ON THE COVERING RADIUS PROBLEM FOR CODES II. CODES OF LOW DIMENSION; NORMAL AND ABNORMAL CODES*

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Abstract. In this two-part paper we introduce the notion of a stable code and give a new upper bound on the normalized covering radius of a code. The main results are that, for fixed k and large n , the minimal covering radius $t[n, k]$ is realized by a normal code in which all but one of the columns have multiplicity 1; hence $t[n + 2, k] = t[n, k] + 1$ for sufficiently large n . We also show that codes with $n \leq 14$, $k \leq 5$ or $d_{\min} \leq 5$ are normal, and we determine the covering radius of all proper codes of dimension $k \leq 5$. Examples of abnormal nonlinear codes are given. In Part I [this Journal, 8 (1987), pp. 604–618] we investigated the general theory of normalized covering radius, while in Part II we study codes of dimension $k \leq 5$, and normal and abnormal codes.

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6. Codes of dimension $k \leq 5$. The inequivalent projective $[n, k]$ codes were enumerated by Slepian [8]. The numbers of these codes for $k \leq 5$ are shown in Table 1, as well as the numbers of inequivalent, projective, *indecomposable* $[n, k]$ codes. The codes of dimension $k \leq 4$ were studied in detail in [10], where it was shown that, with four exceptions, all such codes are tame and stable. The exceptions are described in Theorem 16 of [10], and are also listed below. We have made a similar investigation of all the proper codes of dimension 5. It would take too much space to list all the codes, and we just give a summary. The main result is the following.

THEOREM 13. *For each projective code B of dimension 5,*

$$(57) \quad CR(B) \leq \rho_{\infty}(B) \leq CR(B) + 1.$$

Equivalently, for any code C for which the contracted code $\tilde{C}$ has dimension 5,

$$(58) \quad CR(\tilde{C}) \leq \rho(C) \leq CR(\tilde{C}) + 1.$$

Every code of dimension ≤ 5 is normal, and in fact every code C for which the contracted code $\tilde{C}$ has dimension ≤ 5 is normal.

Proof. The first assertion was established by direct calculation using a computer, the pivoting bound (26) of Part I, and the results of [10]. The normality of these codes then follows from Theorem 9 of [10] (see the remarks in § 4 of Part I), and the final assertion is a consequence of Theorem 1(b) of Part I.

Remark. While proving Theorem 13 we reconfirmed the assertions required for the proof of Theorem 22 of [3] that there are no proper codes with parameters $[11, 5] 2$, $[13, 5] 3$, $\dots$, $[23, 5] 8$ or $[25, 5] 9$.

Table 2(a) gives upper and lower bounds on $\rho(C)$ for any code C for which $\tilde{C}$ is an $[n, k]$ code with $k \leq 5$. Table 2(b) gives $t[n, k]$ for $k \leq 5$ (a much more extensive table appears in [3]). For large n (indicated by an asterisk) $t[n, k]$ cannot be realized by a projective code.

The four wild codes of dimension 4 are the $[7, 4] 1$ Hamming code H_7 , the $[8, 4] 2$ extended Hamming code, the $[11, 4] 3$ code obtained by omitting the columns of an

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TABLE 1
Number of inequivalent projective $[n, k]$ codes. (In parentheses, the number of inequivalent, projective, indecomposable $[n, k]$ codes.)

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	1 (1)														
2		1 (0)	1 (1)												
3			1 (0)	2 (1)	1 (1)	1 (1)	1 (1)								
4				1 (0)	3 (1)	4 (2)	5 (4)	6 (5)	5 (5)	4 (4)	3 (3)	2 (2)	1 (1)	1 (1)	1 (1)
5					1 (0)	4 (1)	8 (3)	15 (9)	29 (22)	46 (40)	64 (60)	89 (86)	112 (110)	128 (127)	144 (143)

$k \backslash n$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5	145 (144)	129 (129)	113 (113)	91 (91)	67 (67)	50 (50)	34 (34)	21 (21)	14 (14)	9 (9)	5 (5)	3 (3)	2 (2)	1 (1)	1 (1)	1 (1)

$E_3 \oplus \mathbb{F}$ from S_4 , and the $[12, 4]$ 4 code obtained by omitting an E_3 from S_4 . For all four codes, $\rho_\infty = R + 1$. (See Theorem 16 of [10].)

