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SPHERE PACKINGS AND ERROR-CORRECTING CODES

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Error-correcting codes are used in several constructions for packings of equal spheres in n-dimensional Euclidean spaces E^n. These include a systematic derivation of many of the best sphere packings known, and construction of new packings in dimensions 9–15, 36, 40, 48, 60, and 2^m for $m \geqq 6$. Most of the new packings are nonlattice packings. These new packings increase the previously greatest known numbers of spheres which one sphere may touch, and, except in dimensions 9, 12, 14, 15, they include denser packings than any previously known. The density Δ of the packings in E^n for $n = 2^m$ satisfies

$$\log \Delta \sim -\tfrac{1}{2}n \log \log n \quad \text{as} \quad n \to \infty.$$

1. PRELIMINARIES

1.1. Introduction. In this paper we make systematic use of error-correcting codes to obtain sphere packings in E^n, including several of the densest packings known and several new packings. By use of cross-sections we then obtain packings in spaces of lower dimension, and by building up packings by layers we obtain packings in spaces of higher dimension. Collectively, these include all of the densest packings known, and further new packings are also constructed.

Part 1 of the paper is devoted to groundwork for the constructions. § 1.2 introduces sphere packings, and §§ 1.3–1.8 survey the error-correcting code theory used in the later Parts. Part 2 describes and exploits Construction A, which is of main value in up to 15 dimensions. Part 3 describes Construction B, of main value in 16–24 dimensions. Part 4 digresses to deal with packings built up from layers, while Part 5 gives some special constructions for dimensions 36, 40, 48 and 60. Part 6 deals with Construction C, applicable to dimensions $n = 2^m$ and giving new denser packings for $m \geqq 6$. We conclude with tables summarizing the results. Table I, for all $n \leqq 24$, supersedes the tables of [**18**; **19**], and Table II gives results for selected $n > 24$. The tables may be used as an index giving references to the sections of the paper in which the packings are discussed.

Partial summaries of this work have appeared in [**22**; **23**]. General references for sphere packing are [**18**; **19**; **31**] and for coding theory [**4**; **25**].

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1.2. The sphere packing problem. Let S be an open n-dimensional sphere of radius ρ and content $V_n\rho^n$, where

$$V_n = \frac{\pi^{\frac{1}{2}n}}{\Gamma(\frac{1}{2}n + 1)}.$$

If $\mathbf{a}_1, \mathbf{a}_2, \ldots$ is an infinite sequence of vectors in E^n, then the translated spheres $\mathbf{a}_1 + S$, $\mathbf{a}_2 + S, \ldots$ form a *packing* if they are pairwise disjoint.

The packing is said to be a *lattice packing* if the centres $\mathbf{a}_1, \mathbf{a}_2, \ldots$ are a set of all vectors of the form

$$u_1\mathbf{b}_1 + u_2\mathbf{b}_2 + \ldots + u_n\mathbf{b}_n,$$

where $\mathbf{b}_1, \ldots, \mathbf{b}_n$ are linearly independent vectors in E^n and $u_1, \ldots, u_n$ are arbitrary integers. The *density* of the packing is the fraction of space covered by the spheres (see [**31**, Chapter 1]), and the *centre density*, denoted by δ, is the density divided by V_n, i.e., the number of centres of unit spheres per unit content. The sphere packing problem is to choose the centres so as to maximize the density.

A related problem is to find the greatest number of spheres that can touch an equal sphere in E^n. The number of spheres touching a given sphere in a packing is the *contact number*, denoted by τ, of that sphere. For large n the two problems may have different solutions. For example in E^9 the packing Λ_9 having the highest known density is different from the packing $P9a$ having the highest known contact number. Neither packing has been proved to be optimal, although it is known (see [**36**] and § 2.7 below) that no lattice packing in E^9 can have as high a contact number as $P9a$.

Frequent use will be made of the *coordinate array* of a point in E^n having integer coordinates. This is formed by setting out in columns the values of the coordinates in the binary scale. The 1's row of the array comprises the 1's digits of the coordinates, and thus has 0's for even coordinates and 1's for odd coordinates. The 2's, 4's, 8's, ... rows similarly comprise the 2's, 4's, 8's, ... digits of the coordinates (see [**19**, § 1.42]). Complementary notation is used for negative integers. For packings based on codes the coordinates are restricted only in the first few rows, later rows being arbitrary except for being ultimately identical.

Let $\mathbf{x}$ and $\mathbf{y}$ be binary vectors. Then $\mathbf{x}$ is said to *contain* $\mathbf{y}$ if $\mathbf{x}$ has a 1 in each coordinate where $\mathbf{y}$ has a 1.

1.3. Error-correcting codes.

Definitions. (i) The *Hamming distance* between two vectors is the number of components where they differ. The *weight* of a vector $\mathbf{x}$ is the number of nonzero components it contains, and is denoted by $wt(\mathbf{x})$.

(ii) An (n, M, d) *code* over GF(q) is a set of M vectors (called *codewords*) of length n with symbols from GF(q) such that the Hamming distance between any two codewords is at least d.

If $\mathbf{c}$ is a codeword, $A_i(\mathbf{c})$ will denote the number of codewords at a Hamming distance of i from $\mathbf{c}$. Of course

$$\sum_i A_i(\mathbf{c}) = M.$$

For group codes (see below) and some nongroup codes, $A_i(\mathbf{c})$ is independent of $\mathbf{c}$ and will be denoted by A_i.

An (n, M, d) code can correct at least $[\frac{1}{2}(d - 1)]$ errors when used as an error-correcting code. A basic (unsolved) problem of coding theory is to maximize M given n and d. For binary codes this is equivalent to choosing as many vertices as possible of an n-dimensional unit cube while maintaining a Euclidean distance of at least $\sqrt{d}$ between any two of the chosen vertices. It is not surprising therefore that coding theory can be of assistance in sphere packing, provided that the packing in the cube can be extended to a packing in the full space E^n. This paper describes three general constructions (§§ 2.1, 3.1, 6.1) and a number of special constructions for doing this.

If the componentwise sum of any two codewords is a codeword, and if the componentwise product of any codeword and any element of GF(q) is a codeword, then the code is a *group* code. In this case $M = q^k$ for some integer k. A code is *cyclic* if whenever $c_0c_1 \dots c_{n-1}$ is a codeword so is $c_1c_2 \dots c_{n-1}c_0$.

It is convenient to represent a codeword $\mathbf{c} = c_0c_1 \dots c_{n-1}$ by the polynomial $c(x) = c_0 + c_1x + \dots + c_{n-1}x^{n-1}$ in the ring $R_n(x)$ of polynomials modulo $x^n - 1$ with coefficients from GF(q). A cyclic shift of $\mathbf{c}$ is then represented by $xc(x)$, and so a cyclic group code is represented by an ideal in $R_n(x)$. Let $g(x)$ be the generator of this ideal. It is easily seen that $g(x)$ divides $x^n - 1$ over GF(q) and that the number of codewords is $q^{n-\deg g(x)}$.

Let n be odd and let α be a primitive nth root of unity, so that

$$x^n - 1 = \prod_{i=0}^{n-1} (x - \alpha^i).$$

If m is the multiplicative order of q modulo n, $\alpha \in \text{GF}(q^m)$. Also,

$$g(x) = \prod_{i \epsilon K} (x - \alpha^i),$$

for some set $K \subseteq \{0, 1, \dots, n - 1\}$, where since $g(x)$ divides $x^n - 1$ over GF(q), $i \in K$ implies $qi \in K$.

An (n, M, d) code over GF(q) can be made into an $(n + 1, M, d')$ code, called an *extended* code, by appending an overall parity check

$$c_n \equiv - \sum_{i=0}^{n-1} c_i \text{ (modulo } q)$$

to each codeword $c_0c_1 \dots c_{n-1}$. If $q = 2$ and d is odd the extended code has $d' = d + 1$.

The following examples of codes are specified in terms of the set K.

1.4. *BCH* codes.

Definition. The *Bose-Chaudhuri-Hocquenghem* (*BCH*) binary code of length $n = 2^m - 1$ and designed distance d is the cyclic code whose generator polynomial has as roots exactly $\alpha, \alpha^2, \alpha^3, \ldots, \alpha^{d-1}$ and their conjugates. (See, for example, [**4**, Chapters 7 and 12].)

The actual minimum Hamming distance between codewords is at least the designed distance, and may exceed it.

1.5. Reed-Muller codes.

Definitions. (i) The weight $W(k)$ of a non-negative integer k is the number of ones in the binary expansion of k.

(ii) For $1 \leqq r \leqq m - 2$, the rth order binary *punctured Reed-Muller* code of length $n = 2^m - 1$ is the cyclic code whose generator polynomial has as roots those α^k such that $1 \leqq k \leqq 2^m - 2$ and $1 \leqq W(k) \leqq m - r - 1$.

(iii) The rth order *Reed-Muller* (*RM*) code of length 2^m is formed by appending an overall parity check to the rth order punctured Reed-Muller code, for $1 \leqq r \leqq m - 2$. The 0th, $(m - 1)$th and mth order Reed-Muller codes are the trivial $(2^m, 2, 2^m)$, $(2^m, 2^{2^m-1}, 2)$ and $(2^m, 2^{2^m}, 1)$ codes, consisting respectively of the all-zeros and all-ones codewords, of all codewords of even weight, and of all codewords, of length 2^m. (See, for example, [**4**, § 15.3].)

An alternative definition of *RM* codes is given in [**25**], and the equivalence of the two definitions is shown in [**4**], for example. The relation between the k-parity of a vector as defined in [**18**] and *RM* codes is that a binary vector of length 2^m has k-parity if and only if it is in the $(m - k)$th order *RM* code as defined in [**25**].

The rth order *RM* code has

$$d = 2^{m-r}, M = \exp_2 \sum_{i=0}^{r} \binom{m}{i},$$

and the number of codewords of minimum weight is

$$A_d = 2^r \cdot \prod_{i=0}^{m-r-1} \frac{2^{m-i} - 1}{2^{m-r-i} - 1}.$$

THEOREM 1.5.1. *The codewords of weight* 4 *in the* $(2^m, 2^{2^m-m-1}, 4)$ $(m - 2)$*th order RM code form the tetrads of a Steiner system* $S(3, 4, 2^m)$.

