$$CR(\tilde{C}) \leq \rho(C) \leq CR(\tilde{C}) + 1$$

(see Theorem 13 of Part II). This implies that all codes of dimension ≤ 5 are normal.

Table 2(a) of Part II gives upper and lower bounds on ρ , and enables one to write down the covering radius of any code C for which the contracted code $\tilde{C}$ has dimension ≤ 5 , with an error of at most 1, when only the length and dimension of $\tilde{C}$ are known. For example, suppose C is a $[3000, 12]$ code for which $\tilde{C}$ is a $[20, 5]$ code. From Table 2(a) of Part II we see that $7 \leq \rho(C) \leq 9$, or in other words (using (2))

$$CR(C) = \frac{3000}{2} - \frac{20}{2} + 8 + \theta = 1498 + \theta,$$

where $\theta = -1, 0$ or 1 .

Section 6 presents a summary of the projective codes of dimension 5 and length $5 \leq n \leq 31$, and gives one or two examples of the best covering codes of each length (see especially Fig. 3 of Part II). This list of codes should be useful, since the investigation of the subject has been hampered by a shortage of examples of good covering codes. The precise determination of ρ for some of these codes requires a separate analysis, as illustrated in Theorem 10, and we have only carried this out in certain cases. At the end of § 6 of Part II we construct an infinite family of (normal) codes with unacceptable coordinates.

Sections 7 and 8 of Part II show that if $d_{\min} \leq 5$ or if $\tilde{C}$ has $d_{\min} \leq 2$ then C is normal. Theorem 18 of Part II summarizes the known conditions on the parameters of a code that imply normality. Finally, § 9 of Part II gives Peter Frankl's construction of an abnormal nonlinear code.

Definitions and notation. The minimal distance of a code is denoted by $d_{\min}$, and the order of its automorphism group by g . We use $\mathbf{F}$ to denote both the Galois field $GF(2)$ and the code $\{0, 1\}$. The empty code of length zero will be denoted by 0, and E_n denotes the $[n, n-1]R = 1$ even weight code. The $[n, 1]R = \lfloor \frac{1}{2}n \rfloor$ repetition code T_n contains 0^n and 1^n . The $[n = 2^k - 1, k]R = 2^{k-1}$ simplex code S_k is defined by a generator matrix in which the columns comprise all distinct nonzero binary k -tuples (see (14) and Fig. 3 of Part II). In particular, $S_0 = 0$, $S_1 = \mathbf{F}$, $S_2 = E_3$. The $2^k - 1$ columns of a generator matrix for S_k , for $k \geq 3$, may be regarded as representing the points of a projective geometry $PG(2, k-1)$ of dimension $k-1$ over $\mathbf{F}$. In such a geometry every line contains exactly three points; three points are collinear if and only if the corresponding vectors sum to zero. We shall occasionally use this geometrical language even when k is less than 3.

Normal codes. Let C be a linear or nonlinear code of length n and covering radius R . For $i = 1, \dots, n$ and $a = 0, 1$ let $C_a^{(i)}$ denote the subset of codewords $(c_1, \dots, c_n)$ of C with $c_i = a$, and for an arbitrary $x \in \mathbf{F}^n$ let

$$f_a^{(i)}(x) = d(x, C_a^{(i)}),$$

if $C_a^{(i)}$ is nonempty, and let $f_a^{(i)}(x) = n$ otherwise. Then

$$N^{(i)} = \max_x \{f_0^{(i)}(x) + f_1^{(i)}(x)\}$$

is called the *norm of C with respect to the i th coordinate*. If

$$(3) \quad N^{(i)} \leq N$$

for at least one coordinate i , we say that C has *norm*¹ N , and coordinates i for which (3) holds are called *acceptable*, the other coordinates being *unacceptable*. Finally, C is *normal* if it has norm N satisfying

$$(4) \quad N \leq 2R + 1,$$

and is otherwise *abnormal*. It follows from the definition that if C has norm N , it also has norm $N+1$, $N+2$, $\dots$. We take N as small as possible. For any code,

$$(5) \quad 2R \leq N.$$

Many other properties of the norm will be found in [3].

2. Normalized covering radius ρ . Let C be an $[n, k]R$ code (assumed throughout to have no coordinate position that is identically zero). Then [1, Thm. 6]

$$(6) \quad R \leq \left\lceil \frac{n}{2} \right\rceil.$$

The normalized covering radius $\rho(C)$, defined in (1), satisfies (see [10]) $\rho(C) \geq 0$; if all m_i are even, $\rho(C) = 0$; if all m_i are 1, $\rho(C) = R$. (Note that $\rho(C)$ does not depend on the

¹ The meanings of three terms in this paper differ from those used in certain earlier works, as will be indicated in footnotes. The new definitions seem preferable. This definition of norm is the one used in [2], [10] and the manuscript of [3]. The definition given in the published version of [3] turns out to be less satisfactory.