Projective codes of dimension 5. In the remainder of this section we briefly describe the $[n, 5]$ projective codes for each length n , and give one or two of the best covering codes. The examples given were selected by applying (in order) the following criteria: (1) minimize R , (2) choose tame rather than wild codes, (3) maximize the order g of the automorphism group, and (4) maximize $d_{\min}$. Generator matrices for some of these codes are displayed in Fig. 3. Although many of these codes can be obtained from later codes by deleting appropriate coordinates, there is no single sequence of nested codes that includes all our best examples.

For each code B in the following list we give upper and lower bounds (differing by at most 1) on $\rho(C)$ for any code C for which $\tilde{C} = B$. All tame codes of dimension 5 satisfy (37) of Part I and are therefore stable. However we have not been able to check all the wild codes (not even all the codes on the following list) to see if they are unstable and the upper bound on $\rho(C)$ is actually attained.

- $n = 5$. One code: $\mathbb{F}^5$, $\rho = 0$, tame, $g = 5!$, $d_{\min} = 1$.
- $n = 6$. All four are tame, with $R = 1$. Example: E_6 , $\rho = 1$, tame, $g = 6!$, $d_{\min} = 2$.
- $n = 7$. All are tame, six have $R = 1$, two have $R = 2$. Example: C_7 (Fig. 3), $\rho = 1$, tame, $g = 72$, $d_{\min} = 2$.
- $n = 8$. Two wild codes, with $\rho = 1$ or 2, thirteen tame, with $R = 2$. Example: $H_7 \oplus \mathbb{F}$, $\rho = 1$ or 2, wild, $g = 168$, $d_{\min} = 1$. (This code is unstable. It can be shown (compare Theorem 10 of Part I) that the precise value of ρ is as follows. Consider any code C for which $\tilde{C} = H_7 \oplus \mathbb{F}$. The columns of H_7 span a $PG(2, 3)$, in which there is a unique further point X that is neither a column nor the sum of two columns of H_7 . If

TABLE 2

(a) Upper and lower bounds on normalized covering radius $\rho(C)$, given that $\tilde{C}$ is an $[n, k]$ code. (b) $t[n, k]$, the smallest possible covering radius R for any $[n, k]$ code. An asterisk indicates that this value of R cannot be realized by a projective code.

(a)

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
1	0														
2		0	1												
3			0	1	1	2	3								
4				0	1	1-2	1-2	2-3	3	3-4	3-4	4	5	6	7
5					0	1	1-2	1-2	2-3	2-4	3-4	3-5	4-5	4-5	5-6

$k \backslash n$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
5	5-7	6-7	6-7	7-8	7-9	8-9	8-9	9-10	9-11	10-11	10-11	11-12	12-13	13	14	15

(b)

$k \backslash n$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
3			0	1	1	2	2*	3*	3*	4*	4*	5*	5*	6*	6*
4				0	1	1	1	2	2*	3	3	4	4*	5*	5*
5					0	1	1	1	2	2	3	3	4	4	5

$k \backslash n$	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31
3	7*	7*	8*	8*	9*	9*	10*	10*	11*	11*	12*	12*	13*	13*	14*	14*
4	6*	6*	7*	7*	8*	8*	9*	9*	10*	10*	11*	11*	12*	12*	13*	13*
5	5	6	6	7	7	8	8	9	9	10	10	11	11*	12*	12*	13*

any of the seven points $PG(2, 3) \setminus \{X \cup H_7\}$ has positive even multiplicity in C , $\rho(C) = 2$; otherwise $\rho(C) = 1$.)

$n = 9$. Twenty-three have $R = 2$ (2 are wild), six have $R = 3$ (all tame). Example: C_9 , $\rho = 2$, tame, $g = 384$, $d_{\min} = 2$.

$n = 10$. Seven have $R = 2$ (all wild), 38 have $R = 3$ (all tame), one has $R = 4$ (tame). Example: C_{10} , $\rho = 2$ or 3, wild, unstable ($\rho = 3$ if any column not in C_{10} has positive even multiplicity, otherwise $\rho = 2$), $g = 1920$, $d_{\min} = 4$.

$n = 11$. Fifty-five have $R = 3$ (2 are wild), nine have $R = 4$ (all tame). Example: C_{11} , $\rho = 3$, tame, $g = 120$, $d_{\min} = 4$.

