An improvement to the Minkowski-Hlawka bound for packing superballs*

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ABSTRACT

The Minkowski-Hlawka bound implies that there exist lattice packings of n-dimensional ''superballs'' $| x_1 |^\sigma + \cdots + | x_n |^\sigma \le 1$ ($\sigma = 1, 2, \ldots$) having density Δ satisfying $\log_2 \Delta \ge -n(1 + o(1))$ as $n \to \infty$. For each $n = p^\sigma$ (p an odd prime) we exhibit a finite set of lattices, constructed from codes over $GF(p)$, that contain packings of superballs having $\log_2 \Delta \ge -c\, n(1 + o(1))$, where $c = 1 + 2 e^{-2\pi^2} \log_2 e + \cdots = 1.000000007719\ldots$ for $\sigma = 2$ (the classical sphere packing problem), worse than but surprisingly close to the Minkowski-Hlawka bound, and $c = 0.8226\ldots$ for $\sigma = 3$, $c = 0.6742\ldots$ for $\sigma = 4$, etc., improving on that bound.

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1. Introduction

The n-dimensional body

$$\mathsf{S}_{n,\sigma} = \left\{ (x_1, \ldots, x_n) \in \mathbf{R}^n : \sum_{j=1}^{n} |x_j|^\sigma \le 1 \right\}$$

may be called a ''superball'', following Piet Hein (who called $\mathsf{S}_{3,\sigma}$ for $\sigma \approx 2.5$ a ''supersphere'' [6, Chap. 18]). The Minkowski-Hlawka bound and its generalizations guarantee that there exist lattice packings of $\mathsf{S}_{n,\sigma}$ with density Δ satisfying

$$\log_2 \Delta \ge -n(1 + o(1)), \qquad \text{as } n \to \infty \tag{1}$$

[4, p. 184], [10, p. 148], [15, p. 4]. These bounds are established by averaging over certain infinite families of lattices.

In the present paper we use instead the average over a certain *finite* set of lattices, obtained from codes of length $n = p^\sigma$ over $GF(p)$, where p is an odd prime. The main result is the following.

Theorem 1. For $\sigma = 1, 2, 3, \ldots$ there exist lattice packings of the superball $\mathsf{S}_{n,\sigma}$ with density Δ satisfying

$$\log_2 \Delta \ge -c_\sigma n + O(\log_2 n) \tag{2}$$

as $n \to \infty$, where

$$c_1 = 1.10041156...,$$

$$c_2 = 1.00000000771923332183...,$$

$$c_3 = 0.82260038...,$$

$$c_4 = 0.67424266...,$$

$$c_5 = 0.56924054...,$$

$$\cdots$$

The exact values of the constant c_σ in (2) are:

$$c_1 = \log_2 \left((3 + 2\sqrt{2})/e \right) ;$$

$$c_2 = \log_2 \left[\sqrt{\frac{2}{\pi e}} \, \frac{\theta_2(r)}{r} \right] ,$$

$$= 1 + (\log_2 e) \left[\frac{2}{e^{2\pi^2}} - \frac{24\pi^4 + 2}{e^{4\pi^2}} + \cdots \right] , \tag{3}$$

where

$$\theta_\sigma(x) = \sum_{m=-\infty}^{\infty} x^{|m|^\sigma} , \tag{4}$$

and

$$r = \frac{1}{\sqrt{e}} \left[1 + \frac{4\pi^2}{e^{2\pi^2}} - \frac{64\pi^6 - 48\pi^4 + 8\pi^2}{e^{4\pi^2}} + \cdots \right]$$

$$= 0.60653072377180785899... \tag{5}$$

is the solution to

$$r\,\theta_2'(r) = \theta_2(r), \qquad 0 < r < 1 ; \tag{6}$$

and for $\sigma = 3, 4, \ldots,$

$$c_\sigma = \log_2 \left\{ \left(\ln \frac{1}{r} \right)^{1/\sigma} \frac{\theta_\sigma(r)}{\Gamma\left(1 + \frac{1}{\sigma}\right)} \right\}, \tag{7}$$

where r is the solution to

$$\sigma \ln \frac{1}{r} = \frac{\theta_\sigma(r)}{r\, \theta_\sigma'(r)}, \qquad 0 < r < 1 . \tag{8}$$

Remarks.

