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Low-dimensional lattices. III. Perfect forms

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The perfect lattices or quadratic forms in dimensions $n \leqslant 7$ found by Barnes, Stacey and others are studied, and their automorphism groups, orbits of minimal vectors and eutactic coefficients are determined. It is shown that just 30 of the 33 known seven-dimensional perfect lattices are extreme. A simple and self-contained proof is given of the classification of perfect lattices in dimensions $n \leqslant 4$ and hence of the determination of the densest lattice packings in these dimensions.

1. INTRODUCTION

The main aim of this paper is to describe the perfect lattices (or quadratic forms) of dimension up to 7. We present a simple proof (believed to be new) of the classification of perfect lattices in two, three and four dimensions and of the fact that the lattices $A_2, A_3 \cong D_3$ and D_4 are the densest lattice packings in these dimensions, and we give a detailed table of the known n-dimensional perfect lattices for $n \leqslant 7$. We have made a number of computations concerning these lattices. In particular we describe their automorphism groups and orbits of minimal vectors, and determine (apparently for the first time) just which of them are extreme.

These calculations were made as the first step of a projected definitive enumeration of the seven-dimensional perfect lattices. Because the tables give much valuable information, we have decided to publish them in advance. It is very likely that the eventual enumeration will confirm the completeness of the known list.

Progress in the study of perfect lattices has been steady but slow. It is remarkable that although the three-dimensional problem was solved in 1831, the solution to the four- and five-dimensional problems had to wait until 1877 and the six-dimensional one until 1957, while the problem in seven dimensions is still not fully solved.

Perfect lattices arise in determining the densest lattice packing of spheres. A lattice, or the corresponding quadratic form, is called extreme if its density is a local maximum, and absolutely extreme if its density is a global maximum (see §3). Gauss (1831) showed that the face-centred cubic lattice $A_3 \cong D_3$ is the unique absolutely extreme lattice in three dimensions.

A lattice or quadratic form is perfect if it is uniquely determined by the

knowledge of which vectors are minimal (the formal definition is given in §3). Korkine & Zolotareff (1877) proved that every extreme lattice is perfect, and enumerated all perfect lattices in dimensions $n \leqslant 5$. Voronoi (1907) showed that a lattice is extreme if and only if it is perfect and eutactic (see §3). He also gave an algorithm for enumerating perfect lattices, and used it to simplify the proof of Korkine & Zolotareff's enumeration. Voronoi seems also to have attempted the six-dimensional enumeration, but many years had to pass before this was completed.

In the meantime Blichfeldt (1925, 1929, 1935) proved that E_6, E_7 and E_8 are absolutely extreme lattices in dimensions 6, 7 and 8, without, however, enumerating all the corresponding perfect lattices. Mordell (1944) (see also Oppenheim 1946) showed that the highest attainable density of a lattice packing in eight dimensions can be deduced from knowledge of the corresponding result in seven dimensions.

The complete enumeration of perfect lattices in six dimensions was obtained by Barnes (1957*a*, *b*), using Voronoi's method. (Incomplete lists were given by Hofreiter (1933) and Vladimirov (1958).)

All these papers are rather complicated. Watson (1972) characterized Korkine & Zolotareff's paper as prolix, Barnes's work as deep and difficult, and remarked that Blichfeldt left very heavy calculations to the reader. In a series of papers Watson (1969, 1971*a*, *b*, 1972) provided several new proofs and a new technique for enumerating perfect lattices (see also Davenport & Watson 1954).

Watson (1966) also reports that he was able to check Blichfeldt's calculations in six, seven and eight dimensions and to verify the uniqueness of the absolutely extreme lattices in these dimensions. Vetchinkin (1980) gives a long and detailed proof of this result.

In seven dimensions there are 33 known perfect lattices. Their history is rather complicated. The work of Korkine & Zolotareff (1877), Barnes (1959), Scott (1964, 1966) and Larmouth (1971) led to 22 lattices. The remaining 11 lattices were discovered by Stacey (1973, 1975, 1976), who used Watson's new method to enumerate them. Stacey's computer in fact generated all perfect septenary lattices, but unfortunately it produced thousands of Gram matrices, and Stacey was unable to decide just which pairs of these were equivalent. Instead, she partly classified the lattices by their determinants and numbers of minimal vectors, and sometimes some other invariants. It is therefore conceivable that there are perfect lattices not in Stacey's list of 33. However, any such lattice must have the same determinant and number of minimal vectors as one of the 33.