choice of a generator matrix.) By reordering the coordinates (if necessary) we may assume that the first m_1 columns of C are identical, then the next m_2 columns, and so on. We partition the codewords $c \in C$ as

$$(7) \quad c = (c^{(1)}, c^{(2)}, \dots, c^{(a)}), \quad c^{(i)} = (c_i, c_i, \dots, c_i)$$

where length $(c^{(i)}) = m_i$. Correspondingly, we partition an arbitrary vector $x \in \mathbb{F}_2^n$ as

$$(8) \quad x = (x^{(1)}, \dots, x^{(a)}),$$

where length $(x^{(i)}) = m_i$. The *height* of $x^{(i)}$ is defined to be

$$(9) \quad h_i = ht(x^{(i)}) = wt(x^{(i)}) - \left\lfloor \frac{m_i}{2} \right\rfloor,$$

and the *height* of x is

$$(10) \quad ht(x) = \sum_{i=1}^a h_i.$$

Then we have ([10, (15)])

$$(11) \quad \rho(C) = \max_x \min_{c \in C} ht(x + c).$$

A vector x such that $\min_{c \in C} ht(x + c) = \rho(C)$, or equivalently $d(x, C) = R$, is called a *deep hole* in C . It is shown in Theorem 1 of [10] that, using (11), we can express the problem of finding $\rho(C)$ as an integer programming problem.

The following result will turn out to be very useful.

THEOREM 1. Suppose a code C is the row space of a matrix of the form $\begin{bmatrix} G_1 & G_2 \\ 0 & 0 \end{bmatrix}$, where G_1 and G_2 have no columns of 0's, and let codes A and B be the row spaces of G_1 and G_2 , respectively. (a) If $G_3 = 0$, so that C is a direct sum $C = A \oplus B$, then

$$(12) \quad \rho(C) = \rho(A) + \rho(B),$$

and if either A or B is normal so is C . (b) If all columns in $\begin{bmatrix} G_2 \\ 0 \end{bmatrix}$ occur an even number of times, then $\rho(C) = \rho(A)$, and if A is normal so is C .

Proof. (a) Clearly

$$(13) \quad CR(C) \leq CR(A) + CR(B)$$

(cf. [7]), with equality if $G_3 = 0$, which implies (12). Suppose $G_3 = 0$, A is normal and coordinate r is acceptable. Let A, B, C have lengths n_A, n_B, n_C , respectively. For an arbitrary $x = |y|z| \in \mathbb{F}^{n_C}$, $y \in \mathbb{F}^{n_A}$, $z \in \mathbb{F}^{n_B}$ we have, in the notation of § 1,

$$d(x, C_a^{(r)}) = d(y, A_a^{(r)}) + d(z, B), \quad a = 0, 1,$$

$$\begin{aligned} \text{Norm}(C) &= \text{Norm}(A) + 2CR(B) \\ &\leq 2CR(A) + 1 + 2CR(B) \\ &= 2CR(C) + 1, \end{aligned}$$

and C is normal. (b) Now $CR(C) = CR(A) + \frac{1}{2}n_B$, so $\rho(C) = \rho(A)$. If A is normal, then $d(x, C_a^{(r)}) \leq d(y, A_a^{(r)}) + \frac{1}{2}n_B$, and again $\text{Norm}(C) \leq 2CR(C) + 1$.

3. The effect on ρ of varying the multiplicities; stable codes. We now investigate how $\rho(C)$ changes as the multiplicities m_i of the columns of C vary. Any code C of dimension $\leq k$ can be obtained by assigning suitable multiplicities to the columns of the simplex code S_k . Let us arrange the columns of S_k in some fixed order, for example, the

binary order illustrated in (14) for S_3 (see also Fig. 3 of Part II):

$$(14) \quad S_3: \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

We choose a generator matrix G for C , and let $m_i \geq 0$ be the number of times the i th column of S_k appears in G , for $i = 1, \dots, 2^k - 1$. Then we let

$$\rho^{(k)}(m_1, m_2, \dots, m_{2^k-1})$$

equal the normalized covering radius $\rho(C)$. Choosing a different generator matrix for C permutes the m_i 's but does not change the value of $\rho^{(k)}(m_1, \dots, m_{2^k-1})$. (However most permutations of the arguments do change the value of $\rho^{(k)}(m_1, \dots, m_{2^k-1})$.)

It turns out that the function $\rho^{(k)}$ is best studied by allowing the m_i to vary while fixing their *parity*, or in other words by investigating how $\rho^{(k)}$ changes when pairs of identical columns are added to or deleted from the generator matrix G . It is an elementary fact that when two identical columns are adjoined to G (columns which may or may not already be present in G), the covering radius of C increases by either 1 or 2, so the normalized covering radius is either unchanged or increases by 1. This establishes the *monotonicity property* [10, Thm. 2]: if $m_i \leq m'_i$ and $m_i \equiv m'_i \pmod{2}$ for all i , then

$$(15) \quad \rho^{(k)}(m_1, \dots, m_{2^k-1}) \leq \rho^{(k)}(m'_1, \dots, m'_{2^k-1}).$$

The earliest codes to be considered are therefore the projective codes. Given an arbitrary code C , with parameters $[n, k]R$ and generator matrix G , the corresponding *contracted code*² $\tilde{C}$ is the projective code which is the row space of the matrix formed by taking one copy of each column of G that has odd multiplicity. $\tilde{C}$ is independent of the choice of G . (If all m_i are even we set $\tilde{C}$ equal to the empty code 0.) We denote the parameters of $\tilde{C}$ by $[\tilde{n}, \tilde{k}]\tilde{R}$, so $\tilde{n} \leq n$, $\tilde{k} \leq k$, $\tilde{R} \leq R$ and $\rho(\tilde{C}) \leq \rho(C)$.