(i) For $\sigma = 1$ and 2 Theorem 1 is weaker than (1), although for $\sigma = 2$ it is remarkably close. For $\sigma \geq 3$ Theorem 1 is stronger than the Minkowski-Hlawka bound.

(ii) For the classical sphere packing problem (the case $\sigma = 2$) and the cross-polytope packing problem (the case $\sigma = 1$), these lattices are closer to the Minkowski-Hlawka bound than any restricted family of lattices hitherto considered. For example, it is known that the best lattice packings of spheres that can be obtained from class field towers only achieve $\log_2 \Delta = -cn(1 + o(1))$ with $1.694 < c < 2.218$ [11], [5, Chap. 8, §7.5].

(iii) In the other direction it is known that all packings of $S_{n,\sigma}$ satisfy

$$\log_2 \Delta \leq -0.397\ldots\, n \;\; (1 + o(1)), \;\; \text{if } \sigma = 1 ,$$

$$\log_2 \Delta \leq -0.599\ldots\, n \;\; (1 + o(1)), \;\; \text{if } \sigma = 2 ,$$

and

$$\log_2 \Delta \leq -\frac{n}{\sigma} + \log_2 \left(\frac{n}{\sigma} + 1 \right), \;\; \text{if } \sigma \geq 3 .$$

(For $\sigma = 1$ see [10, p. 326, Theorem 4], for $\sigma = 2$ see Kabatiansky and Levenshtein [7], [17], or [5, Chap. 9], and for $\sigma \geq 3$ see [10, p. 323, Theorem 2].)

2. Codes over GF(p)

To prove Theorem 1 we give a new version of the Gilbert-Varshamov bound for codes over $GF(p)$, classifying codewords according to their ''$\sigma -$ norm'' instead of their Hamming weight.

By a slight abuse of notation we use the same symbol α to represent an element of $GF(p)$ (where p is an odd prime) and the corresponding element in $\mathbf{Z}$ which satisfies

$$- \frac{p-1}{2} \leq \alpha \leq \frac{p-1}{2} .$$

Then the ''absolute value'' $|\alpha|$ maps $GF(p)$ to $\{ 0, 1, \ldots, (p-1)/2 \}$, and we define the $\sigma -$ *norm* of a vector $u = (u_1, \ldots, u_n) \in GF(p)^n$ to be

$$\| u \|_\sigma = | u_1 |^\sigma + \cdots + | u_n |^\sigma ,$$

for $\sigma = 0, 1, 2, \ldots,$ with the convention that $|0|^0 = 0$. $\| u \|_0$ is the Hamming weight of u and $\| u \|_1$ is its Lee weight [8], [12, Chap. 5].

We define an $[n, k, w, p, \sigma]$ code C to be a k-dimensional subspace of $GF(p)^n$ such that $\| c \|_\sigma \geq w$ for every $c \neq 0$ in C. The following theorem is an analog of the Gilbert-Varshamov bound [2, §13.7], [12, Chap. 1, Theorem 12] for such codes.

Theorem 2. Let $n, w \geq 1$ and $k, \sigma \geq 0$. If

$$k < n+1 - \log_p \left\lceil \frac{p-1}{2} B_{n,p} (w-1) \right\rceil \tag{9}$$

where the integers $B_{n,p}(i)$ are defined by

$$\sum_{i=0}^{\infty} B_{n,p}(i)\, x^i = \frac{\hat{\theta}_\sigma(x)^n}{1-x} , \tag{10}$$

$$\hat{\theta}_\sigma(x) = \sum_{m=-(p-1)/2}^{(p-1)/2} x^{|m|^\sigma} , \tag{11}$$

for $0 < x < 1$, then an $[n, k, w, p, \sigma]$ code exists.