THEOREM 1.5.2. (See [**4**, p. 362].) *The BCH code of designed distance d is contained in that of designed distance d* $-$ 1. *The rth order RM code is contained in the* $(r + 1)$*th order RM code. The rth order RM code is a subcode of the code obtained by appending an overall parity check to the BCH code of designed distance* $2^{m-r} - 1$.

1.6. Quadratic residue codes. The *quadratic residue* code of odd prime length n over GF(q) is the cyclic code whose generator polynomial has roots $\{\alpha^r: r =$ quadratic residue modulo $n\}$ (see [**4**, § 15.2]). This is an $(n, q^{\frac{1}{2}(n+1)}, d)$ code for some d. We shall be concerned only with extended quadratic residue codes.

The most important code of this type is the $(24, 2^{12}, 8)$ binary Golay code (see [**13**; **4**, p. 359]), having weight distribution $A_0 = 1 = A_{24}$, $A_8 = 759 = A_{16}$, and $A_{12} = 2576$. The codewords of weight 8 form the octuples of a Steiner system $S(5, 8, 24)$. For later use we observe that if the eight coordinates belonging to an octuple are deleted from all the codewords, the truncated codewords form two copies of a $(16, 2^{11}, 4)$ 2nd order *RM* code (see [**5**]).

Other extended quadratic residue codes are the $(12, 3^6, 6)$ ternary Golay code (see [**13**; **4**, p. 359]), and the $(24, 3^{12}, 9)$ and $(48, 3^{24}, 15)$ ternary codes studied by Assmus and Mattson [**1**; **2**]. In these three codes the codewords of maximum weight consist exactly of the rows of an $n \times n$ Hadamard matrix and its negative.

1.7. Pless codes. Pless [**27**; **28**] has constructed $(2q + 2, 3^{q+1}, d)$ codes over GF(3) for every odd prime power $q \equiv -1$ (modulo 3). The first five are the $(12, 3^6, 6)$ Golay code again, and $(24, 3^{12}, 9)$, $(36, 3^{18}, 12)$, $(48, 3^{24}, 15)$, $(60, 3^{30}, 18)$ codes. Pierce [**26**] has found the weight distributions of the last two. The codes of length 24 and 48 have the same weight distribution as the corresponding quadratic residue codes of § 1.6, but have different symmetry groups and are not equivalent.

The following properties of these codes will be used later.

1.7.1 The all-ones codeword is in the code.

1.7.2 The numbers of -1's, 0's or $+1$'s in any codeword are each multiples of 3.

1.7.3 The codewords of maximum weight contain among them the rows of an Hadamard matrix and its negative, and for lengths 12, 24 and 48 these are the only codewords of maximum weight.

1.7.4 For lengths $n = 12$, 24 and 48, the codewords containing only 0's and $+1$'s are of the types 0^n, 1^n, and $0^{\frac{1}{2}n}\, 1^{\frac{1}{2}n}$.

1.7.5 The weight distribution of the $(48, 3^{24}, 15)$ code is in part $A_{15} = 415104$, $A_{18} = 20167136$, $A_{45} = 6503296$, $A_{48} = 96$ (Pierce [**26**]).

2. CONSTRUCTION *A*

2.1. The construction. Let $\mathscr{C}$ be an (n, M, d) binary code. The following construction specifies a set of centres for a sphere packing in E^n.

Construction A. $\mathbf{x} = (x_1, \ldots, x_n)$ is a centre if and only if $\mathbf{x}$ is congruent (modulo 2) to a codeword of $\mathscr{C}$.

Thus a point $\mathbf{x}$ with integer coordinates is a centre if and only if the 1's row of the coordinate array of $\mathbf{x}$ is in $\mathscr{C}$.

A lattice packing is obtained if and only if $\mathscr{C}$ is a group code. Construction A is a generalization of the construction of Λ_4 as given in [**18**, § 1.1].

2.2. Centre density. On the unit cube at the origin,

$$\{0 \leqq x_i \leqq 1 : i = 1, \ldots, n\},$$

the centres are exactly the M codewords. All other centres are obtained by adding even integers to any of the coordinates of a codeword. This corresponds to shifting the unit cube by two in any direction. Thus all the centres may be obtained by repeating a building block consisting of a $2 \times 2 \times \ldots \times 2$ cube with the codewords marked on the vertices of the $1 \times 1 \times \ldots \times 1$ cube in one corner.

Each copy of the $2 \times 2 \times \ldots \times 2$ cube contributes M spheres of radius ρ (say), so the centre density obtained from Construction A is

$$\delta = M\rho^n 2^{-n}.$$

If two centres are congruent to the same codeword their distance apart is at least 2. If they are congruent to different codewords then they differ by at least 1 in at least d places and so are at least $\sqrt{d}$ apart. Thus we may take the radius of the spheres to be

$$\rho = \tfrac{1}{2} \min(2, \sqrt{d}).$$

2.3. Contact numbers. Let S be a sphere with centre $\mathbf{x}$, where $\mathbf{x}$ is congruent to the codeword $\mathbf{c}$. Candidates for centres closest to $\mathbf{x}$ are as follows. (a) There are $2n$ centres of the type $\mathbf{x} + (\pm 2)0^{n-1}$ at a distance of 2 from $\mathbf{x}$. (b) Since there are $A_d(\mathbf{c})$ codewords at a distance of d from $\mathbf{c}$, there are $2^d A_d(\mathbf{c})$ centres of the type $\mathbf{x} + (\pm 1)^d 0^{n-d}$ at a distance of $\sqrt{d}$ from $\mathbf{x}$.

Therefore, the number of spheres touching S is

$$\begin{aligned} 2^d A_d(\mathbf{c}) &\text{ if } d < 4, \\ 2n + 16A_4(\mathbf{c}) &\text{ if } d = 4, \\ 2n &\text{ if } d > 4. \end{aligned}$$

2.4. Dimensions 3–6. Let $\mathscr{C}$ be the $(n, 2^{n-1}, 2)$ group code consisting of all codewords of even weight. Then $A_2 = \frac{1}{2}n(n-1)$, and from the construction we obtain a lattice packing D_n in E^n, with $\rho = 2^{-\frac{1}{2}}$, $\delta = 2^{-\frac{1}{2}n-1}$ and $\tau = 2n(n-1)$. The centres of D_n are alternate vertices of the regular cubic lattice.

For $n = 3, 4, 5$, D_n is the densest possible lattice packing in E^n, and is denoted by Λ_n, since each is a section of the 24-dimensional packing Λ_{24} (see § 4.5). In E^3 and E^5 equally dense nonlattice packings exist (see [**20**] and § 4.2 below).

The densest six-dimensional lattice packing, Λ_6, is not directly given by Construction A, but will be obtained by stacking layers of Λ_5 in § 4.2, and as a section of Λ_7 in § 4.5.

2.5. Dimensions 7 **and** 8. From now on we apply Construction A to codes with minimum distance $d = 4$. The packing obtained is of spheres of unit radius, and has centre density $\delta = M2^{-n}$ and contact numbers $\tau = 2n + 16A_4(\mathbf{c})$.

Let H_n denote the binary matrix obtained from an $n \times n$ Hadamard matrix upon replacing $+1$'s by 0's and -1's by 1's. We assume that H_n has been normalized so that the first row and column are all zeros. The rows of H_8 with the first column deleted form a (7, 8, 4) group code with $A_4 = 7$, the codewords being the vertices of a seven-dimensional simplex. From this code we obtain Λ_7, the densest lattice packing in E^7, with $\delta = 2^{-4}$ and $\tau = 2\cdot 7 + 16\cdot 7 = 126$.

The rows of H_8 together with their complements form an (8, 16, 4) 1st order Reed-Muller code, with $A_4 = 14$. From this we obtain Λ_8, the densest lattice packing in E^8, with $\delta = 2^{-4}$ and $\tau = 2\cdot 8 + 14\cdot 16 = 240$.

2.6. Dimensions 9–12. Let $\mathscr{D}$ be the 11×11 matrix consisting of the vector 11011100010 (with 1's at position zero and at the quadratic residues modulo 11) together with its cyclic shifts. Then the modified Hadamard matrix H_{12} may be taken as

$$\begin{pmatrix} 0 & \mathbf{0}^T \\ \mathbf{0} & \mathscr{D} \end{pmatrix}.$$

The sums modulo 2 of pairs of rows of H_{12} and the complements of such sums form the 132 vectors of a Steiner system $S(5, 6, 12)$ (see [**18**], for example). Since no two of these vectors can overlap in more than four places, they form a (12, 132, 4) code, every codeword having weight six. This code may be increased to a (12, 144, 4) nongroup code by adding six codewords of type $0^{10}\,1^2$ and six of type $0^2\,1^{10}$, the six 1^2 and the six 0^2 being disjoint sets.

Several different versions of this code are possible, depending on the relationship between the positions of the ones in the 12 "loose" codewords and the vectors of the Steiner system.

By shortening the (12, 144, 4) code we obtain (11, 72, 4), (10, 38, 4) and (9, 20, 4) codes. Codes equivalent to these four were first given by Golay [**14**] and Julin [**16**], and alternative constructions and generalizations are given in [**32**; **33**].

Applying Construction A to the various versions of the (12, 144, 4) code we obtain nonlattice packings in E^{12} with centre density $\delta = 144\cdot 2^{-12} = 2^{-8}\cdot 3^2 = .03516$, which is less than that of K_{12} (see [**12**]), but in the most favourable versions with some spheres touching as many as 840 others, as we now show.

Let $\mathbf{c}$ be a codeword of weight 6. Any vector of weight 4 and length 12 is contained in exactly 8 vectors of weight 5, and therefore in exactly 4 codewords of weight 6 (since any vector of weight 5 is contained in a unique codeword of weight 6). So the number of codewords of weight 6 that are at a Hamming distance of 4 from $\mathbf{c}$ is $3\binom{6}{4} = 45$.