We say that two codes C, D are *congruent* (written $C \equiv D$) if $\tilde{C} = \tilde{D}$. If the multiplicities of the columns in C do not exceed the multiplicities of the same columns in D (but with no constraints on their parities), we write $C \leq D$. For example, the contracted code $\tilde{C}$ satisfies

$$(16) \quad \tilde{C} \equiv C \quad \text{and} \quad \tilde{C} \leq C.$$

The monotonicity property (15) states that

$$(17) \quad C \equiv D, \quad C \leq D \quad \Rightarrow \quad \rho(C) \leq \rho(D).$$

We shall need the following corollary to Theorem 19 of [3].

THEOREM 2. *Let C be normal and suppose the r th coordinate is acceptable. Let D be formed by adjoining $2m$ copies of the r th coordinate to C . Then $CR(D) = CR(C) + m$, $\text{Norm}(D) = \text{Norm}(C) + 2m$, D is normal, and any copy of the r th coordinate is an acceptable coordinate for D .*

Proof. D is an amalgamated direct sum [3] of C with the repetition code T_{2m+1} . By the remarks made earlier in this section, $CR(D) \geq CR(C) + m$. From Theorem 19(ii) of [3], $CR(D) = CR(C) + m$. Then D is normal by Theorem 19(iii) of [3].

The determination of the covering radius of codes of low dimension is greatly facilitated by the observation that for many of these codes ρ does not increase when pairs of identical columns are adjoined to the generator matrix. We call C *stable* if it has this

² A different definition of contracted code was used in [2].

property, or more precisely if

$$(18) \quad C \equiv D, \quad C \leq D \Rightarrow \rho(C) = \rho(D).$$

We shall see, for example, that all codes of dimension $k \leq 3$ are stable.

In view of Theorem 1(b), adjoining pairs of identical columns from outside the subspace of $PG(2, k-1)$ spanned by the columns of C has no effect on ρ , so such columns can be ignored when investigating the stability of C .

THEOREM 3. *Let an $[n, k]$ code C be the row space of a matrix G , and let Π be the subspace of $PG(2, k-1)$ spanned by the columns of G . Then C is stable if and only if $\rho(C)$ does not increase when any number of pairs of identical columns representing points in Π are adjoined to G .*

Examples. If all the multiplicities m_i are even, then $\rho = 0$. But $\tilde{C}$ is the empty code 0, with $\rho = 0$. We deduce that 0 is stable. (This can also be deduced directly from the theorem.)

If C has dimension 1 then (since no coordinate may be identically zero) $C = T_n$, $R = [n/2]$, $\rho = 0$. But $\tilde{T}_n = 0$ if n is even, $\tilde{T}_n = F$ if n is odd, both having $\rho = 0$. We deduce that F is stable. In fact all the codes F^n and E_n are stable (see the examples preceding Theorem 6).

Any code C of dimension 2 has a generator matrix containing (say) a columns $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$, b columns $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and c columns $\begin{pmatrix} 1 \\ 1 \end{pmatrix}$. As stated on page 388 of [3], the covering radius of C is given by

$$(19) \quad \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{c}{2} \right\rfloor + 1 \quad \text{if } a, b, c \text{ are odd,}$$

$$(20) \quad \left\lfloor \frac{a}{2} \right\rfloor + \left\lfloor \frac{b}{2} \right\rfloor + \left\lfloor \frac{c}{2} \right\rfloor \quad \text{otherwise.}$$

This now has a very short proof. We calculate $\tilde{C}$, which is

$$\begin{aligned} &0, \text{ with } \rho = 0, \quad \text{if } a, b, c \text{ are even,} \\ &F, \text{ with } \rho = 0, \quad \text{if one of } a, b, c \text{ is odd,} \\ &F^2, \text{ with } \rho = 0, \quad \text{if two of } a, b, c \text{ are odd,} \\ &E_3, \text{ with } \rho = 1, \quad \text{if } a, b, c \text{ are odd.} \end{aligned}$$

All four codes are stable, and (19), (20) follow immediately.

THEOREM 4. *A stable code is normal, and all coordinates are acceptable.*

LEMMA 5. *Let C be any $[n, k]$ code such that for some i ($1 \leq i \leq n$), and all $l = 0, 1, 2, \dots$, adjoining $2l$ copies of column i to C does not increase the normalized covering radius ρ . Then C is normal and coordinate i is acceptable.*

Proof. Suppose coordinate i is unacceptable. Therefore there is a vector x such that $f_0^{(i)}(x) + f_1^{(i)}(x) \geq 2R + 2$. Without loss of generality $f_0^{(i)}(x) \leq f_1^{(i)}(x)$, say $f_0^{(i)}(x) = R - \theta$, $f_1^{(i)}(x) \geq R + \theta + 2$, where $0 \leq \theta \leq R$. We construct D by adjoining $2R + 2$ copies of column i to C . Then $\rho(D) = \rho(C)$, so

$$(21) \quad CR(D) = CR(C) + R + 1 = 2R + 1.$$

Let $x^* = |x|u$, where u is a vector of length $2R + 2$ and weight $w = R + \theta + 2$. Then for D ,

$$\begin{aligned} f_0^{(i)}(x^*) &= f_0^{(i)}(x) + w = 2R + 2, \\ f_1^{(i)}(x^*) &= f_1^{(i)}(x) + 2R + 2 - w \geq 2R + 2, \end{aligned}$$

which contradicts (21).

Theorem 4 now follows because a stable code satisfies the hypothesis of the lemma for every i .