Proof. We begin by showing that $B_{n,p}(i)$ is the number of points in a ''$\sigma -$ ball'' of maximal norm i, the discrete analog of a superball. For $u \in GF(p)^n$ we define the $\sigma -$ *ball* around u to be

$$\mathsf{V}_{n,p,i}(u) = \left\{ v \in GF(p)^n : \| u - v \|_\sigma \le i \right\} .$$

A $\sigma -$ ball around a subset of $GF(p)^n$ is understood to mean the union of the $\sigma -$ balls around the individual points. Let

$$T_{n,p,i} = \mathrm{card} \left\{ v \in GF(p)^n : \| v \|_\sigma = i \right\} ,$$

so that, for any $u \in GF(p)^n$,

$$\mathrm{card}\ \mathsf{V}_{n,p,i}(u) = \sum_{j=0}^{i} T_{n,p,j} .$$

It is now easy to check (compare [2, p. 298]) that for $\sigma \ge 0$ the numbers $T_{1,p,i}$ are given by

$$\sum_{i=0}^{\infty} T_{1,p,i}\, x^i = \hat{\theta}_\sigma(x) ,$$

and hence that

$$\sum_{i=0}^{\infty} T_{n,p,i}\, x^i = \hat{\theta}_\sigma(x)^n ,$$

$$\sum_{i=0}^{\infty} \text{card } V_{n,p,i}(u) \cdot x^i = \frac{1}{1-x}\, \hat{\theta}_\sigma(x)^n , \tag{12}$$

valid for $0 < x < 1$ and all $u \in GF(p)^n$. By comparing (10) and (12) we see that

$$B_{n,p}(i) = \text{card } V_{n,p,i}(u) . \tag{13}$$

We prove the theorem by induction on k, the case $k = 0$ being trivial. Suppose a code C_i of type $[n, i, w, p, \sigma]$ exists, and that (9) holds for $k = i+1$, i.e.

$$\frac{p-1}{2} B_{n,p}(w-1)\, p^i < p^n . \tag{14}$$

We shall show that an $[n, i+1, w, p, \sigma]$ code C_{i+1} exists, completing the proof. We have

$$\text{card } GF(p)^n > \frac{p-1}{2} B_{n,p}(w-1)\, p^i$$

$$= \frac{p-1}{2} \text{ card } \mathsf{V}_{n,p,w-1}(0) \cdot \text{card } C_i$$

$$\geq \frac{p-1}{2} \text{ card } \left\{ \bigcup \mathsf{V}_{n,p,w-1}(c) \text{ for } c \in C_i \right\}$$

$$= \frac{p-1}{2} \text{ card } \mathsf{V}_{n,p,w-1}(C_i)$$

$$\geq \text{card } \left\{ \alpha u \colon 1 \leq \alpha \leq \frac{p-1}{2},\ u \in \mathsf{V}_{n,p,w-1}(C_i) \right\}$$

$$= \text{card } \left\{ \alpha u \colon \alpha \in GF(p),\ u \in \mathsf{V}_{n,p,w-1}(C_i) \right\}$$

$$= \text{card } \left\{ GF(p) \cdot \mathsf{V}_{n,p,w-1}(C_i) \right\} .$$

Therefore there exists $u \neq 0$ in $GF(p)^n$ such that

$$u \notin GF(p) \cdot \mathsf{V}_{n,p,w-1}(C_i) . \tag{15}$$

Let C_{i+1} be the $[n, i+1, w', p, \sigma]$ code generated by u and C_i. It remains to show that $w' \geq w$. For $y \in C_{i+1}$, $y \neq 0$, we have $y = \alpha u + c$, $\alpha \in GF(p)$, $c \in C_i$. If $\alpha = 0$, $\| y \|_\sigma \geq w$ by the induction hypothesis. So suppose $\alpha \neq 0$ and, seeking a contradiction, that $\| y \|_\sigma < w$. Then