$n = 12$. Twenty have $R = 3$ (all wild), 68 have $R = 4$ (all tame), one has $R = 5$ (tame). Example: C_{12} (Fig. 3, or omit first, second and last two columns of C_{16}), $\rho = 3$ or 4, wild, unstable, $g = 192$, $d_{\min} = 4$.

$n = 13$. One hundred four have $R = 4$ (6 are wild), eight have $R = 5$ (all tame). Example: C_{13} (omit first, second and last columns of C_{16}), $\rho = 4$, tame, $g = 576$, $d_{\min} = 5$.

$n = 14$. Forty-four have $R = 4$ (all wild), 88 have $R = 5$ (all tame). Example: C_{14} (omit first and last columns of C_{16}), $\rho = 4$ or 5, wild, unstable, $g = 2688$, $d_{\min} = 6$.

$n = 20$. Thirty-nine have $R = 7$ (all wild), 28 have $R = 9$ (1 is wild). Example: C_{20} (omit columns 1–4 of C_{24}), $\rho = 7$ or 8, wild, stability not known,⁴ $g = 384$, $d_{\min} = 9$.

$n = 21$. Forty-seven have $R = 8$ (6 are wild), three have $R = 9$ (all tame). Example: C_{21} (omit columns 1–3 of C_{24}), $\rho = 8$, tame, $g = 384$, $d_{\min} = 9$. (Alternatively, the cyclic code C'_{21} with generator polynomial $x^{16} + x^{12} + x^{11} + x^8 + x^6 + x^4 + x^3 + x^2 + x + 1$, $\rho = 8$ or 9, wild, stability not known, $g = 1008$, $d_{\min} = 10$.)

$n = 22$. Nineteen have $R = 8$ (all wild), 15 have $R = 9$ (all tame). Example: C_{22} (omit columns 1 and 2 of C_{24}), $\rho = 8$ or 9, wild, $g = 768$, $d_{\min} = 10$.

$n = 23$. Nineteen have $R = 9$ (3 are wild), two have $R = 10$ (both tame). Example: C_{23} (omit columns 1–4, 6, 8, 9, 12 of S_5), $\rho = 9$, tame, $g = 128$, $d_{\min} = 10$. (Alternatively, C'_{23} : omit columns 1–8 of S_5 , or column 1 of C_{24} , $\rho = 9$ or 10, wild, $g = 2688$, $d_{\min} = 11$.)

$n = 24$. Six have $R = 9$ (all wild), eight have $R = 10$ (1 wild). Example: C_{24} (Fig. 3, or omit columns 1–6, 8 of S_5), $\rho = 9$ or 10, wild, $g = 384$, $d_{\min} = 11$.

$n = 25$. Eight have $R = 10$ (2 are wild), one has $R = 11$ (tame). Example: C_{25} (omit columns 1, 2, 4, 7–9 of S_5), $\rho = 10$, tame, $g = 192$, $d_{\min} = 11$. (Alternatively C'_{25} : omit columns 1–6 of S_5 , $\rho = 10$ or 11, wild, $g = 9216$, $d_{\min} = 12$.)

$n = 26$. Three have $R = 10$ (all wild), two have $R = 11$ (both tame). Example: C_{26} (omit first 5 columns of S_5), $\rho = 10$ or 11, wild, $g = 3072$, $d_{\min} = 12$.

$n = 27$. Three codes with $R = 11$ (two are wild). Example: C_{27} (omit F^4 from S_5), $\rho = 11$, tame, $g = 384$, $d_{\min} = 12$.

$n = 28$. Two codes. C_{28} (omit F^3 from S_5), $\rho = 12$, tame, $g = 2304$, $d_{\min} = 13$. (Alternatively, C'_{28} : omit E_3 from S_5 , $\rho = 12$ or 13, wild, $g = 64512$, $d_{\min} = 14$.) For $n \geq 28$ we can achieve a smaller R by using codes that are not projective (see Table 2(b)).

$n = 29$. One code: C_{29} (omit any 2 columns of S_5), $\rho = 13$, tame, $g = 21504$, $d_{\min} = 14$.

$n = 30$. One code: C_{30} (omit any column of S_5), $\rho = 14$, tame, $g = 322560$, $d_{\min} = 15$.