If the 12 loose codewords are chosen so that there are three words $0^{10}\,1^2$ whose 1's coincide with pairs of 1's from $\mathbf{c}$, and three words $0^2\,1^{10}$ whose 0's coincide with pairs of 0's from $\mathbf{c}$, then there are an additional 6 codewords at a Hamming distance of 4 from $\mathbf{c}$ and $A_4(\mathbf{c}) = 45 + 6 = 51$. Based on these codes we obtain packings in E^{12} with a maximum contact number of $2 \cdot 12 + 16 \cdot 51 = 840$ for those spheres whose centres are congruent to $\mathbf{c}$.

There are still several inequivalent versions of these packings, depending on the choice of the remaining loose codewords, but they all have the same maximum contact number of 840, and are collectively called $P12a$ in Table I.

Other choices for the 12 codewords give packings with maximum contact numbers of 824 or 808.

It can be shown that, regardless of the choice of the 12 loose codewords, the average contact number obtained is equal to $770\frac{2}{3}$. (This is true even for those packings where the maximum contact number is less than 840.)

In E^{11} we equate $x_{12} = 1$ in $P12a$, and find the maximum contact number to be 566 in the most favourable cases, collectively called $P11a$, but it can be only 550 or 534 in other cases (if $0^{10}\,1$ is altered to 0^{11}). Except in this last case the average is $519\frac{7}{9}$.

In E^{10}, if we equate $x_{11} = x_{12} = 1$, where x_{11}, x_{12} are a pair of both $0^{10}\,1^2$ and $0^2\,1^{10}$, we obtain packings $P10b$ with $\delta = 2^{-8} \cdot 3^2$ and a maximum contact number of 500. If instead we equate $x_{12} = 1$, $x_{10} = 0$ where x_{10}, x_{12} is not a pair of either form, we obtain packings with $\delta = 2^{-9} \cdot 19$. In the most favourable cases, called $P10a$, the maximum contact number is 372. The average contact number is $353\frac{9}{19}$ for $P10a$ and $340\frac{1}{3}$ for $P10b$.

In E^9 we equate $x_{10} = 0$, $x_{11} = x_{12} = 1$ and obtain packings $P9a$ with $\delta = 2^{-7} \cdot 5$ and a maximum contact number of 306; the average contact number is $235\frac{3}{5}$.

An alternative derivation of $P10b$ may be given. The tetrads of a Steiner system $S(3, 4, 10)$ (see [**37**]) form 30 codewords of length 10, weight 4 and Hamming distance at least 4 apart. For example the tetrads may be taken to be the cyclic permutations of 1110001000, of 1101100000 and of 1010100100. By including the zero codeword and five codewords of weight 8, several different (10, 36, 4) codes are obtained. Construction A then gives the packings $P10b$.

Similarly, the packings $P9a$, $P10a$ and $P11a$ may be obtained by applying Construction A to the (9, 20, 4), (10, 38, 4) and (11, 72, 4) codes. All these are nonlattice packings.

2.7. Comparison of lattice and nonlattice packings. It is still an open question whether there exists any nonlattice packing with density exceeding that of the densest lattice packing. But the nonlattice packings $P10a$, $P11a$, and $P13a$ of § 4.3 below, have density greater than that of the densest ***known*** lattice packings. It is hoped that this will stimulate further investigation of whether these are the densest possible lattice packings in these dimensions.

In this connection it should be mentioned that G. L. Watson [**36**] has recently proved that $\Lambda_1 - \Lambda_9$ have the maximum possible contact numbers for lattice packings. Therefore the nonlattice packing $P9a$ has some spheres touching more spheres than is possible in any lattice packing.

3. CONSTRUCTION B

3.1. The construction. Let $\mathscr{C}$ be an (n, M, d) binary code with the property that the weight of each codeword is even. A sphere packing in E^n is given by:

Construction B. $\mathbf{x} = (x_1, \ldots, x_n)$ is a centre if and only if $\mathbf{x}$ is congruent (modulo 2) to a codeword of $\mathscr{C}$, and $\sum_{i=1}^{n} x_i$ is divisible by 4.

Thus a point $\mathbf{x}$ with integer coordinates is a centre if and only if the 1's row of the coordinate array of $\mathbf{x}$ is a codeword $\mathbf{c} \in \mathscr{C}$ and the 2's row has either even weight if the weight of $\mathbf{c}$ is divisible by 4, or odd weight if the weight of $\mathbf{c}$ is divisible by 2 but not by 4.

As in Construction A a lattice packing is obtained if and only if $\mathscr{C}$ is a group code. Construction B is a generalization of the construction of Λ_8 as given in [**18**, § 1.1].

3.2. Centre density and contact numbers. The number of centres is half that of Construction A, so $\delta = M\rho^n 2^{-n-1}$.

Let S be a sphere with centre $\mathbf{x}$, where $\mathbf{x}$ is congruent to a codeword $\mathbf{c}$. Candidates for centres closest to $\mathbf{x}$ are (a) the $2^2\binom{n}{2}$ centres of the type $\mathbf{x} + (\pm 2)^2 0^{n-2}$, and (b) the $2^{d-1}A_d(\mathbf{c})$ centres congruent to the codewords differing minimally from $\mathbf{c}$. Therefore the contact number of S is

$$\begin{aligned} 2^{d-1}A_d(\mathbf{c}) &\text{ if } d < 8, \\ 2n(n-1) + 128A_8(\mathbf{c}) &\text{ if } d = 8, \\ 2n(n-1) &\text{ if } d > 8; \end{aligned}$$

and S has radius $\rho = \frac{1}{2}\min(\sqrt{d}, \sqrt{8})$.

3.3. Dimensions 8, 9 **and** 12. In E^8 we apply Construction B to the trivial code $\{0^8, 1^8\}$ to obtain the packing Λ_8 of § 2.5 again. In E^9 we use the code $\{0^9, 1^8 0\}$ and obtain the lattice packing Λ_9 with $\rho = \sqrt{2}$, $\delta = 2^{-4\frac{1}{2}}$ and $\tau = 272$.

In E^{12} we use the (12, 24, 6) code formed from the rows of H_{12} and their complements to obtain the nonlattice packing L_{12} (see [**18**]) with $\delta = 2^{-16} \cdot 3^7$

and $\tau = 704$. The $(12, 4, 8)$ code $\{0^{12}, 0^4 1^8, 1^4 0^4 1^4, 1^8 0^4\}$ gives the lattice packing Λ_{12}, with $\delta = 2^{-5}$ and $\tau = 648$.

3.4. Dimensions 15–24. In E^{16} the $(16, 32, 8)$ 1st order *RM* code gives the lattice packing Λ_{16} having $\delta = 2^{-4}$ and $\tau = 4320$. Shortening this code by equating a coordinate to zero we obtain the $(15, 16, 8)$ simplex code, which gives the lattice packing Λ_{15} in E^{15} having $\delta = 2^{-4\frac{1}{2}}$ and $\tau = 2340$.

Let $\mathscr{C}_i$, $i = 0, 1, \ldots, 5$, denote the shortened code obtained from the $(24, 2^{12}, 8)$ Golay code by setting i coordinates equal to zero, and let a_i denote the number of codewords of weight 8 in $\mathscr{C}_i$. Then $a_0 = 759$, $a_1 = 506$, $a_2 = 330$, $a_3 = 210$, $a_4 = 130$ and $a_5 = 78$.

The sequence of lattice packings in E^{24-i}, $i = 0, 1, \ldots, 5$, obtained from C_i has $\delta_i = 2^{-\frac{1}{2}(i+2)}$, and was given in [**18**, § 2.4]. In $E^{19} - E^{21}$ these are the best known packings $\Lambda_{19} - \Lambda_{21}$, but in $E^{22} - E^{24}$ they can be improved as shown in [**19**, §§ 2.31 and 2.41] and in §§ 4.4–4.5 below.

3.5. Remark. Constructions A and B cannot be successful for large n because for any (n, M, d) code, $A_8 \leqq \binom{n}{8}$ and so from these constructions $\tau = O(n^8)$ at most. But Construction C applied to *RM* codes gives already

$$\tau \geqq \text{constant} \cdot n^{\frac{1}{2}(\log_2 n - 1)}$$

(see [**18**, § 3.4] or § 6.5 below).

4. PACKINGS BUILT UP BY LAYERS

4.1. Packing by layers (see [**20**; **21**]). The basic idea is very simple. Let Λ be a lattice sphere packing in E^n. Let $\bar{a}$ be the maximum distance of any point in E^n from a centre of Λ, and let $D(\Lambda)$ be the set of all points at a distance of at least $\bar{a}$ from every centre. ($D(\Lambda)$ is also well-defined for periodic but nonlattice packings.)

A *layer* of spheres in E^{n+1} is a set of spheres whose centres lie in a hyperplane, and whose cross section in the hyperplane is Λ. In the hyperplane of centres of such a layer there are two distinguished sets of points, the set C of centres of Λ, and the set D of the points most distant from C.

We shall try to build up a dense sphere packing in E^{n+1} by stacking such layers as closely as possible. We therefore place adjacent layers so that the set C of one layer is opposite to some or all of the set D of the next. Since the layers are lattice packed, if one point of C is opposite to a point of D, then so are all the points of C.

It may happen that D is more numerous (in any finite region) than C, in which case several inequivalent packings may be produced in E^{n+1}. These may be lattice or nonlattice packings or both. Examples will be found below and in succeeding sections. (So far we have not found an example of only a nonlattice packing being formed, but it does not seem impossible.)

For example let Λ be the square lattice in E^2. Then C comprises the vertices of the squares and D their centres. The layers are placed so that the vertices of the squares in one layer are opposite to the centres of the squares in the next. Since C and D are equally numerous, this arrangement is unique, and the lattice packing Λ_3 in E^3 is obtained.