$$y = \alpha u + c \in \mathsf{V}_{n,p,w-1}(0) ,$$
$$\alpha u \in \mathsf{V}_{n,p,w-1}(-c) \subseteq \mathsf{V}_{n,p,w-1}(C_i) ,$$
$$u \in \alpha^{-1} \mathsf{V}_{n,p,w-1}(C_i) \subseteq GF(p) \cdot \mathsf{V}_{n,p,w-1}(C_i) ,$$

which contradicts (15) and completes the proof.

Remark. Let N_m be the number of codewords of $\sigma -$ norm m in an $[n, k, w, p, \sigma]$ code

C. The generating function $\sum_m N_m x^m$ can be obtained from the Lee enumerator of C [12, p. 145] by replacing x_i by x^{i^σ} for $i = 0, \ldots, (p-1)/2$. Therefore, by the MacWilliams identity [12, p. 145, Theorem 12], if the Lee enumerator of the dual code $C^\perp$ is known we can determine the minimal $\sigma -$ weight w of C.

3. The construction of lattice packings of superballs from codes over GF(p)

The construction is essentially Construction A of [5], [9], [13], [16]. If C is an $[n, k, w, p, \sigma]$ code, where $\sigma \geq 1$, let $\Lambda(C)$ consist of the points $u \in \mathbf{Z}^n$ such that $u (\bmod\ p) \in C$. It is easily verified that $\Lambda(C)$ is a lattice for which the determinant (the square of the volume of a fundamental region) is

$$\det \Lambda(C) = p^{2(n-k)} ,$$

and the minimal value of $\sum |u_j|^\sigma$ for $u = (u_1, \ldots, u_n) \in \Lambda(C), u \neq 0$, is

$$\mu = \min \{p^\sigma , w\} .$$

Thus $\Lambda(C)$ is an admissible lattice for the superball $\mu^{1/\sigma} \cdot \mathsf{S}_{n,\sigma}$. Therefore, by [10, p. 165, Theorem 1], we may place copies of the superball $\frac{1}{2} \mu^{1/\sigma} \cdot \mathsf{S}_{n,\sigma}$ at the points of $\Lambda(C)$ and obtain a lattice packing of these superballs. This packing has density

$$\Delta = \frac{S_{n,\sigma} (\frac{1}{2} \mu^{1/\sigma})^n}{\sqrt{\det \Lambda(C)}} , \tag{16}$$

where

$$S_{n,\sigma} = \frac{2^n \Gamma(1 + \frac{1}{\sigma})^n}{\Gamma(1 + \frac{n}{\sigma})} \tag{17}$$

is the volume of $S_{n,\sigma}$ [10, p. 321, Eq. (7)].

Once n and w have been chosen we set k equal to the largest integer permitted by (9), and deduce from Theorem 2 that lattices exist with density given by (16), (17). We shall take $n = p^{\sigma}$ (which seems to produce the best results), and then adjust w to maximize the density. In this way we find by computer that the codes and lattices shown in Table I exist. (It is easy to prove from (16) that if $n = p^{\sigma}$ one should always choose $w \le n$, so that $\mu = w$.) For comparison we also give the density $\Delta = \Delta_{MH}$ guaranteed by the Minkowski-Hlawka bound, in the form

$$\Delta \ge \frac{\zeta(n)}{2^{n-1}} \ , \tag{18a}$$

for $\sigma \ne 2$ [15, p. 4], where ζ is the Riemann zeta-function, and in Rogers' strengthened version

$$\Delta \ge \frac{n\,\zeta(n)}{e(1 - e^{-n})\, 2^{n-1}} \ , \tag{18b}$$

for $\sigma = 2$ [14], [15, p. 4]. The asymptotic behaviour of the density of these lattices is studied in the following section.