4. Upper bounds on ρ . Suppose C is an $[n, k]R$ code, and the contracted code $\tilde{C}$ is an $[\tilde{n}, \tilde{k}]R$ code. It follows immediately from (1), (6) that

$$(22) \quad \rho(C) \leq \left\lceil \frac{\tilde{n}}{2} \right\rceil.$$

A stronger upper bound was given in Theorem 8 of [10]. The main goal of the present section is to prove Theorem 7, which strengthens both of these results. We first quote (from Corollary 6 of [7]) the result that the *maximal* covering radius of any $[n, k]$ code is given by

$$(23) \quad T[n, k] = \begin{cases} \left\lceil \frac{n}{2} \right\rceil & \text{for } 1 \leq k \leq \left\lceil \frac{n}{2} \right\rceil, \\ n - k & \text{for } \left\lceil \frac{n}{2} \right\rceil \leq k \leq n \end{cases}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

The method of *pivoting*, introduced in § VII of [10], is a useful technique for getting upper bounds on $\rho(C)$ (which are often tight), and leads to Theorem 6. Consider C to be formed from the simplex code S_k with appropriate multiplicities m_i , with length $n = \sum m_i$. We partition vectors of F^n into blocks as in (7), (8). We choose a coordinate Q ($1 \leq Q \leq 2^k - 1$), called the *pivot*, such that $m_Q \neq 0$. For an arbitrary vector x we first make $ht(x^{(Q)}) \leq 0$ by (if necessary) adding a codeword $c = (c^{(1)}, \dots, c^{(2^k-1)}) \in C$ for which $c^{(Q)} \neq 0$.

Let $C_a^{[Q]}$ denote the set of all codewords of C for which $c^{(Q)} = a$, with the Q th block of coordinates deleted, for $a = 0, 1$. $C_0^{[Q]}$ is a code of length $n - m_Q$ and dimension $k - 1$. $C_1^{[Q]}$ is a translate of $C_0^{[Q]}$ and has the same covering radius. In particular, $C_0^{[Q]}$ is obtained by assigning multiplicities m'_P (say) to S_{k-1} . The m'_P are related to the original multiplicities m_P as follows. The m_P ($1 \leq P \leq 2^k - 1$) are nonnegative integers assigned to the points $P \in PG(2, k - 1)$. When we form the subcode $C_0^{[Q]}$, the m_P are combined in pairs to give the new multiplicities m'_P . The multiplicities m_R and m_S are combined if and only if QRS is a line in $PG(2, k - 1)$. Thus

$$(25) \quad m'_P = m_R + m_S \quad \text{for } QRS \text{ a line in } PG(2, k - 1).$$

In particular, the number of distinct columns in $C_0^{[Q]}$ with odd multiplicity, ν say, is equal to the number of lines QTU for which one of m_T and m_U is odd and the other even.

We return to the problem of reducing the distance from x to C . By adding a suitable codeword of $C_0^{[Q]}$ we can make $ht(x) \leq \rho^{(k-1)}(m'_1, \dots, m'_{2^{k-1}-1})$. This leads to the *pivoting bound* [10, Thm. 7]: if $m_Q \neq 0$,

$$(26) \quad \rho(C) \leq \eta + \rho(C_0^{[Q]}),$$

i.e.,

$$(27) \quad \rho^{(k)}(m_1, \dots, m_{2^k-1}) \leq \eta + \rho^{(k-1)}(m'_1, \dots, m'_{2^{k-1}-1}),$$

where η is the number of lines QRS for which m_R and m_S are odd, and the m'_P are given

by (25). In the other direction, we have (from (17))

$$(28) \quad \rho(\tilde{C}) \leq \rho(C).$$

It is often convenient to be able to refer directly to the code obtained by contracting $C_0^{[Q]}$, which we shall denote by $\hat{C}$. In particular, if $\hat{C}$ is stable, we have

$$(29) \quad \rho(C) \leq \eta + \rho(\hat{C}).$$

Remarks. (1) Different choices for the pivot Q may give different bounds, so we can replace the right side of (27) by

$$(30) \quad \min_Q \{ \eta + \rho^{(k-1)}(m'_1, \dots, m'_{2^{k-1}-1}) \}.$$

It appears best to choose Q so that m_Q is odd. Even so, (30) may not be tight: there may be no Q for which equality holds in (27). If there is such a Q we call C and $\tilde{C}$ *tame*, otherwise *wild*. (2) It was shown in [10] (see especially Theorem 9) that if $\dim C = \dim \tilde{C}$ and the bounds (26) and (28) differ by at most 1, i.e., if, for some choice of the pivot Q , either $\rho(\tilde{C}) = \eta + \rho(C_0^{[Q]})$, or $\rho(\tilde{C}) = \eta + \rho(C_0^{[Q]}) - 1$, then C is normal. (It is easy to show in fact that the hypothesis $\dim C = \dim \tilde{C}$ is unnecessary.)

Examples. Suppose C is an $[n, k]$ code such that $\tilde{C} = \mathbb{F}^k$. We show by induction on k that $\rho(C) = 0$. Without loss of generality C has a generator matrix $[IG_1]$ where I is an identity matrix and all the columns of G_1 have even multiplicity. By pivoting on the first coordinate, we obtain $\rho(C) = 0$. Therefore $\mathbb{F}^k$ is stable for all k , and $\rho(F^k) = 0$. A similar argument shows that E_k is stable and $\rho(E_2) = 0$, $\rho(E_k) = 1$ for $k \geq 3$.