The situation is different if Λ is chosen to be Λ_2, the triangular lattice in E^2. Now D is the set of centres of triangles, and is twice as numerous as C. If we regard the triangles as coloured black and white alternately, the spheres of each layer can be placed opposite to either all the black triangles or all the white triangles of the adjacent layer. If the layers are so stacked that the spheres in the two layers adjacent to any layer are opposite to triangles of different colours, then the lattice Λ_3 is obtained. But if the adjacent spheres on both sides of each layer are opposite to triangles of the same colour, then a nonlattice packing, the *hexagonal close packing*, is obtained, having the same density as Λ_3. If we require uniform packings there is no further choice, as all layers have to be fitted alike, and this construction gives precisely these two packings.

In general, the layers are placed just far enough apart that the smallest distance between centres in adjacent layers equals the smallest distance between centres in the same layer. There are four cases which can arise here. The spheres of one layer may (i) not reach, (ii) just touch, or (iii) penetrate, the central hyperplane of an adjacent layer, or (iv) it may be possible to fit the spheres of one layer into the spaces between the spheres of the adjacent layer and so merge the two layers.

Examples of case (i) are $n = 2$ (as above) and 3–5 (in § 4.2).

In case (ii), extra contacts may occur because of spheres on opposite sides of a layer touching. Examples are the case $n = 6$ in § 4.2, and the packing $P13a$ in § 4.3.

In case (iii), the layers on each side of a given layer must be staggered to avoid overlapping. Examples are the construction of Λ_8 from layers of D_7 as given in [**21**], and the local arrangements $P14b$, $P15a$ in § 4.3.

Case (iv) doubles the density of the original packing. Examples of this in E^8, E^{12}, E^{24} and E^{48} are given in § 4.4 and § 5.6.

It is sometimes advantageous to stack layers which are not lattices. Examples are the local arrangements of spheres in E^{11}, E^{14} and E^{15} and the nonlattice packing $P13a$ in E^{13} to be described in § 4.3.

4.2. Dimensions 4–7. We consider packings in E^{n+1} formed by stacking layers of the packing D_n (defined in § 2.4). The centres of D_n form a lattice consisting of alternate vertices of the regular cubic lattice. For $n \geqq 3$ the cells of D_n are of two kinds: each omitted vertex of the cubic lattice is the centre of a cell β_n, while inscribed in each cube of the lattice is a cell $h\gamma_n$. The latter cells are twice as numerous as the former. We shall regard the cells $h\gamma_n$ as being coloured black and white alternately.

For $n = 3$ the cells β_3 (octahedra) are larger than the cells $h\gamma_3$ (tetrahedra), and so we place the spheres of each layer opposite to octahedra of the adjacent layer. As the octahedra and spheres are equally numerous we arrive uniquely at the lattice packing Λ_4.

For $n = 4$ the cells β_4 and $h\gamma_4$ are congruent, the lattice D_4 being the regular honeycomb $\{3, 3, 4, 3\}$. Thus there is a threefold choice for placing each layer: the spheres of each layer may be placed opposite to the omitted vertices or the black cells or the white cells. In this case we obtain either the lattice packing Λ_5 or three distinct uniform nonlattice packings, all having the same density and contact numbers (see [**20**]).

For $n > 4$ the cells $h\gamma_n$ are larger than the cells β_n, so for maximum density in E^{n+1} we stack the layers with the spheres of each layer opposite to the cubes of the adjacent layers. At each stage the spheres may be placed opposite to the black or the white cubes. For $n = 5$ or 6 we obtain in this way two uniform packings, the lattice packing Λ_{n+1} and a nonlattice packing of equal density. For $n = 5$ both have the same contact numbers. For $n = 6$ each sphere in the lattice packing Λ_7 has two more contacts than in the nonlattice packing, because it touches spheres in layers two away from it. This is an example of case (ii) above. For $n = 7$ only the lattice packing Λ_8 is produced. This is an example of case (iii) above, where the adjacent layers have to be staggered to avoid overlap.

4.3. Dimensions 11 **and** 13–15. In E^{11} we construct a local arrangement $P11b$ of 576 spheres touching one sphere, by stacking three partial layers. The central layer consists of the 500 centres of $P10b$ touching one sphere, with eleventh coordinate zero. The two outer layers consist of all points of the form

$$(c_1, \ldots, c_{10}, 0) - (\tfrac{1}{2}, \ldots, \tfrac{1}{2}, \pm \tfrac{1}{2}\sqrt{6}),$$

where $(c_1, \ldots, c_{10})$ runs through the $(10, 38, 4)$ code, and contains 76 points. It does not seem possible to extend this to a dense space packing.

In E^{13} we take any one of the packings $P12a$ and its translates by repetitions of amount $((\frac{1}{2})^{12}, \pm 1)$, and obtain a family of nonlattice packings collectively called $P13a$. Any of these packings has $\delta = 2^{-8} \cdot 3^2$ and a maximum contact number of $840 + 2 \cdot 144 + 2 = 1130$, the last 2 being caused by a sphere touching spheres two layers away. The average contact number is $1060\frac{2}{3}$.

The tetrads of a Steiner system $S(3, 4, 14)$ (see [**15**]) form 91 codewords of length 14, weight 4 and Hamming distance at least 4 apart. For example, if the fourteen objects are numbered 1 2 3 4 5 6 7 and 1′ 2′ 3′ 4′ 5′ 6′ 7′, the tetrads may be taken to be 1 2 3 6, 2′ 5′ 6′ 7′, 1 2 4 2′, 6 4′ 6′ 7′, 2 7 1′ 6′, 2 4 4′ 6′, 5 7 1′ 3′, 1 5 3′ 7′, 1 4 4′ 7′, 4 7 1′ 4′, 6 7 1′ 2′, 5 6 2′ 3′, 4 5 3′ 4′, and their transforms under the permutation (1 2 3 4 5 6 7) (1′ 2′ 3′ 4′ 5′ 6′ 7′). Construction A then gives a local arrangement $P14a$ of 1484 spheres touching one sphere. This can be improved, however, by cutting down to thirteen dimensions and rebuilding. The best section of $S(3, 4, 14)$ in E^{13} has 65 codewords, giving a local arrangement $P13b$ with $26 + 16\cdot65 = 1066$ contacts, inferior to $P13a$.

We now form a local arrangement $P14b$ in E^{14} by stacking five partial layers. The central layer is $P13b$. The adjacent layers are $(\mathbf{c}, 0) - ((\frac{1}{2})^{13}, \frac{1}{2}\sqrt{3})$ and $(\mathbf{c}', 0) - ((\frac{1}{2})^{13}, -\frac{1}{2}\sqrt{3})$, where $\mathbf{c}$ runs through the shortened Hamming (13, 256, 4) code and $\mathbf{c}'$ denotes $\mathbf{c} + (1^2, 0^{11})$ reduced modulo 2. The two outer layers contain just 2 spheres each: $(\pm1, 0^{12}, \pm\sqrt{3})$. Thus the central sphere touches $1066 + 2\cdot256 + 2\cdot2 = 1582$ others.

Similarly in E^{15} we form a local arrangement $P15a$ from five partial layers. The central layer is $P14a$. The adjacent layers are $(\mathbf{c}, 0) - ((\frac{1}{2})^{14}, \frac{1}{2}\sqrt{2})$ and $(\mathbf{c}', 0) - ((\frac{1}{2})^{14}, -\frac{1}{2}\sqrt{2})$, where $\mathbf{c}$ runs through the shortened Hamming (14, 512, 4) code and $\mathbf{c}' = \mathbf{c} + (1^2, 0^{12})$ reduced modulo 2. The outer layers are $(x_1, \ldots, x_{14}, \pm\sqrt{2})$, with $x_1, \ldots, x_{14}$ all zero except for one pair x_{2i-1}, x_{2i} which are ±1 in all four combinations. The central sphere touches $1484 + 2\cdot512 + 2\cdot28 = 2564$ others.

4.4. Density doubling and the packing in 24 dimensions. In E^8 two copies of D_8 can be fitted together without overlap to form the lattice packing Λ_8, the second copy being a translation of the first by $((\frac{1}{2})^8)$.

In E^{24}, Leech [**19**] has shown that two copies of the 24-dimensional packing obtained in § 3.4 from the $(24, 2^{12}, 8)$ Golay code may be fitted together without overlap to form the packing Λ_{24} with $\delta = 1$ and $\tau = 196560$. The second copy is a translation of the first by $(-1\frac{1}{2}, (\frac{1}{2})^{23})$. (See [**19**, § 2.31] and §§ 5.6–7 below.) Conway [**7**; **8**] has extensively studied this lattice, especially its associated groups.

4.5. Cross sections of the 24-dimensional packing. All of the densest known lattice packings in fewer than 24 dimensions occur as sections of Λ_{24}. There are two main sequences of sections, Λ_i, $i = 1, 2, \ldots, 23$ and K_i, $i = 6, 7, \ldots, 18$, where the subscript indicates the dimension. Because of the symmetry of Λ_{24}, there are several different ways of describing some of the sections.

The sequence of lattice packings Λ_i is defined as follows (see [**18**, § 2.4; **19**, § 2.41]). Λ_{23}, Λ_{22} and Λ_{21} are obtained from Λ_{24} by equating any two, three or four coordinates to each other. Λ_{21} (again), Λ_{20} and Λ_{19} are obtained by equating any three, four or five coordinates to 0. We recall that codewords of weight 8 in the $(24, 2^{12}, 8)$ code form the octuples of a Steiner system $S(5, 8, 24)$. Because of the fivefold transitivity of the Steiner system, the choice of coordinates in forming $\Lambda_{19} - \Lambda_{23}$ is arbitrary.

Λ_{19} (again), Λ_{18}, Λ_{17} and Λ_{16} are obtained by equating to 0 the sum of eight coordinates forming a Steiner octuple and also any four, five, six or all of them.

The $(16, 2^5, 8)$ *RM* code occurs as a subcode of the $(24, 2^{12}, 8)$ code, and we associate the remaining 16 coordinates with coordinate positions of this subcode. Any two intersecting octads of the (16, 2^5, 8) code split the coordinates into four tetrads.