4. Proof of Theorem 1

In this section we fix $\sigma = 1, 2, \ldots,$ set $n = p^{\sigma}$, and let $w = [n/a]$ be (essentially) a fixed fraction of n, where [ ] is the greatest integer function. We prove Theorem 1 by applying the construction of the previous section to the $[n, k, w, p, \sigma]$ codes that are guaranteed to exist by Theorem 2, and letting $p \to \infty$. In view of the observation in the previous paragraph the densest packings are obtained by taking $a \ge 1$. It will be shown that asymptotically the optimal choice for a is 1 (if $\sigma = 1$ or 2), 7.8539... (if $\sigma = 3$), 12.5643... (if $\sigma = 4$), etc. — see Table II.

Lemma 3. Let integers $A_n(m)$ be defined by

$$\sum_{m=0}^{\infty} A_n(m)\, x^m = \frac{\theta_\sigma(x)^n}{1-x}\,, \qquad 0 < x < 1\,.$$

Then, for fixed $a > 0$,

$$A_n\left(\left[\frac{n}{a}\right]\right) = \frac{c_1(a)\, C_2}{\sqrt{n}}\, F(a)^n \left(1 + O\left(\frac{1}{\sqrt{n}}\right)\right) \tag{19}$$

as $n \to \infty$, where $c_1(a)$ is independent of n, $1 \le C_2 \le r^{-1}$,

$$F(a) = \frac{\theta_\sigma(r)}{r^{1/a}}\,, \tag{20}$$

and r is given by

$$a\, r\, \theta_\sigma{}'(r) = \theta_\sigma(r)\,, \qquad 0 < r < 1\,. \tag{21}$$

Proof. Let $m = [n/a]$. From Cauchy's theorem

$$A_n(m) = \frac{1}{2\pi i} \int_{|z| = t} \frac{\theta_\sigma(z)^n}{(1-z)\, z^{m+1}}\, dz\,,$$

for any $0 < t < 1$. The integrand has the form $g(z)\, e^{nh(z)}$, where

$$h(z) = \ln\, \theta_\sigma(z) - \frac{1}{a} \ln z\,.$$

There is a saddle-point on the real axis at $z = r$ where $h'(r) = 0$, i.e. at the solution to (21). Eq. (19) now follows from the saddle point method — see for example [3, p. 88, Eq. (5.7.2)].

Lemma 4. Let $\sigma \geq 1$, and fix a in the range

$$0 < a < 2^{\sigma} \ . \tag{22}$$

For $n = p^{\sigma}$ and $w = [n/a]$,

$$B_{n,p}(w) = A_n(w)\,(1 + o(1)) \tag{23}$$

as $n \to \infty$.

Proof. $E_w = A_n(w) - B_{n,p}(w) > 0$ is the coefficient of x^w in

$$\frac{1}{1-x}\left\{\theta_\sigma(x)^n - \hat{\theta}_\sigma(x)^n\right\}$$

$$= \left\{2 \sum_{m = (p+1)/2}^{\infty} x^{m^\sigma}\right\} \cdot \sum_{s=0}^{n-1} \theta_\sigma(x)^{n-1-s}\,\hat{\theta}_\sigma(x)^s / (1-x) \ .$$

Then for n sufficiently large we may bound E_w by the coefficient of $x^{w'}$ in

$$\frac{2n\,\theta_\sigma(x)^{n-1}}{(1-x)^2} \ ,$$

where

$$w' = w - \left(\frac{p+1}{2}\right)^{\sigma} > 0 \ ,$$

using (22). The saddle point method now shows that

$$E_w \leq c_2(a')\ C_2 \sqrt{n}\,F(a')^n \left\{1 + O\left(\frac{1}{n}\right)\right\}$$

where $a' = n/w' > a$. Since

$$\frac{dF}{da} = \frac{\theta_\sigma(r)}{a^2 r^{1/a}} \ln r < 0 ,$$

it follows that $E_w = o(A_n(w))$ as $n \to \infty$, which implies (23).