THEOREM 6. Let C be an $[n, k]$ code and let the contracted code $\tilde{C}$ be an $[\tilde{n}, \tilde{k}]$ code. Then

$$(31) \quad \rho(C) \leq T[\tilde{n}, \tilde{k}].$$

Proof. The proof is by induction on k , the result being immediate for $k = 0$ and 1. Let C have multiplicities m_P , and choose Q so that m_Q is odd (if all m_P are even, $\rho(C) = 0$). Then $C_0^{[Q]}$ has dimension $k - 1$. Let $\hat{C}$, the contracted code of $C_0^{[Q]}$, have parameters $[\nu, \kappa]$. Without loss of generality we may assume the generator matrix for C has the form shown in Fig. 1. The parities of the multiplicities $m_Q, m_R, m_S, \dots$ are indicated below the matrix. QRS is a typical line with m_R, m_S odd; QTU is a typical line with m_T odd and m_U even; and QVW is a typical line with m_V, m_W even. There are η such pairs R, S , and ν such pairs T, U . x, y, z denote column vectors of length $k - 1$.

Then $\tilde{k}$ is the rank of the subspace of $\mathbb{F}^k$ spanned by the columns of types Q, R, S and T , while κ is the rank of the subspace of $\mathbb{F}^{k-1}$ spanned by the vectors y . The difference in rank $\Delta = \tilde{k} - \kappa$ therefore satisfies $1 \leq \Delta \leq 1 + \eta$. By the pivoting bound, $\rho(C) \leq \eta + \rho(C_0^{[Q]})$, and by the induction hypothesis $\rho(C) \leq \eta + T[\nu, \kappa]$. The desired result will

	Q	R	S	T	U	V	W
	I	O	I	O	I	O	I
C:	O	x	x	y	y	z	z
	O						
	⋮						
	⋮						
	O						
	ODD	ODD	ODD	ODD	EVEN	EVEN	EVEN

$C_0^{[Q]}$ →

FIG. 1. Pivoting (see proof of Theorem 6).

follow if we show that

$$(32) \quad \eta + T[\nu, \kappa] \leq T[\tilde{n}, \tilde{k}].$$

We also know (by counting the odd columns in Fig. 1) that $\tilde{n} = 1 + 2\eta + \nu$. By (23), (24) we can find a code A with parameters $[2\eta + 1, \Delta]R = \eta$ (using (23) if $1 \leq \Delta \leq \eta$, and (24) if $\Delta = \eta + 1$). Suppose B is a code with parameters $[\nu, \kappa]$ and covering radius $R = T[\nu, \kappa]$. Then $A \oplus B$ has length $1 + 2\eta + \nu = \tilde{n}$, dimension $\kappa + \Delta = \tilde{k}$, and covering radius $\eta + T[\nu, \kappa]$. This implies (32) and completes the proof.

By combining Theorem 8 of [10] with Theorem 6, we obtain our final bound.

THEOREM 7. *Given a code C , let the contracted code $\tilde{C}$ have parameters $[\tilde{n}, \tilde{k}]$. Then*

- (a) $\rho(C) \leq [\tilde{n}/2] - 1$ if $\tilde{k} \leq [\tilde{n}/2]$ and $\tilde{C}$ is not a simplex code;
- (b) $\rho(C) = [\tilde{n}/2]$ if $\tilde{k} \leq [\tilde{n}/2]$ and $\tilde{C}$ is a simplex code (this requires $\tilde{n} = 2^i - 1$ for some i);
- (c) $\rho(C) \leq \tilde{n} - \tilde{k}$ if $\tilde{k} \geq [\tilde{n}/2]$.

Remark. We conjecture, but have not been able to prove, that Theorem 6 can be replaced by $\rho(C) \leq T^*[\tilde{n}, \tilde{k}]$, where $T^*[\tilde{n}, \tilde{k}]$ is the largest covering radius of any projective $[\tilde{n}, \tilde{k}]$ code.

5. Behavior of ρ as $n \rightarrow \infty$; structure of best covering codes. In this section we investigate the behavior of ρ as the length $n \rightarrow \infty$ (while the dimension k is held fixed). There are only a finite number of projective codes of a given dimension k (for their length cannot exceed $2^k - 1$). Previously we began with an arbitrary code C and considered the contracted code $\tilde{C}$. Now we reverse the process and begin with a projective code B , with parameters $[n_B, k]$, and consider an arbitrary $[n, k]R$ code C , with multiplicities m_i say, for which $\tilde{C} = B$.

Note that R and $\rho(C)$ are related via (2):

$$(33) \quad \rho(C) = R - \frac{n}{2} + \frac{n_B}{2}.$$

Therefore, if n_B and n are fixed, by minimizing $\rho(C)$ we minimize R .

If $\tilde{C} = B$ and B is stable (§ 3), then $\rho(C) = \rho(B)$; in general, however, $\rho(C) > \rho(B)$. We define³

$$(34) \quad \rho_\infty(B) = \max \{ \rho(C) : C \equiv B, C > B \}.$$

Note that in view of Theorem 1(b), adjoining pairs of identical columns from outside the subspace of $PG(2, k-1)$ has no effect on ρ , and so there is no loss of generality in (34) in assuming that $\dim C = \dim B$. We also note that by (22), $\rho_\infty(B)$ is finite:

$$(35) \quad \rho_\infty(B) \leq \left\lceil \frac{n_B}{2} \right\rceil.$$

From the pivoting bound (26),

$$(36) \quad \rho_\infty(B) \leq \eta + \rho_\infty(\tilde{B}).$$

³ This definition of ρ_∞ differs from that in [10] in allowing even multiples of columns not present in B to be adjoined to B .