Λ_{15}, Λ_{14}, Λ_{13} and Λ_{12} are obtained by equating to 0 the sum of the coordinates forming a tetrad and none, any one, any two or all of them. Λ_{11}, Λ_{10} and Λ_9 are obtained from Λ_{12} by equating any two, any three or all four coordinates to each other in one of the remaining tetrads. Λ_9 (again) and Λ_8 are obtained by equating to 0 any three or all four coordinates in a tetrad.

$\Lambda_7 - \Lambda_4$ are obtained as sections of Λ_8 in the same way that $\Lambda_{11} - \Lambda_8$ are obtained from Λ_{12}, defining any four coordinates of the eight to be a tetrad. Finally, $\Lambda_3 - \Lambda_1$ are obtained from Λ_4 (calling the four coordinates a tetrad) in the same way that $\Lambda_{15} - \Lambda_{13}$ were obtained from Λ_{16}.

The sequence $\Lambda_1 - \Lambda_{24}$ includes the densest known lattice packings in $E^1 - E^{10}$ and $E^{14} - E^{24}$.

In E^{12} there is a denser section of Λ_{24} than Λ_{12}, called K_{12}, obtained as follows. The 24 coordinates may be set out in a 6 × 4 array so that the coordinates in any row together with those in any column but omitting that at the intersection form a Steiner octuple (see [**35**]). K_{12} is found by equating to 0 all six of a column and also all six sums of three remaining in the rows. Thus K_{12} is a section of Λ_{18}. An alternative construction for K_{12} using 18-dimensional coordinates x_{ij}, $i = 1$ to $6, j = 1$ to 3, is given in [**19**, § 2.11]. It is shown there that K_{12} has $\delta = 3^{-3}$ and $\tau = 756$.

Λ_6 may be obtained as a section of K_{12} (in the 18-dimensional coordinates) by setting

$$x_{11} - x_{12} = x_{21} - x_{22} = x_{31} - x_{32} = x_{41} - x_{42} = x_{51} - x_{52} = x_{61} - x_{62}$$

and also $x_{13} + x_{23} + x_{33} + x_{43} + x_{53} + x_{63} = 0$.

THEOREM 4.5.1 (Conway [**9**]). *Let* Λ *be a packing in* E^n *which is one of* Λ_4, Λ_8, K_{12}, Λ_{16} *or* Λ_{24} *rescaled so as to have one sphere per unit content. To every section of* Λ *in* E^i *there is a corresponding section in* E^{n-i} *with an equal number of spheres per unit content.*

THEOREM 4.5.2 (Conway [**9**]). (a) *There is a sequence of sections of* Λ_{24}: $K_6 = \Lambda_6, K_7, \ldots, K_{17}, K_{18} = \Lambda_{18}$, *including* K_{12} *above, with the property that if* Λ_{24} *is normalized as in Theorem* 4.5.1, *the number of centres per unit content of* K_i *is equal to that of* K_{24-i}, *for* $6 \leqq i \leqq 18$.

(b) *Each* Λ_i *is the densest section of* Λ_{i+1}, *for* $1 \leqq i \leqq 23$.

(c) *Each* K_i *is the densest section of* K_{i+1} *containing or contained by* K_{12}, *for* $6 \leqq i \leqq 17$.

Proof. $\Lambda_1 - \Lambda_8$ are known to be the densest possible lattice packings. Therefore from Theorem 4.5.1, so are $\Lambda_9 - \Lambda_{15}$ as sections of Λ_{16}, $\Lambda_{16} - \Lambda_{23}$ as sections of Λ_{24}, $K_6 - K_{11}$ as sections of K_{12}, and $K_{13} - K_{18}$ as sections of Λ_{24} containing K_{12}.

Note that it is not proved that Λ_9, Λ_{10}, K_{11}, K_{12}, K_{13}, Λ_{14}, Λ_{15} are the densest sections of Λ_{24}, but any denser sections cannot form part of a sequence including any of Λ_8, K_{12} or Λ_{16}.

K_{13} may be specified in terms of the 6×4 array by equating all six coordinates of a column to 0 and also equating four of the sums of three remaining in the rows to zero and equating the other two sums of three to each other. K_{13} has $\delta = 2^{-1}\cdot 3^{-2\frac{1}{2}}$ and $\tau = 918$, and K_{11} has $\delta = 2^{-1}\cdot 3^{-2\frac{1}{2}}$ and $\tau = 432$.

K_{11}, K_{12} and K_{13} are the densest known lattice packings in these dimensions, although $P11a$ and $P13a$ are denser nonlattice packings. Also the nonlattice packing $P10a$ is denser than Λ_{10}.

Remarks. (a) Equally dense nonlattice packings are known for all the K_i and Λ_i sequences except for Λ_1, Λ_2, Λ_4, Λ_8, K_{11}, K_{12}, $\Lambda_{22} - \Lambda_{24}$.

(b) Some of the K_i's are not unique. For instance there are K_{10}'s with the same density and 270 or 276 contacts.

(c) There is a close analogy between the packings $D_3 - D_8$ of § 2.4 and the packings in $E^{19} - E^{24}$ of § 3.4, in that the first three are the densest known, the last can be doubled in density, and the densest intermediate packings can be derived either as sections of the doubled packing or by building up in layers.

5. OTHER CONSTRUCTIONS FROM CODES

5.1. A code of length 40. We construct a new binary code of length 40 which will be used in § 5.2 to obtain a sphere packing.

Let $\mathscr{C}_1$ be the $(16, 2^{11}, 4)$ 2nd order *RM* code. The 140 codewords of weight 4 form the blocks of a Steiner system $S(3, 4, 16)$. Let $\mathscr{C}_2$ be the $(24, 2^{12}, 8)$ Golay code, with coordinates arranged so that $1^8 0^{16}$ is a codeword. There are 759 codewords of weight 8, forming the blocks of a Steiner system $S(5, 8, 24)$.

Let $\mathscr{C}$ be the code of length 40 consisting of all codewords of the form $(\mathbf{x}, \mathbf{y}, \mathbf{z})$ where $\mathbf{y} \in \mathscr{C}_1$, $\mathbf{z} \in \mathscr{C}_1$, and $(\mathbf{x}, \mathbf{y} + \mathbf{z}) \in \mathscr{C}_2$.

THEOREM 5.1.1. *$\mathscr{C}$ is a* $(40, 2^{23}, 8)$ *group code.*

Proof. $\mathbf{y}$ and $\mathbf{z}$ can each be chosen in 2^{11} ways, and then, by § 1.6, there are two ways of choosing $\mathbf{x}$ so that $(\mathbf{x}, \mathbf{y} + \mathbf{z}) \in \mathscr{C}_2$. Therefore $\mathscr{C}$ contains 2^{23} codewords. Since

$$(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}, \mathbf{y} + \mathbf{z}, \mathbf{0}) + (\mathbf{0}, \mathbf{z}, \mathbf{z}), \quad wt(\mathbf{x}, \mathbf{y}, \mathbf{z}) \geqq wt(\mathbf{x}, \mathbf{y} + \mathbf{z}, \mathbf{0}) \geqq 8$$

by construction, and so $\mathscr{C}$ has minimum weight 8.

THEOREM 5.1.2. *The number of codewords of minimum weight in $\mathscr{C}$ is* 2077.

Proof. Let $(\mathbf{x}, \mathbf{y}, \mathbf{z}) = (\mathbf{x}, \mathbf{y} + \mathbf{z}, \mathbf{0}) + (\mathbf{0}, \mathbf{z}, \mathbf{z})$ be a codeword of weight 8. There are five cases. (1) If $\mathbf{y} = \mathbf{z} = \mathbf{0}$, then $\mathbf{x} = 1^8$. (2) If $\mathbf{y} \neq \mathbf{0}$, $\mathbf{z} = \mathbf{0}$, then there are 758 codewords of the form $(\mathbf{x}, \mathbf{y}) \in \mathscr{C}_2$, $\mathbf{y} \neq \mathbf{0}$. (3) If $\mathbf{y} = \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$, again there are 758 possibilities. (4) If $\mathbf{y} = \mathbf{z} \neq \mathbf{0}$, then $wt(\mathbf{y}) = 4$, $\mathbf{x} = \mathbf{0}$, and there are 140 choices for $\mathbf{y}$. (5) If $\mathbf{y} \neq \mathbf{0}$, $\mathbf{z} \neq \mathbf{0}$, $\mathbf{y} + \mathbf{z} \neq \mathbf{0}$, then $\mathbf{x} = \mathbf{0}$, $wt(\mathbf{y} + \mathbf{z}) = 8 = wt(\mathbf{y}) + wt(\mathbf{z})$. Therefore $\mathbf{y}$ and $\mathbf{z}$ have disjoint sets of 1's.

$\mathbf{y}$ can be chosen in 140 ways and for each of these there are 3 choices for $\mathbf{z}$ so that $(\mathbf{0}, \mathbf{y} + \mathbf{z}) \in \mathscr{C}_2$, giving 420 codewords. The total of (1)–(5) is 2077.

Remark. If $(\mathbf{0}, \mathbf{y}, \mathbf{z}) \in \mathscr{C}$, then $(\mathbf{0}, \mathbf{y} + \mathbf{z}) \in \mathscr{C}_2$ and so $\mathbf{y} + \mathbf{z}$ is in the $(16, 2^5, 8)$ 1st order *RM* code and therefore $(\mathbf{y}, \mathbf{z})$ is in the $(32, 2^{16}, 8)$ 2nd order *RM* code.

5.2. A lattice packing in E^{40}. A lattice packing in E^{40} can be obtained from the code $\mathscr{C}$ of § 5.1 as follows. The 40 coordinates are divided $8 + 16 + 16$ corresponding to the division of the codewords of $\mathscr{C}$.

Then $\mathbf{X} = (\mathbf{x}, \mathbf{y}, \mathbf{z})$ is a centre if and only if the 2's row of $\mathbf{X}$ is in $\mathscr{C}$, the 1's row is all 0's or is 0 on one of the sets 8, 16, 16 and 1 on the other two, and the 4's row has even weight if the set of 8 is even or has odd weight if the set of 8 is odd.