Proof of Theorem 1. We now assume $a \geq 1$. Then

$$\log_2 S_{n,\sigma} = -\frac{n}{\sigma} \log_2 \left\{ \frac{n}{e \sigma 2^\sigma} \Gamma \left(1 + \frac{1}{\sigma} \right)^{-\sigma} \right\} + O(\log_2 n) \tag{24}$$

from (17),

$$\log_2 B_{n,p}(w-1) = n \log_2 \frac{\theta_\sigma(r)}{r^{1/a}} + O(\log_2 n) \tag{25}$$

from Lemmas 3, 4, where r is given by (21), and so from Theorem 2 an $[n, k, w, p, \sigma]$ code exists with $n = p^\sigma$, $w = [n/a]$ and

$$k = n - \frac{\sigma n}{\log_2 n} \log_2 \frac{\theta_\sigma(r)}{r^{1/a}} + O(1) . \tag{26}$$

From (16), (24)-(26) the resulting lattice has density Δ satisfying

$$\log_2 \Delta \geq -n \log_2 \left\{ \frac{(e\sigma)^{-1/\sigma}}{\Gamma(1 + \frac{1}{\sigma})} \frac{a^{1/\sigma} \theta_\sigma(r)}{r^{1/a}} \right\} + O(\log_2 n) . \tag{27}$$

We must therefore choose a to minimize

$$f(a) = \frac{a^{1/\sigma} \theta_\sigma(r)}{r^{1/a}} , \qquad a \geq 1 . \tag{28}$$

Note that

$$f'(a) = \frac{df}{da} = \frac{f(a)}{a} \left[\frac{1}{\sigma} - \frac{1}{a} \ln \frac{1}{r} \right] . \tag{29}$$

We treat the cases $\sigma = 1$, $\sigma = 2$ and $\sigma \geq 3$ separately.

Case $\sigma = 1$. We have $\theta_1(x) = (1 + x)/(1 - x)$, $r = \sqrt{a^2 + 1} - a$, and it can be checked that $df/da \geq 0$ for $a \geq 1$. Thus the right-hand size of (27) is maximized by taking $a = 1$, at which point $r = \sqrt{2} - 1$, $\theta_1(r) = \sqrt{2} + 1$ and

$$\log_2 \Delta \geq - n \log_2 \frac{3 + 2\sqrt{2}}{e} + O(\log_2 n) ,$$

which establishes Theorem 1 in the case $\sigma = 1$.

Case $\sigma = 2$. For $\sigma = 2$, $r = r(a)$ (given by Eq. (21)) is a monotonic decreasing function of a for $a \geq 0$. (To see this, consider instead $a(r) = \theta_2(r) / r\, \theta_2'(r)$ as a function of r. Then

$$\frac{da}{dr} = \frac{r\, \theta_2'(r)^2 - \theta_2(r)\, \theta_2'(r) - r\, \theta_2(r)\, \theta_2''(r)}{r^2\, \theta_2'(r)^2} ,$$

and the numerator is

$$- 2 \sum_{m=1}^{\infty} m^4 r^{m^2 - 1} - 4 \sum_{m=1}^{\infty} \sum_{i = m+1}^{\infty} (m^2 - i^2)^2 r^{m^2 + i^2 - 1} ,$$

which is negative.) To investigate the behaviour as $r \to 1$, we set $r = e^{-t}$ $(t > 0)$ and use the Jacobi identity ([1, p. 11], [18, p. 475])

$$\sum_{m=-\infty}^{\infty} e^{-t m^2} = \sqrt{\frac{\pi}{t}} \sum_{m=-\infty}^{\infty} e^{-\pi^2 m^2/t} \ . \tag{30}$$

From this it can be shown that, as $r \to 1$, $a \to 0$ and also that $f(0) = \sqrt{2 \pi e}$, $f'(0) = 0$. Furthermore, $f'(a) \geq 0$ for $a \geq 0$. Figure 1 shows the graphs of $r(a)$ and $f(a)$ for $a \geq 0$. The graph of $f(a)$ is quite flat; $f'(1)$ is only 0.00000043648... .