This gives a useful criterion for stability. If, for some choice of the pivot,

$$(37) \quad \rho(B) = \eta + \rho_\infty(\hat{B}),$$

then B is stable. For (17) and (37) assume that $\rho_\infty(B) = \rho(B)$. Also, if $\tilde{C} = B$ and (37) holds, C is tame and stable.

The problem of finding $\rho_\infty(B)$ can also be expressed as an integer programming problem:

$$(38) \quad \rho_\infty(B) = \max \{h_1 + \cdots + h_{2^k-1}\},$$

subject to the constraints

$$(39) \quad h_i \in \mathbb{Z} \quad \text{for } i = 1, \dots, 2^k - 1,$$

$$(40) \quad \sum_{i=1}^{2^k-1} h_i c_i \leq \frac{1}{2} \sum_{i=1}^{2^k-1} \pi_i c_i \quad \text{for } c = (c_1, \dots, c_{2^k-1}) \in S_k,$$

where the c_i ($=0$ or 1) in (40) are interpreted as real numbers, and $\pi_i = 0$ if m_i is even, $\pi_i = 1$ if m_i is odd (cf. [10, Thm. 1]).

If C is such that $\tilde{C} = B$, $\dim C = \dim B$, and $\rho(C) = \rho_\infty(B)$, we say that C has been obtained by *saturating* B . Obviously a saturated code is stable.

It is sometimes useful to know how long it would take to saturate B , if pairs of identical columns were added to B so as to drive ρ up to $\rho_\infty(B)$ as quickly as possible. The next theorem gives an upper bound on the answer, and also on the values of the m_i and h_i that are required.

THEOREM 8. *Let B be a projective code of dimension k . (a) There is an $[n, k]$ code C with multiplicities $m_i \leq 2^k$ and length*

$$(41) \quad n \leq 2^k(2^k - 1),$$

satisfying $\tilde{C} = B$ and $\rho(C) = \rho_\infty(B)$. (b) Consider any $[n, k]$ code C with $\tilde{C} = B$ and $\rho(C) = \rho_\infty(B)$. Let x be a deep hole in C for which 0 is a closest codeword. Then the heights h_i of the blocks of x satisfy

$$(42) \quad |h_i| \leq 2^{k-1}, \quad i = 1, \dots, 2^k - 1.$$

Proof. We prove (b) first. We know that

$$(43) \quad h_1 + \cdots + h_{2^k-1} = \rho_\infty(B),$$

so we shall maximize h_i (and later $-h_i$) subject to the constraints (39), (40), (43). Inequality (40) implies that

$$(44) \quad \sum_{i=1}^{2^k-1} h_i c_i \leq 2^{k-2} \quad \text{for } c \in S_k,$$

since all the nonzero codewords in S_k have weight 2^{k-1} , and $\pi_i \leq 1$. If the inequalities (44) corresponding to those codewords $c = (c_1, \dots, c_{2^k-1}) \in S_k$ with $c_1 = 1$ are summed we obtain $2^{k-1}h_1 + 2^{k-2}(h_2 + \cdots + h_{2^k-1}) \leq 2^{2k-3}$, which by (43) becomes $h_1 \leq 2^{k-1} - \rho_\infty(B) \leq 2^{k-1}$. Similarly, by using the codewords with $c_1 = 0$, we obtain $-h_1 \leq 2^{k-1}$, so $|h_1| \leq 2^{k-1}$. The same argument applies to the other h_i . (a) Knowing the possible range of the individual h_i we can work out how large the multiplicities m_i must be. From (9), $m_i = 2^k$ is enough to permit $-2^{k-1} \leq h_i \leq 2^{k-1}$. Therefore there is a code C of length $n = \sum m_i \leq 2^k(2^k - 1)$ with $\rho(C) = \rho_\infty(B)$.

Theorem 8 implies that we can add the extra inequalities

$$(45) \quad |h_i| \leq 2^{k-1}, \quad i = 1, \dots, 2^k - 1,$$

to the integer program (38)–(40) defining $\rho_\infty(B)$. Stronger bounds than (45) can often be obtained in particular cases by detailed examination of (40).

The quantity $\rho_\infty(B)$ tells us the *maximal* value of $\rho(C)$ over all codes C with $\tilde{C} = B$. We now investigate the *minimal* value. Let

$$(46) \quad \rho_n^*(B) = \min \{ \rho(C) : C \text{ is an } [n, k] \text{ code with } C \equiv B, C \geq B \}.$$

In view of (33), this also minimizes the covering radius of C over all $[n, k]$ codes with $\tilde{C} = B$. From $\rho_n^*(B) \leq \rho_\infty(B)$ and (15), for sufficiently large n $\rho_n^*(B)$ is independent of n , and will be denoted by $\rho_*(B)$. Then

$$(47) \quad \rho(B) \leq \rho_*(B) \leq \rho_\infty(B).$$

THEOREM 9. *Let B be an $[n_B, k]$ projective code. For sufficiently large n of the same parity as n_B there is an $[n, k]$ normal code C with $C \equiv B$, $C \geq B$, and $\rho(C) = \rho_*(B)$, which is obtained by adjoining $n - n_B$ copies of a single column to B .*

Proof. Let us arrange the coordinates of S_k so that the coordinates of B appear first. The set of vectors $(h_1, \dots, h_{2^k-1})$ satisfying (39), (40) (with $\pi_i = 1$ for $1 \leq i \leq n_B$, $\pi_i = 0$ otherwise) and (45) is a finite set H (depending on B but not on n).