$n = 31$. One code: S_5 (Fig. 3), $\rho = 15$, tame, $g = 31 \cdot 30 \cdot 28 \cdot 24 \cdot 16 = 9999360$, $d_{\min} = 16$.

Codes with unacceptable coordinates. We have established certain facts about codes having unacceptable coordinates (the definition is given in § 1 of Part I), which we record here. All coordinates of all projective codes of dimension $k \leq 4$ are acceptable. The first instance of a projective code with an unacceptable coordinate is the $[10, 5]$ 2 code C'_{10} given in (4) of [3]. The second is the $[11, 5]$ 3 code C'_{11} of (50) in Part I, obtained by adding an overall parity check to C'_{10} . The first coordinate of C'_{11} is unacceptable. These two are the only projective codes of dimension 5 and length $n \leq 13$ with unacceptable coordinates. By adjoining an even number of copies of any acceptable coordinate to either C'_{10} or C'_{11} , we obtain $[n, 5]$ codes with unacceptable coordinates for all $n \geq 10$. Hence by taking direct sums with F there are $[n, k]$ codes with unacceptable coordinates for all $n \geq k + 5 \geq 10$.

7. Codes with minimal distance 4 or 5 are normal. It is shown in [2] that all codes with minimal distance $d_{\min} \leq 3$ are normal, and any coordinate in the support of a codeword of minimal weight is acceptable. In this section we extend this result to codes with minimal distance 4 and 5.

We begin with two lemmas, the first of which is elementary. For any $x \in F^n$, let x' denote the vector obtained by deleting the first coordinate of x . Suppose C is an $[n, k]$ R code, and let D be the $[n - 1, k']$ R' code $D = \{c' : c \in C\}$.

⁴ Preliminary calculations suggest this code may be stable, with $\rho_{\infty} = 7$.

LEMMA 14. For any $x \in \mathbb{F}^n$, $d(x, C) \geq d(x', C')$.

LEMMA 15. Suppose C and D have the same covering radius R , and let $u \in \mathbb{F}^{n-1}$ be such that $d(u, D) = R$. Then there are codewords c_0, c_1 in C such that

$$d(u, c'_0) = d(u, c'_1) = R,$$

the first coordinate of c_0 is 0, and the first coordinate of c_1 is 1.

Proof. Let $x = 0u$. Then $d(x, C) = R$ by Lemma 14, say $d(x, c_0) = R$. If $c_0 = 1c'$ then $d(c', u) = R - 1$, a contradiction, so $c_0 = 0c'$, with $d(c', u) = R$ as required. Similarly $x = 1u$ leads to c_1 .

THEOREM 16. A code with minimal distance 4 or 5 is normal, and all coordinates in the support of a codeword of minimal weight are acceptable.

Proof. Suppose C is an $[n, k]R$ code containing the word $s = 11 \dots 100 \dots 0$ of weight $d_{\min} = 4$ or 5. We present the proof so that it applies simultaneously to both cases.

Let primes indicate that the first coordinate has been deleted, and set $D = \{c' : c \in C\}$. Since D has minimal distance 1 less than C , we may assume by induction that D is normal and the first coordinate of D is acceptable. The covering radius of D is either R or $R - 1$.

We shall show that the first coordinate of C is acceptable; i.e.,

$$d(x, C_0^{(1)}) + d(x, C_1^{(1)}) \leq 2R + 1,$$

for all $x \in \mathbb{F}^n$, or equivalently that, for each x , there exists $c_0 \in C_0^{(1)}$, $c_1 \in C_1^{(1)}$ such that

$$(59) \quad d(x', c'_0) + d(x', c'_1) \leq 2R.$$

Let $d(x', D) = R - m$, where $m = 0, 1, 2, \dots$.

Case $m = 0$. Then (59) follows immediately from Lemma 15.

Case $m = 1$. Without loss of generality we suppose there is $c_0 = .0 \dots \in C$ such that $d(x', c'_0) = R - 1$. If $x = .1 \dots$ then $g = s + c_0$ satisfies $d(x', g') \leq R + 1$ and (59) holds (with $c_1 = g$). So we may assume $x = .0 \dots$, say $x = \alpha 0 \beta \gamma \delta \dots$. Since D is normal there is $c_1 = .1 \dots \in C$ such that $d(x', c'_1) \leq R + 2$. We choose c_1 to minimize $d(x', c'_1)$. There are four subcases.