The centres closest to the origin are

x	y	z	Type	Number
zero	odd	odd	$0^8(\pm 1)^{32}$	$2^{16} = 65536$
odd	zero	odd	$(\pm 1)^8 0^{16} (\pm 1)^{16}$ with ∓ 3 for one ± 1	$24 \cdot 2^{12} = 98304$
odd	odd	zero	$(\pm 1)^8 (\pm 1)^{16} 0^{16}$ with ∓ 3 for one ± 1	$24 \cdot 2^{12} = 98304$
	all even		$0^{32}(\pm 2)^8$	$2^7 \cdot 2077 = 265856$
	all even		$0^{38}(\pm 4)^2$	$40 \cdot 39 \cdot 2 = 3120$
				Total $= 531120$

(The centres described in the first line have 1's row $0^8 1^{32}$, 2's row equal to any codeword $(\mathbf{0}, \mathbf{y}, \mathbf{z}) \in \mathscr{C}$, and 4's row chosen to make all the nonzero coordinates equal to ± 1, and by the remark at the end of § 5.1 the number of such centres is 2^{16}. The other types of centres are easily counted.)

Thus we have a lattice packing Λ_{40} with $\rho = 2\sqrt{2}$, $\delta = 2^4$ and $\tau = 531120$.

5.3. Cross sections of the 40-dimensional packing. Λ_{40} contains Λ_{24} as a section, and for $1 \leqq n \leqq 8$ has a section Λ_{40-n} in E^{40-n} of centre density 16 times that of Λ_{24-n}. For instance in E^{36} there is a section Λ_{36} with $\delta = 2$ and $\tau = 234456$. Λ_{36} is obtained from Λ_{40} by equating to zero four coordinates from a tetrad contained in one of the sets of 16. Λ_{32} is obtained by equating to zero all coordinates from an octad contained in one of the sets of 16, and has $\delta = 1$ and $\tau = 208320$.

5.4. Packings based on ternary codes. Let $\mathscr{C}$ be an (n, M, d) ternary code. By analogy with the constructions based on binary codes we obtain sphere packings in E^n from the following constructions.

Construction A^*. $\mathbf{x} = (x_1, \ldots, x_n)$ is a centre if and only if $\mathbf{x}$ is congruent (modulo 3) to a codeword of $\mathscr{C}$.

Construction B^*. In addition, $\sum_{i=1}^{n} x_i$ is divisible by 2.

From Construction A^* we obtain a packing of spheres of radius $\rho = \min(\frac{3}{2}, \frac{1}{2}\sqrt{d})$ and centre density $\delta = \rho^n \cdot 3^{-\frac{1}{2}n}$, and from Construction B^*,

$$\rho = \begin{cases} \min(2^{-\frac{1}{2}} \cdot 3, \frac{1}{2}\sqrt{d}) & \text{if } d \text{ is even} \\ \min(2^{-\frac{1}{2}} \cdot 3, \frac{1}{2}\sqrt{(d+3)}) & \text{if } d \text{ is odd,} \end{cases}$$

and $\delta = \rho^n\, 2^{-1} \cdot 3^{-\frac{1}{2}n}$.

5.5. Packings obtained from the Pless codes. By applying Construction B^* to the Pless codes of § 1.7, we obtain the lattice packings shown in Table 5.5.1.

Table 5.5.1

E^n	Code	Radius	Center density	Name
E^{12}	$(12, 3^6, 6)$	$2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}}$	2^{-7}	$D_{12} = \frac{1}{4}\Lambda_{12}$
E^{24}	$(24, 3^{12}, 9)$	$3^{\frac{1}{2}}$	$\frac{1}{2}$	$\frac{1}{2}\Lambda_{24}$
E^{36}	$(36, 3^{18}, 12)$	$3^{\frac{1}{2}}$	$\frac{1}{2}$	$P36p$
E^{48}	$(48, 3^{24}, 15)$	$2^{-\frac{1}{2}} \cdot 3$	$2^{-25} \cdot 3^{24}$	$\frac{1}{2}P48p$
E^{60}	$(60, 3^{30}, 18)$	$2^{-\frac{1}{2}} \cdot 3$	$2^{-31} \cdot 3^{30}$	$P60p$

Applying Construction A^* to the $(12, 3^6, 6)$ code gives the packing $D_{12}{}^2$ with $\rho = 2^{-\frac{1}{2}} \cdot 3^{\frac{1}{2}}$ and $\delta = 2^{-6}$. It is known that $\boldsymbol{D}_{12}$ can be doubled in density to give $D_{12}{}^2$, and in fact quadrupled to give Λ_{12} (see [**18**, § 2.1]).

We show in § 5.7 that the packings in E^{24} and E^{48} of Table 5.5.1 can also be doubled in density, to give Λ_{24} and a new packing $P48p$.

On the other hand we have not found doublings of the packings $P36p$ and $P60p$, the technique of § 5.7 failing because of the existence of codewords of maximum weight with odd numbers of each sign (Pless [**29**]).

In $P36p$ the centres closest to the origin are those of the type $(\pm 1)^{12}0^{24}$ and correspond to codewords of minimum weight, so $\tau = A_{12} = 42840$. In $P60p$, $\tau = A_{18} + 2 \cdot 60 \cdot 59 = 3908160$.

5.6. Packings obtained from quadratic residue codes. We observed in § 1.6 that there are $(24, 3^{12}, 9)$ and $(48, 3^{24}, 15)$ ternary quadratic residue codes with the same weight distributions as the corresponding Pless codes. Applying Construction B^*, we obtain packings $\frac{1}{2}\Lambda_{24}$ and $\frac{1}{2}P48q$. It will now be shown that these can be doubled in density to give Λ_{24} in E^{24} and another new packing $P48q$ in E^{48}.

5.7. Density doubling in E^{24} and E^{48}. Any of the packings $\frac{1}{2}\Lambda_{24}$, $\frac{1}{2}P48p$ and $\frac{1}{2}P48q$ may be doubled in density, by adding a second copy which is a translation of the first by $(-2\frac{1}{2}, (\frac{1}{2})^{n-1})$.

To show this, we must verify that no point of the original lattice is closer to $(-2\frac{1}{2}, (\frac{1}{2})^{n-1})$ than the origin. Every point of the original lattice differs from this point by at least $\frac{1}{2}$ in every coordinate. If any coordinates are congruent to -1 (modulo 3) then at least three are, by 1.7.2, and so at least three coordinates differ by $1\frac{1}{2}$. On the other hand if all coordinates are congruent to 0 or 1 (modulo 3), then by 1.7.4 they are congruent to one of the types 0^n, 1^n or $0^{\frac{1}{2}n}\,1^{\frac{1}{2}n}$. Since the sum is even, at least one coordinate differs by $2\frac{1}{2}$.

In E^{24} the doubled packing has centre density 1 and so must be Λ_{24} (regardless of whether it came from the Pless code or the quadratic residue code) since Conway [**7**] has shown this packing to be unique.

In E^{48} we obtain a packing which we call $P48p$ by doubling that given by the $(48, 3^{24}, 15)$ Pless code, and one which we call $P48q$ from the $(48, 3^{24}, 15)$ quadratic residue code. Both $P48p$ and $P48q$ have $\delta = 2^{-24} \cdot 3^{24}$.

The contact number for $P48p$ can be obtained from the partial weight distribution of the $(48, 3^{24}, 15)$ code given in 1.7.5. The centres closest to the origin are

Coordinates	Number
$(\pm 1)^{14}(\mp 2)^{1}0^{33}$	$15 \cdot 415104$
$(\pm 1)^{18}0^{30}$	20167136
$(\pm 3)^{2}0^{46}$	$48 \cdot 47 \cdot 2$
$(\pm \frac{1}{2})^{47}(\mp 2\frac{1}{2})$	$48 \cdot 96$
$(\pm \frac{1}{2})^{45}(\pm 1\frac{1}{2})^{3}$	$4 \cdot 6503296$

The total is $\tau = 52416000$.

$P48q$ was first found by Thompson [**34**]. By modular form theory, $P48p$ and $P48q$ have the same contact numbers. However Thompson has shown that they have different symmetry groups and are not equivalent.

6. CONSTRUCTION *C*

6.1. The construction. Let $\mathscr{C}_i = (n, M_i, d_i)$, $i = 0, 1, \ldots, a$, be a family of codes with $d_i = \gamma \cdot 4^{a-i}$, where $\gamma = 1$ or 2. A sphere packing in E^n is given by:

Construction C. A point $\mathbf{x}$ with integer coordinates is a centre if and only if the 2^i's row of the coordinate array of $\mathbf{x}$ is in $\mathscr{C}_i$, for $i = 0, 1, \ldots, a$.

This is a generalization of the constructions for E^{2^m} given in [**18**, § 1.6]. Construction *C* follows the trend of *A* and *B* in successively imposing more restrictions on the coordinate array of a centre. In general a nonlattice packing is obtained.

6.2. Distance between centres. If two centres differ in the 2^i's row, then (i) if $i > a$ their distance apart is at least 2^{a+1}, and (ii) if $0 \leqq i \leqq a$ they differ by at least 2^i in at least d_i places, and so are at least $(d_i \cdot 4^i)^{\frac{1}{2}} = \sqrt{\gamma} \cdot 2^a$ apart. Thus we may take the radius of the spheres to be $\rho = \sqrt{\gamma} \cdot 2^{a-1}$.