We have made an intensive study of the known perfect lattices in up to seven dimensions, as the first stage in a projected definitive solution of the seven-dimensional problem. In fact, we found it quite difficult even to recover the complete list of 33 seven-dimensional lattices from the literature; no previous publication attempts to give such a list. Barnes, Scott and Larmouth give only the forms discovered by them. Scott's and Larmouth's lists contain misprints that were corrected by Stacey, and a correct definition of Stacey's lattice K_5 does not appear in any of Stacey (1973, 1975, 1976). We were able to recover K_5 only by applying Voronoi's algorithm. It is also hard to obtain the automorphism groups of these lattices from the literature.

For all these reasons we have felt it desirable to present a complete list of the known perfect lattices in dimensions $n \leqslant 7$ in this paper, along with various information about their minimal vectors, automorphism groups, eutactic coefficients and Voronoi neighbours. The lattices are summarized in table 1 below.

The main problem in the systematic enumeration of perfect lattices is the production of efficient algorithms for rigorously identifying these lattices. We have completed this step, and have used it to determine the Voronoi neighbours of the seventeen known seven-dimensional perfect lattices for which $s = 28$ (see §6). The results agree with the (sometimes putative) lists of neighbours given by Stacey (1973), P. R. Scott (quoted in Stacey 1973) and Shushbaev (1985), who together have studied the Voronoi neighbours for 27 of the seven-dimensional perfect lattices.

For background information and terminology the reader is referred to Barnes (1957*b*), Cassels (1971, 1978), Conway & Sloane (1988*a*, *b*, *c*), Coxeter (1951), Lekkerkerker (1969) and Rogers (1964).

2. Observations on perfect and extreme forms

H. S. M. Coxeter lectured on perfect and extreme forms to the International Congress of Mathematicians in Cambridge, Massachusetts on 2 September 1950 (Coxeter 1951). In particular, he made several empirical observations concerning these forms. Because Coxeter (1951) is still a standard reference, it is appropriate to mention here the present status of some of these questions.

'*We do not know whether every perfect form is extreme*' (Coxeter 1951, p. 392). As Barnes (1957*a*, *b*) showed, the form P_6^4 is perfect but not eutactic, and hence not extreme. P_6^4 is in fact 'semi-eutactic', as is p_7^{26} (see §6). We can now exhibit stronger counter-examples: any system of eutactic coefficients for either p_7^{18} or p_7^{29} must contain a strictly negative number.

'*It seems to be a reasonable conjecture that every eutactic form with* $s \geqslant \frac{1}{2}n(n+1)$ *should be extreme*' (Coxeter 1951, p. 392). Because $A_2 \otimes A_2 = L_4$ is not extreme, it can be distorted into a denser lattice L_4'. So the 32-dimensional lattice $L_{32} = L_4 \otimes E_8$ is not extreme because it can be distorted into the denser lattice $L_{32}' = L_4' \otimes E_8$. But it is easy to see that L_{32} has

$$s = 3 \cdot 3 \cdot 120 = 1080$$

$$> \tfrac{1}{2}32 \cdot 33 = 528$$

pairs of minimal vectors, and is eutactic because its automorphism group is irreducible. Barnes (1957*a*) gave a 20-dimensional example, and in a letter to the authors he points out the still simpler example $E_7 \oplus E_7$.

'*Every known perfect form in less than nine variables* [*has a system of eutactic coefficients that are*] *all equal*' (Coxeter 1951, p. 396). We now know many forms that do not have this property. The unique system of eutactic coefficients for p_7^{24} for example contains 16 different numbers.

In a similar vein we remark that in dimensions $n \leqslant 6$

the perfect lattices are generated by their minimal vectors,

and that this is also true for the known seven-dimensional lattices. But as Coxeter (1951, p. 430) remarks, there are higher-dimensional counter-examples, such as A_9^{+2} (this is A_9^2 in Coxeter's notation).