For fixed large n with $n \equiv n_B \pmod{2}$, we consider all $[n, k]$ codes C for which $\tilde{C} = B$. Such a code C is defined by its multiplicities $m_1, \dots, m_{2^k-1}$, where

$$(48) \quad \begin{aligned} m_1, \dots, m_{n_B} & \text{ are odd,} \\ m_{n_B+1}, \dots, m_{2^k-1} & \text{ are even, and} \\ \sum_{i=1}^{2^k-1} m_i &= n. \end{aligned}$$

Then $\rho(C) = \rho^{(k)}(m_1, \dots, m_{2^k-1})$ is equal to the maximum of $h_1 + \dots + h_{2^k-1}$ subject to (39), (40) and

$$(49) \quad -\left\lceil \frac{m_i}{2} \right\rceil \leq h_i \leq \left\lfloor \frac{m_i}{2} \right\rfloor \quad \text{for } i = 1, \dots, 2^k - 1$$

(cf. [10, Thm. 1]). Also

$$\rho_n^*(B) = \min \{ \rho^{(k)}(m_1, \dots, m_{2^k-1}) : \text{the } m_i \text{ satisfy (48)} \}.$$

Let us define

$$f(i) = \rho^{(k)}(1, \dots, 1, n - n_B + 1, 1, \dots, 1, 0, \dots, 0),$$

for $i = 1, \dots, n_B$, where $n - n_B + 1$ is in position i and there are $n_B - 1$ ones, and

$$f(i) = \rho^{(k)}(1, \dots, 1, 0, \dots, 0, n - n_B, 0, \dots, 0)$$

for $i = n_B + 1, \dots, 2^k - 1$, where $n - n_B$ is in position i and there are n_B ones. Since H is a finite set, for n sufficiently large $f(i)$ is independent of n and is equal to either

$$\rho^{(k)}(1, \dots, 1, a_i, 1, \dots, 1, 0, \dots, 0)$$

for $i = 1, \dots, n_B$, or

$$\rho^{(k)}(1, \dots, 1, 0, \dots, 0, a_i, 0, \dots, 0)$$

for $i = n_B + 1, \dots, 2^k - 1$, where $a_1, \dots, a_{2^k-1}$ are constants (depending only on B).

Finally, let

$$M = \min \{f(1), \dots, f(2^k - 1)\}$$

and choose an r such that $f(r) = M$.

After these preliminaries we come to the heart of the proof. Suppose n is large (specifically, we need $n > 2^k \max \{a_i\}$), and $\rho_n^*(B) = \rho^{(k)}(m_1, \dots, m_{2^k-1})$, where the m_i satisfy (48). At least one of the m_i must be large, say m_j (and in particular $m_j > a_j$). By the monotonicity property (15),

$$\begin{aligned} \rho_n^*(B) &\geq \begin{cases} \rho^{(k)}(1, \dots, m_j, \dots, 1, 0, \dots, 0) & (j \leq n_B) \\ \rho^{(k)}(1, \dots, 1, 0, \dots, m_j, \dots, 0) & (j > n_B) \end{cases} \\ &= \begin{cases} \rho^{(k)}(1, \dots, a_j, \dots, 1, 0, \dots, 0) & (j \leq n_B) \\ \rho^{(k)}(1, \dots, 1, 0, \dots, a_j, \dots, 0) & (j > n_B) \end{cases} \\ &\geq M. \end{aligned}$$

In fact $\rho_n^*(B) = M$, since if C has multiplicities $m_i = 1$ for $i \leq n_B$ and $m_i = 0$ for $i > n_B$, except for $m_r = n - n_B + 1$ (if $r \leq n_B$) or $n - n_B$ (if $r > n_B$), then $\rho(C) = M$. Therefore $\rho_n^*(B) = \rho_*(B) = M$.

Since $\rho(C) = \rho_*(B)$, we can adjoin $2l$ copies of the special column to C without increasing ρ , for $l = 0, 1, \dots$. Therefore C is normal by Lemma 5. This completes the proof of Theorem 9.

Remark. The single column mentioned in the theorem need not be a column of B .

Example. As an illustration of the preceding notions we analyze the $[11, 5]R = 3$ code C'_{11} having generator matrix

$$(50) \quad C'_{11}: \begin{array}{c|cccccccccccc} & 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \\ \hline & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 \\ & 0 & 1 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 \\ & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 \\ & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 1 & 1 \end{array},$$

which is an interesting code for several reasons (see § 6). It has weight distribution $0^1 4^{10} 6^{16} 8^5$, and its automorphism group has order 1920, with two orbits on coordinates, $\{1\}$ and $\{2, \dots, 11\}$. Suppose C is an $[n, 5]$ code with $\tilde{C} = C'_{11}$, having odd multiplicities $m_1, \dots, m_{11}$ on the columns of (50), and even multiplicities on the remaining 20 nonzero columns of length 5. From (17), $\rho(C) \geq \rho(C'_{11}) = 3$.