(a) $d(x', c'_1) = R + 2$. Let $y = \alpha 1 \beta \gamma \delta \dots$ (differing from x in one coordinate), and let $c'_2 \in D$ be a closest codeword to y' , with $d(y', c'_2) \leq R$. Then $d(y', c'_2) = R$. (For suppose $d(y', c'_2) \leq R - 1$. If $c_2 = .0 \dots$, $d(x', c'_2) \leq R - 2$, contradicting the definition of c_0 , and if $c_2 = .1 \dots$, $d(x', c'_2) \leq R$, contradicting the definition of c_1 .) By Lemma 15 there are $g_0 \in C_0^{(1)}$, $g_1 \in C_1^{(1)}$ with $d(y', g'_0) = d(y', g'_1) = R$. Also g_0 and g_1 have second digit 0, or else $d(x', g'_i) \leq R + 1$, again contradicting the definition of c_1 . Therefore $d(x', g'_0) = d(x', g'_1) = R - 1$, and (59) holds.

(b) $d(x', c'_1) = R + 1$. Define y and c_2 as in (a). Again $d(y', c'_2) = R$, and $g_0 \in C_0^{(1)}$, $g_1 \in C_1^{(1)}$ exist with $d(y', g'_0) = d(y', g'_1) = R$. If the second digit of either g_0 or g_1 is 0, $d(x', g'_0) + d(x', g'_1) \leq 2R$, and (59) holds. Suppose then that $g_0 = 01 \dots$, $g_1 = 11 \dots$. Choose i so that g_i and c_0 differ in the first digit. Then $d(x', c'_0) + d(x', g'_i) \leq (R - 1) + (R + 1)$, and again (59) holds.

(c) $d(x', c'_1) = R$. If $c_0 = \alpha 0 \dots$ and $c_1 = \bar{\alpha} 1 \dots$, or if $c_0 = \bar{\alpha} 0 \dots$ and $c_1 = \alpha 1 \dots$ then we may use c_0 and c_1 in (59). The difficult cases are when c_0 and c_1 begin with the same digit. We define $z = \alpha 0 \bar{\beta} \bar{\gamma} \delta \dots$ (differing from x in two coordinates), let $d_0 = d(z, C) \leq R$, and let U denote the set of codewords $u \in C$ with $d(z, u) = d_0$.

Case (c.1). Suppose $c_0 = \alpha 0 \dots$, $c_1 = \alpha 1 \dots$. If U contains a vector $u = \bar{\alpha} \dots$ then $d(u, x) \leq d(u, z) + 2 \leq R + 2$, $d(u', x') \leq R + 1$, and we use c_0 and u in (59). Otherwise all $u \in U$ begin with α , and $d(z', D') = d(z, D) = d_0$. If $d_0 = R$, then z is a deep

hole in D and by Lemma 15 there is $u = \bar{\alpha} \dots \in U$, a contradiction. Therefore $d_0 \leq R - 1$. Choose some $u \in U$. If $u = \alpha \cdot \beta \gamma \dots$ then $d(x, u) \leq d(z, u) - 2 \leq R - 3$, a contradiction. Therefore $u = \alpha \cdot \beta \bar{\gamma} \dots$, $\alpha \cdot \bar{\beta} \gamma \dots$ or $\alpha \cdot \bar{\beta} \bar{\gamma} \dots$. In every case $g = s + u$ satisfies $d(x', g') \leq R + 1$, and we may use c_0 and g in (59).

Case (c.2). Suppose $c_0 = \bar{\alpha} 0 \dots$, $c_1 = \bar{\alpha} 1 \dots$. The argument is similar to case (c.1) and is omitted.

(d) $d(x', c') \leq R - 1$. Let $g = s + c_1$. Then $d(x', g') \leq R + 1$ and we may use c_1 and g in (59).

Case $m \geq 2$. Choose $c \in C$ so that $d(x', c') \leq R - 2$, say $c \in C_0^{(1)}$. Then $g = s + c \in C_1^{(1)}$ satisfies $d(x', g') \leq R + 2$, and we use c and g in (59). This completes the proof.

8. Further conditions for a code to be normal. In this section we show that if $\tilde{C}$ has minimal distance 1 or 2 then C is normal. Theorem 18 summarizes the known conditions on the parameters of a code that imply normality.