We conclude that, for $\sigma = 2$, the right-hand side of (27) is maximized by taking $a = 1$, so that r is defined by Eq. (6).

To solve (6), we set $r = e^{-1/u}$ and apply (30) with $t = 1/u$ to both sides, obtaining

$$\sum_{m=-\infty}^{\infty} \left(1 - \frac{u}{2} + \pi^2 m^2 u^2\right) e^{-\pi^2 m^2 u} = 0 \ . \tag{31}$$

Then (31) implies (5) and

$$\theta_2(r) = \sqrt{2 \pi} \left(1 + \frac{4 \pi^2 + 2}{e^{2 \pi^2}} - \frac{64 \pi^6 - 24 \pi^4}{e^{4 \pi^2}} + \cdots\right) ,$$

and hence (3). Since

$$e^{2 \pi^2} = 3.73791533224226189756... \cdot 10^8 ,$$

r, $\theta_2(r)$ and c_2 are very close to $1/\sqrt{e}$, $\sqrt{2 \pi}$ and 1, respectively. (The 20-digit decimal expansions for r and c_2 stated in Theorem 1 were obtained by direct computer solution of (6).)

Case $\sigma \geq 3$. For $\sigma \geq 3$, $f(a)$ has a local minimum in the range $a \geq 1$ at the point where $f'(a) = 0$, i.e. at $a = \sigma \ln(1/r)$, which implies (8). This completes the proof of Theorem 1.

The values of a , r and the coefficient c_σ in the bound $\log_2 \Delta \geq - c_\sigma n + O(\log_2 n)$ are summarized in Table II.

For σ large, but fixed, and $n \rightarrow \infty$, we find from Eq. (8) that

$$\ln \frac{1}{r} = \ln (2\sigma) + \ln \ln (2\sigma) + \text{smaller terms} ,$$

$$a = \sigma \log_2 (2\sigma) + \text{smaller terms} ,$$

and so

$$\log_2 \Delta \geq - \frac{n}{\sigma} \frac{\ln \ln \sigma}{\ln 2} + \text{smaller terms} , \tag{32}$$

which should be compared with the upper bound stated at the end of Section 1.

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Table I

Codes $[n, k, w, p, \sigma]$ guaranteed by Theorem 2,
density Δ of corresponding lattice in $\mathbf{R}^n$, and
density Δ_{MH} guaranteed by Minkowski-Hlawka bound.

n	k	w	p	σ	$\log_2 \Delta$	$\log_2 \Delta_{MH}$
3	1	3	3	1	−1.000	−1.734
7	2	7	7	1	−6.684	−5.988
17	7	17	17	1	−19.725	−16.000
31	16	31	31	1	−33.396	−30.000
127	81	127	127	1	−143.078	−126.000
9	7	2	3	2	−5.947	−6.270
25	16	6	5	2	−23.613	−20.799
49	30	15	7	2	−47.511	−43.828
121	75	46	11	2	−121.137	−114.524
169	107	68	13	2	−167.979	−162.042
27	21	3	3	3	−18.123	−26.000
125	93	15	5	3	−100.034	−124.000
81	63	6	3	4	−49.836	−80.000

Table II

σ	a	r	c_σ
1	1	0.41421356...	1.10041157...
2	1	0.60653072...	1.0000000077...
3	7.8539...	0.07295125...	0.82260038...
4	12.5643...	0.04323663...	0.67424266...
5	17.4749...	0.03034913...	0.56924054...

Figure Caption

The case $\sigma = 2$: graphs of $r(a)$ and $f(a)$ for $0 \le a \le 6$.