6.3. Centre density. How many integer points $\mathbf{x}$ satisfy the conditions of Construction C? The fraction of $\mathbf{x}$'s with 1's row in $\mathscr{C}_0$ is $M_0 2^{-n}$, of these a fraction $M_1 2^{-n}$ has 2's row in $\mathscr{C}_1$, and so on. Thus the fraction of integer points which are accepted as centres is $M_0 M_1 \ldots M_a 2^{-(a+1)n}$. The radius is $\rho = \sqrt{\gamma} \cdot 2^{a-1}$, so the centre density of the packing is

$$\begin{aligned}\delta &= M_0 M_1 \ldots M_a 2^{-(a+1)n} \rho^n \\ &= M_0 M_1 \ldots M_a \gamma^{\frac{1}{2}n} 2^{-2n}.\end{aligned}$$

6.4. Contact numbers. We shall calculate the number of spheres touching the sphere at the origin. From the discussion in § 6.2 this means that we must find the number of centres of type $(\pm 2^r)^{d_r} 0^{n-d_r}$ for each $r = 0, 1, \ldots, a$. A coordinate equal to $+2^r$ contributes to the coordinate array a column with a single one in the 2^r's row; while a coordinate equal to -2^r has ones in the 2^i's row for all $i \geqq r$. The 1's in the 2^r's row form a codeword $\mathbf{c}$ (say) in $\mathscr{C}_r$, and the minus signs must be at the locations of the ones in a codeword in $\mathscr{C}_{r+1} \cap \mathscr{C}_{r+2} \cap \ldots \cap \mathscr{C}_a$. The 2^r's row can be chosen in A_{d_r} ways, and for each of these the number of ways of choosing the signs is equal to the number of codewords in $\mathscr{C}_{r+1} \cap \mathscr{C}_{r+2} \cap \ldots \cap \mathscr{C}_a$ which are contained in the codeword $\mathbf{c}$ of $\mathscr{C}_r$. Let the latter number be $N_r(\mathbf{c})$.

Then the number of centres of the desired type is $\sum N_r(\mathbf{c})$, and

$$\tau = \sum_{r=0}^{a} \sum N_r(\mathbf{c}),$$

where $\sum$ denotes the sum over all $\mathbf{c} \in \mathscr{C}_r$. If $N_r(\mathbf{c}) = N_r$ is independent of $\mathbf{c}$, this becomes $\tau = \sum_{r=0}^{a} A_{d_r} N_r$.

6.5. Packings obtained from Reed-Muller codes. As an example of Construction C we take $\mathscr{C}_r$ to be the $(2r)$th order RM code of length $n = 2^m$ to obtain a packing $P2^m r$ in E^{2^m}. This and similar packings were given in [**18**].

If m is even, $\gamma = 1$ and $a = \frac{1}{2}m$, while if m is odd, $\gamma = 2$ and $a = \frac{1}{2}(m - 1)$; in both cases the radius is $\rho = 2^{\frac{1}{2}(m-2)}$. As pointed out in § 1.5, RM codes are nested. Therefore $N_r(\mathbf{c})$ is equal to the number of codewords in the $(2r + 2)$th order RM code of length 2^m which are contained in a codeword of minimum weight in the $(2r)$th order RM code, and this is the same as the number of codewords in the 2nd order RM code of length 2^{m-2r}; that is,

$$N_r(\mathbf{c}) = \exp_2\left\{1 + \binom{m-2r}{1} + \binom{m-2r}{2}\right\}.$$

Using the properties of RM codes given in § 1.5, the centre density and maximum contact number are then found to be

$$\begin{aligned}\delta &= 2^{-5n/4} n^{n/4} \\ \tau &= (2 + 2)(2 + 2^2) \ldots (2 + 2^m) \\ &\sim 4.768 \ldots 2^{\frac{1}{2}m(m+1)}.\end{aligned}$$

(See [**18**] for details.) Examples are given in Table II. These are lattice packings only for $m \leqq 5$. Barnes and Wall [**3**] have described a family of lattice packings in E^{2^m}, which coincide with $P2^m r$ for $m \leqq 5$ and have the same density and maximum contact number for all m. Their packings are called $P2^m a$ in Table II.

6.6. Packings obtained from ***BCH*** **and other codes.** Both low and high order *RM* codes contain the maximum number of codewords for this length and Hamming distance; but intermediate order *RM* codes are poor. For $n \geqq 64$ there are *BCH* codes with more codewords, and these too are known not to be optimal. For length 64 extended cyclic codes are known which are better than *BCH* codes. Unfortunately for lengths greater than 64 cyclic non-*BCH* codes have not been extensively studied. *RM*, *BCH* and cyclic codes of length 2^m, $m \geqq 5$, are collected in Table 6.6.1. For $m \geqq 7$ only the best codes known are given. The trivial $(n, 2^n, 1)$, $(n, 2^{n-1}, 2)$, $(n, 2, n)$ codes have been omitted.

Table 6.6.1

Best codes known of length 2^m (see [**4**; **6**; **25**])

n	$\log_2 M$	d	Type
32	26	4	3rd order *RM*
	16	8	2nd order *RM*
	6	16	1st order *RM*
64	57	4	4th order *RM*
	42	8	3rd order *RM*
	45	8	Extended *BCH*
	46	8	Extended cyclic
	22	16	2nd order *RM*
	24	16	Extended *BCH*
	28	16	Extended cyclic
	7	32	1st order *RM*
128	120	4	5th order *RM*
	106	8	Extended *BCH*
	78	16	" "
	43	32	" "
	8	64	1st order *RM*
256	247	4	6th order *RM*
	231	8	Extended *BCH*
	199	16	" "
	139	32	" "
	55	64	" "
	9	128	1st order *RM*
512	502	4	7th order *RM*
	484	8	Extended *BCH*
	448	16	" "
	376	32	" "
	250	64	" "
	85	128	" "
	10	256	1st order *RM*

Since the centre density in Construction C is proportional to the number of codewords, by using the best codes from this table we obtain considerable improvements over the packings $P2^m r$ of the previous section. In E^{64} we obtain a nonlattice packing $P64c$ with $\delta = 2^{22}$. For $m \geqq 7$ we use extended BCH codes to obtain an infinite family of nonlattice packings $P2^m b$ in E^{2^m}. For $m = 7, 8, 9$ these have $\delta = 2^{85}, 2^{250}, 2^{698}$ respectively, giving improvements over $P2^m r$ by factors of $2^{21}, 2^{58}, 2^{186}$ respectively.

The density of $P2^m b$ is estimated for all m in the next section.

BCH codes are nested (§ 1.5) and so also are the extended cyclic codes of length 64 in the above Table. Therefore calculation of the contact numbers in $P64c$ and $P2^m b$ requires knowing the number of codewords of the minimum weight d in each code used and the number of codewords of the code with distance $\frac{1}{4}d$ which are wholly contained in such minimum weight codewords. This appears to be a difficult problem.

6.7. Density of *BCH* packings. This section contains lower and upper bounds and an asymptotic expansion (Theorem 6.7.1) for the centre density of the packings in E^{2^m} obtained by using extended BCH codes in Construction C. Here "code" will mean "extended BCH code of length $n = 2^m$." Two packings are considered. Packing (a) uses the codes of (actual) Hamming distance $1, 4, 16, \ldots, 4^{[\frac{1}{2}m]}$, and packing (b) uses the codes of (actual) Hamming distance $2, 8, 32, \ldots, 2 \cdot 4^{[\frac{1}{2}(m-1)]}$. Then let $P2^m b$ denote the denser of the two packings.

By the last sentence of Theorem 1.5.2, the code of designed distance 2^λ has actual distance 2^λ. Let $2^{k\lambda}$ denote the number of codewords in this code. But there may be codes of designed distance less than 2^λ also having actual distance 2^λ. Let $2^{K\lambda}$ denote the number of codewords in the largest such code. Then clearly $k_\lambda \leqq K_\lambda < k_{\lambda-1}$.

For example, for codes of length 128 it is known (see [**17**]) that $k_5 = 36$, $K_5 = 43$ and $k_4 = 78$.

If δ_a and δ_b denote the centre densities of the two packings, then from § 6.3 it follows that

$$\log_2 \delta_a = \sum_{i=0}^{[\frac{1}{2}m]} K_{2i} - 2n,$$

$$\log_2 \delta_b = \sum_{i=0}^{[\frac{1}{2}(m-1)]} K_{2i+1} - \tfrac{3}{2}n.$$

An algorithm for calculating k_λ is given in [**4**, § 12.3]. The results of this algorithm may be stated as follows. Let numbers $a_{i,j}$ be defined by

$$\begin{cases} a_{i,j} = 2^j - 1 & \text{for } 1 \leqq j \leqq i \\ a_{i,j} = a_{i,j-i} + a_{i,j-i+1} + \cdots + a_{i,j-1} & \text{for } 1 \leqq i < j. \end{cases}$$

Then $k_0 = n$, $k_m = 1$ and

$$k_\lambda = m + a_{m-\lambda,m} \quad \text{for} \quad 0 < \lambda < m.$$

Let $A_i(x) = \sum_{j=1}^{\infty} a_{i,j}x^j$. From the definition of $a_{i,j}$,

$$A_i(x) = \frac{x + 2x^2 + \cdots + ix^i}{1 - x - x^2 - \cdots - x^i}.$$

Let $1 - x - x^2 - \ldots - x^i = \prod_{\nu=1}^{i}(1 - t_\nu x)$. The partial fraction expansion of $A_i(x)$ is then

$$A_i(x) = \sum_{\nu=1}^{i} \frac{1}{1 - t_\nu x},$$

and so

$$a_{i,j} = \sum_{\nu=1}^{i} t_\nu{}^j.$$

The polynomial $G(y) = y^{i+1} - 2y^i + 1 = (y - 1)(y^i - y^{i-1} - \ldots - y - 1)$ has roots $1, t_1, \ldots, t_i$. A sketch of $G(y)$ shows that one root, t_1 (say), is close to 2. In fact for $i > 1$, $G(2 - 2^{1-i}) < 0$ and $G(2 - 2^{-i}) > 0$, so

$$2 - 2^{1-i} < t_1 < 2 - 2^{-i}, i > 1.$$

It has been shown by Mann [**24**], while calculating the number of codewords in the *BCH* codes of length $2^m - 1$ and distances 2^λ and $2^\lambda + 1$, that $|t_\nu| < 1$ for $\nu = 2, 3, \ldots, i$. Then

$$(2 - 2^{1-i})^m - i + 1 < a_{i,m} < (2 - 2^{-i})^m + i - 1, i > 1.$$

Collecting these results we find that lower and upper bounds to $\log_2 \delta_a$ are respectively

$$2^m \sum_{i=1}^{[\frac{1}{2}m]} (1 - 2^{2i-m})^m - 2^m + O(m^2)$$

and

$$2^m \sum_{i=1}^{[\frac{1}{2}m]} (1 - 2^{2i-m-2})^m - 2^m + O(m^2).$$

The sum in the lower bound may be written as

$$\sum_{i=0}^{[\frac{1}{2}m]-c\ \log_4 m} + \sum_{i=[\frac{1}{2}m]-c\ \log_4 m}^{[\frac{1}{2}m]-d\ \log_4 m} + \sum_{i=[\frac{1}{2}m]-d\ \log_4 m}^{[\frac{1}{2}m]} = \sum\nolimits_I + \sum\nolimits_{II} + \sum\nolimits_{III}$$

(say), where $c > 1$ and $d < 1$ are constants. Then

$$\textstyle\sum_I > \sum_{i=0}^{[\frac{1}{2}m]-c\,\log_4 m} (1 - m4^i \cdot 2^{-m})$$

$$> \tfrac{1}{2}m - \log_4 m + O(1)$$

since c may be made arbitrarily close to 1.