THEOREM 17. *If the contracted code $\tilde{C}$ has $d_{\min} = 1$ or 2 then C is normal.*

Proof. If $\tilde{C}$ has $d_{\min} = 1$ then $\tilde{C}$ is a direct sum $\mathbf{F} \oplus \dots$. Therefore $C = T_m \oplus \dots$ for some repetition code T_m . Since T_m is normal, so is C by Theorem 1(b) of Part I.

Suppose $\tilde{C}$ has $d_{\min} = 2$, and there is a word of weight 2 with ones on the first and last coordinates. By Theorem 19 of [2], $\tilde{C}$ is normal, and the first and last coordinates are acceptable.

We first “blow up” all but the first and last coordinates of $\tilde{C}$, obtaining a code B (say) of length n_B , norm N_B , covering radius R_B , and $d_{\min} = 2$. Again by Theorem 19 of [2], B is normal and the first and last coordinates are acceptable.

Next we blow up the last coordinate of B , giving it odd multiplicity m_Q , and obtaining (by Theorem 2 of Part I) a normal code A (say) of length n_A , norm $N_A = N_B + m_Q - 1$, and covering radius $R_A = R_B + [m_Q/2]$. By Theorem 2 of Part I we know that the last m_Q coordinates of A are acceptable. The difficult part of the proof is to show that the first coordinate of A is acceptable. This is enough to prove the theorem, for then we can obtain C by applying the amalgamated direct sum construction again, blowing up the first coordinate of A , and deduce from Theorem 2 of Part I that C is normal.

Let $w'_2 \in B$ be the codeword with ones on the first and last coordinates of B , and let w_2 be the corresponding codeword of A . (Primes will indicate that the last $m_Q - 1$ coordinates have been deleted.) Take any $x \in \mathbf{F}^{n_A}$, and choose $b_i \in B_i^{(1)}$ so as to minimize $d(x', b_i)$ for $i = 0, 1$. Since the first coordinate of B is acceptable, $d(x', b_0) + d(x', b_1) \leq N_B$. Let r, s be the last digits of b_0, b_1 respectively, and let $a_i \in A$ be obtained by repeating the last digit of b_i $m_Q - 1$ times. (α) If $r \neq s$ then the last digit of x' agrees with either r or s , and

$$\begin{aligned} d(x, a_0) + d(x, a_1) &= d(x', b_0) + d(x', b_1) + m_Q - 1 \\ &\leq N_B + m_Q - 1 = N_A, \end{aligned}$$

and the first coordinate of A is acceptable. (β) If $r = s$, suppose the last digit of x' is $1 - r$. Then $y_0 = b_0 + w'_2$, $y_1 = b_1 + w'_2$ satisfy

$$d(x', y_0) + d(x', y_1) = d(x', b_0) + d(x', b_1) - 2,$$

contradicting the choice of b_0 and b_1 . Therefore x' ends with r .

Let $b = b_0$ or b_1 be a closest codeword in B to x' . If the initial digits of x and b differ, let $c = b + w'_2 \in B$, so that b and c differ in their last digits, $d(x', b) = d(x', c)$, $d(x', b) + d(x', c) \leq N_B$, and by (α) the first coordinate of A is acceptable. Therefore we may assume x and b agree in their initial digits. We now have the situation shown

in Fig. 4. If there are more than $\frac{1}{2}m_Q$ r 's in the last m_Q digits of x , then $d(x, a_0) + d(x, a_1) \leq d(x', b_0) + d(x', b_1) + m_Q - 1 \leq N_A$. If there are fewer than $\frac{1}{2}m_Q$ r 's we add w_2 to a_0 and a_1 and the same inequalities hold. This completes the proof.

THEOREM 18. *Let C be an $[n, k]R$ code, with minimal distance $d_{\min}$ and contracted code $\tilde{C}$. Any of the following conditions implies that C is normal: (a) $n \leq 14$, (b) $k \leq 5$, (c) $R \leq 2$, (d) $d_{\min} \leq 5$, (e) $\tilde{C}$ has dimension ≤ 5 , or (f) $\tilde{C}$ has minimal distance ≤ 2 .*

Proof. (b) and (e) follow from Theorem 13, (c) is Theorem 22 of [2], (d) follows from Theorem 24 of [2] and Theorem 16 above, and (f) is Theorem 17. (a) follows from (b), (d) and the tables of minimal distance [4], [11].