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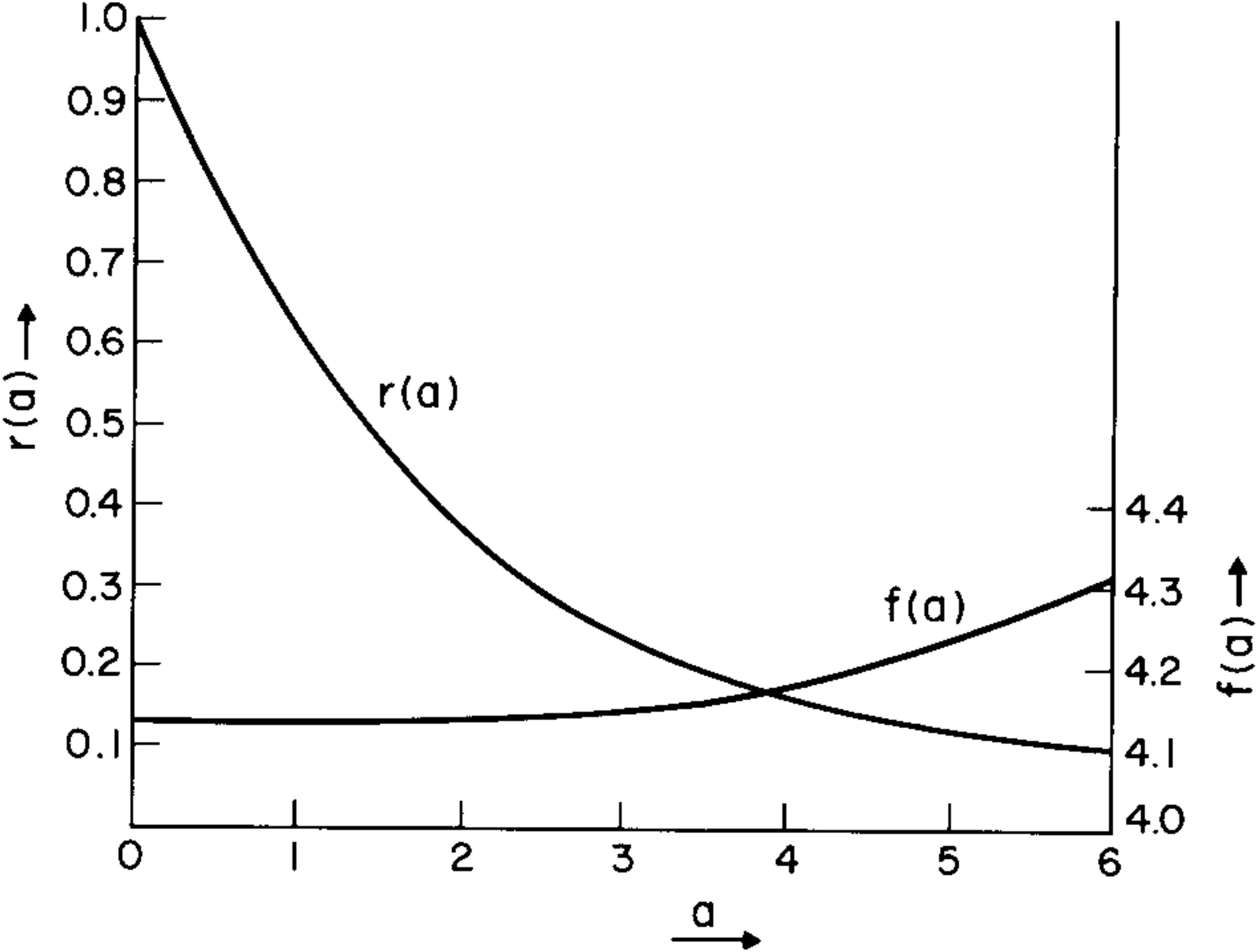
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r(a)
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4.4
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1
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3
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5
6
a