Also $\sum_{II} = o(\log_4 m)$, since c and d can be made arbitrarily close together. Finally since $d < 1$,

$$0 < \textstyle\sum_{III} < (1 - m^{-d})^m (d \cdot \log_4 m + 1) = o(1).$$

Therefore

$$\log_2 \delta_a > m2^{m-1} - 2^m \log_4 m + o(2^m \log_4 m),$$

$$= \tfrac{1}{2}n \log_2 n - \tfrac{1}{2}n \log_2 \log_2 n + o(n \log_2 \log_2 n),$$

since $n = 2^m$. Similar arguments apply to the upper bound and to bounds for $\log_2 \delta_b$. This proves

THEOREM 6.7.1. *Let δ be the centre density of either of the packings in E^n, $n = 2^m$, obtained by using extended BCH codes in Construction C. Then*

$$\log_2 \delta \sim \tfrac{1}{2}n \log_2 n - \tfrac{1}{2}n \log_2 \log_2 n.$$

6.8. Comparison of the densities with the bounds. The density Δ of a packing in E^n is related to the centre density by $\Delta = V_n \delta$, or

$$\log_2 \Delta = \log_2 \delta - \tfrac{1}{2}n \log_2 n + O(n).$$

The highest attainable density in E^n satisfies (see [**31**])

$$-n \lesssim \log_2 \Delta \lesssim -\tfrac{1}{2}n.$$

Whenever n is a power of 2, the packings $P2^m r$ obtained from *RM* codes satisfy

$$\log_2 \Delta \sim -\tfrac{1}{4}n \log_2 n,$$

and for the *BCH* packings $P2^m b$,

$$\log_2 \Delta \sim -\tfrac{1}{2}n \log_2 \log_2 n.$$

So there is still room for improvement, although the *BCH* packings seem to be the densest packings yet constructed in these dimensions.

Table I

Sphere packings in up to 24 dimensions

n	Name of packing	Center density		Contact number			Section	Type
		Attained	Upper bound	Maximum	Average	Upper bound		
1	$\Lambda_1 = A_1$	$2^{-1} = 0.5$	0.5	2	2	2	4.5	L
2	$\Lambda_2 = A_2$	$2^{-1}\cdot 3^{-\frac{1}{2}} = 0.28867$	0.28867	6	6	6	4.5	L
3	$\Lambda_3 = D_3$	$2^{-2\frac{1}{2}} = 0.17677$	0.18612	12	12	12	2.4	B
4	$\Lambda_4 = D_4$	$2^{-3} = 0.125$	0.13127	24	24	26	2.4	L
5	$\Lambda_5 = D_5$	$2^{-3\frac{1}{2}} = 0.08838$	0.09987	40	40	48	2.4	B
6	$\Lambda_6 = E_6$	$2^{-3}\cdot 3^{-\frac{1}{2}} = 0.07216$	0.08112	72	72	85	4.5	B
7	$\Lambda_7 = E_7$	$2^{-4} = 0.0625$	0.06981	126	126	146	2.5	B
8	$\Lambda_8 = E_8$	$2^{-4} = 0.0625$	0.06326	240	240	244	2.5	L
9	$\Lambda_9 = T_9$	$2^{-4\frac{1}{2}} = 0.04419$	0.06007	272	272	401	3.3	B
	$P9a$	$2^{-7}\cdot 5 = 0.03906$		306	235 3/5		2.6	N
10	$\Lambda_{10} = \Phi_{10}$	$2^{-4}\cdot 3^{-\frac{1}{2}} = 0.03608$	0.05953	336	336	648	4.5	B
	$P10a$	$2^{-9}\cdot 19 = 0.03710$		372	353 9/19		2.6	N
	$P10b$	$2^{-8}\cdot 3^2 = 0.03515$		500	340 1/3		2.6	N
11	$\Lambda_{11} = J_{11}$	$2^{-5} = 0.03125$	0.06136	438	438	1035	4.5	B
	K_{11}	$2^{-1}\cdot 3^{-2\frac{1}{2}} = 0.03207$		432	432		4.5	L
	L_{11}	$2^{-14\frac{1}{2}}\cdot 3^6 = 0.03146$		440	440		[18]	N
	$P11a$	$2^{-8}\cdot 3^2 = 0.03515$		566	519 7/9		2.6	N
	$P11b$			576			4.3	A
12	$\Lambda_{12} = J_{12}$	$2^{-5} = 0.03125$	0.06559	648	648	1637	3.3	B
	K_{12}	$3^{-3} = 0.03703$		756	756		4.5	L
	L_{12}	$2^{-16}\cdot 3^7 = 0.03337$		704	704		3.3	N
	$P12a$	$2^{-8}\cdot 3^2 = 0.03515$		840	770 2/3		2.6	N

Table I—*concluded*

		Center density		Contact number				
n	Name of packing	Attained	Upper bound	Maximum	Average	Upper bound	Section	Type
13	Λ_{13}	$2^{-5} = 0.03125$	0.07253	906	906	2569	4.5	*B*
	K_{13}	$2^{-1} \cdot 3^{-2\frac{1}{2}} = 0.03207$		918	918		4.5	*B*
	*P*13*a*	$2^{-8} \cdot 3^{2} = 0.03515$		1130	1060 2/3		4.3	*N*
	*P*13*b*			1066			4.3	*A*
14	Λ_{14}	$2^{-4} \cdot 3^{-\frac{1}{2}} = 0.03608$	0.08278	1422	1422	4003	4.5	*B*
	*P*14*a*			1484			4.3	*A*
	*P*14*b*			1582			4.3	*A*
15	Λ_{15}	$2^{-4\frac{1}{2}} = 0.04419$	0.09735	2340	2340	6198	3.4	*B*
	*P*15*a*			2564			4.3	*A*
16	Λ_{16}	$2^{-4} = 0.0625$	0.11774	4320	4320	9544	3.4	*B*
17	Λ_{17}	$2^{-4} = 0.0625$	0.14624	5346	5346	14628	4.5	*B*
18	$\Lambda_{18} = K_{18}$	$2^{-3} \cdot 3^{-\frac{1}{2}} = 0.07216$	0.18629	7398	7398	22324	4.5	*B*
19	Λ_{19}	$2^{-3\frac{1}{2}} = 0.08838$	0.24308	10668	10668	33940	3.4	*B*
20	Λ_{20}	$2^{-3} = 0.125$	0.32454	17400	17400	51421	3.4	*B*
21	Λ_{21}	$2^{-2\frac{1}{2}} = 0.17677$	0.44289	27720	27720	77664	3.4	*B*
22	Λ_{22}	$2^{-1} \cdot 3^{-\frac{1}{2}} = 0.28867$	0.61722	49896	49896	116965	4.5	*L*
23	Λ_{23}	$2^{-1} = 0.5$	0.87767	93150	93150	175696	4.5	*L*
24	Λ_{24}	$1 = 1.0$	1.27241	196560	196560	263285	4.4	*L*

Table II

Sphere packings in more than 24 dimensions

		$\log_2$ (center density)		Maximum		
n	Name of packing	Attained	Upper bound	contact number	Section	Type
32	Λ_{32}	0	5.52	208320	5.3	L
	$P32r$	0		146880	6.5	L
36	Λ_{36}	1	8.63	234456	5.3	L
	$P36p$	−1		42840	5.5	L
40	Λ_{40}	4	12.04	531120	5.2	L
48	$P48p$	14.039	19.64	52416000	5.7	L
	$P48q$	14.039		52416000	5.7	L
60	$P60p$	16.548	32.70	3908160	5.5	L
64	$P64a$	16	37.44	9694080	6.5	L
	$P64r$	16		9694080	6.5	N
	$P64c$	22			6.6	N
128	$P128a$	64	131.8		6.5	L
	$P128r$	64			6.5	N
	$P128b$	85			6.6	N
256	$P256a$	192	383.3		6.5	L
	$P256r$	192			6.5	N
	$P256b$	250			6.6	N
512	$P512a$	512	1012		6.5	L
	$P512r$	512			6.5	N
	$P512b$	698			6.6	N

Notes on the Tables. 1. For the history of and alternative names for the packings in low dimensions see [**10**; **31**].

2. The upper bounds in Table I are taken from [**19**]. The upper bound to the contact number is Coxeter's conjectured bound [**11**]. This bound in 32 dimensions is 6256830. The upper bound ρ to the centre density in Table II is Rogers' bound [**30**] in the form

$$\log \rho \approx \tfrac{1}{2}n \log\left(\frac{n}{4e\pi}\right) + \tfrac{3}{2}\log n - \log\frac{e}{\sqrt{\pi}} + \frac{5\frac{1}{4}\log 2}{n + 2\frac{1}{2}},$$

the last term being approximate.

3. Decimal expansions have been truncated rather than rounded.

4. The maximum contact numbers for $P2^m a$ and $P2^m r$ are given in § 6.5 for all m. Those for $P2^m c$ are not known for $m \geqq 6$.

5. In the last column, B indicates that both a lattice and a nonlattice packing with these parameters are known. L indicates that at present only a lattice packing is known, and N that only a nonlattice packing is known. A indicates a local arrangement of spheres touching one sphere.

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