9. Abnormal nonlinear codes exist. At present it is not known if an abnormal *linear* code exists. Abnormal *nonlinear* codes were first constructed by Peter Frankl (personal communication), and with his permission we include his construction here.

In order for a code C to be abnormal, for each $i = 1, \dots, n$ there must be a "bad" vector $x^{(i)}$ such that

$$(60) \quad d(x^{(i)}, C_0^{(i)}) + d(x^{(i)}, C_1^{(i)}) \geq 2R + 2$$

(in the notation of § 1 of Part I). The construction begins by choosing $x^{(1)}, \dots, x^{(n)}$.

Let B be any (linear or nonlinear) code of length n , minimal distance $d \geq 6$, and containing at least n codewords. Let $x^{(1)}, \dots, x^{(n)}$ be distinct codewords of B . We may assume (if necessary by complementing coordinates) that the i th coordinate of $x^{(i)}$ is 0, for $i = 1, \dots, n$. Let

$$(61) \quad S_i = \{y \in \mathbb{F}^n : d(y, x^{(i)}) \leq [(d-2)/2] \text{ and the } i\text{th coordinate of } y \text{ is } 0\}$$

and define

$$(62) \quad C = \mathbb{F}^n \setminus (S_1 \cup S_2 \cup \dots \cup S_n).$$

THEOREM 19. *C has covering radius 1 and norm at least $[d/2] + 1 \geq 4$, and is therefore abnormal.*

Proof. By the triangle inequality the sets S_i are disjoint. We first show C has covering radius 1. If $z \in \mathbb{F}^n$, $z \notin C$, then $z \in S_i$ for a unique i . Let c be obtained by changing the i th coordinate of z to 1. Then $d(z, c) = 1$, $c \notin S_i$ (by (61)), and $d(c, x^{(i)}) \leq [d/2]$. Therefore $d(c, x^{(j)}) \geq [d/2]$ for $j \neq i$, so $c \notin S_j$ and thus $c \in C$. Second, by construction $d(x^{(i)}, C_0^{(i)}) \geq [d/2]$ and $d(x^{(i)}, C_1^{(i)}) \geq 1$, and so the norm of C is at least $[d/2] + 1 \geq 4 = 2R + 2$. Thus C is abnormal.

Example. Let B be a Hadamard code of length 11, and take the $x^{(i)}$ to be the eleven cyclic shifts of (01011100010). Then C is an abnormal code of length 11 containing 1432 codewords.

For a smaller example we use the vectors $x^{(1)}, \dots, x^{(10)}$ of length 10 shown in Table 3, let $S_i = \{y \in \mathbb{F}^{10} : d(y, x^{(i)}) \leq 2, \text{ and } i\text{th coordinate of } y \text{ is } 0\}$, and define C by (62). Although the minimal distance between the $x^{(i)}$ is only 5 (so Theorem 19 does not

		$m_Q - 1$										
w'_2			0	0	.	.	.	0				w_2
x'			0	.	.	.	.	.	r		.	x
$b=b_0$			0	.	.	.	.	.	r	r	r	a_0
b_1			.	.	.	.	.	.	r	r	r	a_1

FIG. 4. Vectors used in proof of Theorem 17. Their names in code B are shown in the left, and in A on the right.

TABLE 3
Vectors
 $x^{(1)}, \dots, x^{(10)}$ used
 in construction of an
 abnormal nonlinear
 code.

0	1	0	1	1	1	0	0	0	1
0	0	1	0	1	1	1	0	0	0
1	0	0	1	0	1	1	1	0	0
0	1	0	0	1	0	1	1	1	0
0	0	1	0	0	1	0	1	1	1
0	0	0	1	0	0	1	0	1	1
1	0	0	0	1	0	0	1	0	1
1	1	0	0	0	1	0	0	1	0
1	1	1	0	0	0	1	0	0	1
0	1	1	1	0	0	0	1	0	0

apply), C turns out to be an abnormal code of length 10, with 564 codewords and covering radius 1.

Finally, it is possible to omit many of those 564 codewords and still have an abnormal code. In this way we have constructed an abnormal code of length 10 with 217 codewords and covering radius 1 (we omit the details). This is the smallest abnormal code known at present. It would be interesting to find the smallest possible example.

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