If we pivot on coordinate $Q = 1$, we see that $\eta = 5$ (since columns 2 and 7, 3 and 8, etc. combine), $C_0^{[1]}$ has even multiplicities; i.e., $\tilde{C} = 0$, and (26) states that $\rho(C) \leq 5$. On the other hand, if we pivot at $Q = 2$ we find that $\eta = 1$, $\tilde{C}$ is the $[8, 4]$ extended Hamming code, for which $\rho = 2$ or 3 depending on the multiplicities m'_p [10, Thm. 16], and (26) yields $\rho(C) \leq 4$. We conclude that

$$(51) \quad 3 \leq \rho(C) \leq 4.$$

The following theorem determines which of these two possibilities occurs.

THEOREM 10. *Let C be any code for which the contracted code is the $[11, 5]R = 3$ code C'_{11} defined by (50). If column 1 of (50) has multiplicity ≥ 3 or any of the 20 nonzero*

Case (i): $k \equiv k_B \pmod{2}$. By Theorem 9 there is an $[n - (k - k_B), k_B]$ normal code A with $\tilde{A} = B$ and $\rho(A) = \rho_*(B)$, obtained by adjoining $(n - n_B) - (k - k_B)$ copies (an even number) of a single column β (say) to B . Then $C'' = A \oplus \mathbf{F}^{k-k_B}$ is an $[n, k]$ normal code with $\rho(C'') = \rho_*(B)$, and has $n_B + k - k_B$ distinct columns with odd multiplicity. From (2),

$$CR(C'') = \frac{n}{2} - \frac{n_B + k - k_B}{2} + \rho_*(B)$$

which is less than $CR(C)$ unless $k = k_B$. Therefore $k = k_B$, and $C'' \in \mathcal{C}[n, k]$ has the desired multiplicities.

Case (ii): $k \not\equiv k_B \pmod{2}$. By Theorem 9, for sufficiently large N_0 there is an $[N_0, k_B]$ normal code A_0 with $\tilde{A}_0 = B$ and $\rho(A_0) = \rho_*(B)$, obtained by adjoining $N_0 - n_B$ copies of a single column β to B . Let A_i be obtained by adjoining i further copies of β to A_0 . We know $CR(A_2) = CR(A_0) + 1$, so

$$\begin{aligned} \text{either } CR(A_1) &= CR(A_0), & CR(A_2) &= CR(A_1) + 1, \\ \text{or } CR(A_1) &= CR(A_0) + 1, & CR(A_2) &= CR(A_1). \end{aligned}$$

In the first case we call A_0 *late* and in the second case we call it *early*. Whether A_0 is late or early depends on the solution to a certain integer programming problem. Therefore, in the sequence $A_0, A_2, A_4, \dots$, from a certain point on either all A_{2i} are early or all are late, and similarly in the sequence $A_1, A_3, A_5, \dots$. Thus for sufficiently large i , A_{2i+1} satisfies the hypothesis of Lemma 5 and is normal. In particular, by taking

$$i = \frac{1}{2}\{n - 1 - N_0 - (k - k_B)\},$$

we obtain normal codes A_{2i} and A_{2i+1} of dimension k_B and lengths $n - 1 - (k - k_B)$ and $n - (k - k_B)$, respectively, with $\rho(A_{2i}) = \rho_*(B)$,

$$CR(A_{2i}) = \frac{n - 1 - (k - k_B)}{2} - \frac{n_B}{2} + \rho_*(B),$$

$$CR(A_{2i+1}) \leq CR(A_{2i}) + 1.$$

Finally, let $C'' = A_{2i+1} \oplus \mathbf{F}^{k-k_B}$. Then

$$CR(C'') = CR(A_{2i+1}) \leq \frac{n}{2} - \frac{n_B}{2} + \rho_*(B),$$

since $k \geq k_B + 1$, with equality only if $k = k_B + 1$ and A_{2i} is early. Therefore $k = k_B + 1$, and $C'' \in \mathcal{C}[n, k]$ has the desired multiplicities. This completes the proof of (a) and (b). To prove (c) we observe that the best choice for B in (52) is independent of n , and when n increases to $n + 2$, the right-hand side of (52) increases by 1.

Remarks. (1) This theorem establishes Conjectures A and D of [2] for sufficiently large n . (2) If the optimal code C is obtained by adjoining $2l$ columns to B , then C has length $n = n_B + 2l$ and covering radius $R = \rho_*(B) + l$. We can write this as

$$(54) \quad \frac{n}{2} - R = \frac{n_B}{2} - \rho_*(B)$$

or, if B is normal, as

$$(55) \quad \frac{n}{2} - CR(C) = \frac{n_B}{2} - CR(B).$$

Thus the best seed codes B are normal codes for which the parameter

$$(56) \quad \delta = \frac{n_B}{2} - CR(B)$$

is as large as possible. (3) A possible interpretation of Theorems 9 and 12 is the following. An optimal covering code has the property that the codewords are constructed so as to be not too far from an arbitrary n -tuple x . This is a difficult task for $n \gg k$, since we are using only 2^k vectors to cover 2^n vectors. We may say informally that codes with the structure described in Theorem 12(b) do this by matching x very carefully on a small number (n_B) of coordinates, and just using an average on the rest (see Fig. 2). (4) This special structure also greatly simplifies the process of finding the closest codeword to a given x .

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