We also point out that the information in our table 1 supports

the A_n hypothesis: A_n is the least dense perfect lattice in n dimensions.

Nevertheless, we suspect that this will fail in sufficiently high dimensions.

Finally, we feel that Coxeter's remark (Coxeter 1951, p. 394) '*we now know that the absolutely extreme forms for $n \leqslant 9$ are Korkine and Zolotareff's*

$$U_1, U_2, U_3, V_4, V_5, X, Y, W_8, T_9$$

(*with a modicum of doubt in the last case, because Chaundy's argument is not quite valid*)' deserves some amplification. The mistake in Chaundy (1946) was essentially to suppose that the densest lattices must be nested. We have discussed the resulting 'laminated lattices' Λ_n in detail for $n \leqslant 48$ in Conway & Sloane (1982*b*) (see also Conway & Sloane 1988*a*, chap. 6). Because denser lattices exist at least in dimensions 11, 12, 13 and 30 onwards, Chaundy's argument cannot be said to provide *any* support for the optimality of Λ_9 (Korkine & Zolotareff's T_9), although Λ_9 quite probably is the absolutely extreme lattice in nine dimensions. (There are also non-lattice packings that are denser than Λ_{10} (Leech & Sloane 1971; Best 1980; Conway & Sloane 1988*a*).

3. Perfect and extreme lattices: Voronoi's theorem

In real euclidean space $\mathbb{R}^n$ equipped with inner product $(v, w) = v \cdot w$, a lattice Λ consists of all integral linear combinations

$$v = \zeta_1 v_1 + \ldots + \zeta_n v_n \quad (\zeta_i \in \mathbb{Z})$$

of n linearly independent vectors $v_1, \ldots, v_n$. The vectors $v_1, \ldots, v_n$ form an integral base for Λ, and

$$f(\zeta) = (v, v) = \zeta A \zeta^{\mathrm{tr}}$$

with

$$\xi = (\zeta_1, \ldots, \zeta_n) \in \mathbb{Z}^n,$$

$$A = (a_{ij}), \quad a_{ij} = (v_i, v_j)$$

is the corresponding quadratic form. The various integral bases for Λ yield integrally equivalent quadratic forms (see the references at the end of §1 for further discussion). The number (v, v) is called the norm of v, and A is the Gram matrix of Λ (or f).

From now on we shall usually speak of lattices rather than quadratic forms, although it is to be understood that the same terminology applies to both concepts.

The minimal value μ of (v, v) for $v \in \Lambda, v \neq 0$, is the minimal norm of Λ, and $d = \det A$ is the determinant of Λ. Then Λ has packing density

$$\Delta = V_n \mu^{\frac{1}{2}n} / 2^n d^{\frac{1}{2}}, \tag{1}$$

where $V_n = \pi^{\frac{1}{2}n} / (\frac{1}{2}n)!$ is the volume of a unit sphere.

The lattice Λ is extreme if Δ does not increase when A is perturbed slightly, and is absolutely extreme if Δ is the maximal value for any n-dimensional lattice.

Let the non-zero lattice vectors with the smallest norm μ be

$$\pm m^{(1)}, \ldots, \pm m^{(s)} \in \Lambda,$$

where
$$m^{(k)} = \zeta_1^{(k)} v_1 + \ldots + \zeta_n^{(k)} v_n,$$

so that
$$f(\zeta^{(k)}) = \mu, \quad (k = 1, \ldots, s), \tag{2}$$

where $\zeta^{(k)}$ is the n-tuple $(\zeta_1^{(k)}, \ldots, \zeta_n^{(k)})$. Thus Λ contains $2s$ minimal vectors.

A lattice is called perfect if its Gram matrix is completely determined by the number μ and the s n-tuples $\zeta^{(1)}, \ldots, \zeta^{(s)}$ (by solving the linear equations (2) for the $N = \frac{1}{2}n(n+1)$ entries a_{ij}). Equivalently, Λ is perfect if there is no non-zero quadratic form $g(\zeta)$ satisfying

$$g(\pm\zeta^{(k)}) = 0, \quad (k = 1, \ldots, s). \tag{3}$$

If Λ is perfect then plainly
$$s \geqslant \tfrac{1}{2}n(n+1). \tag{4}$$

Real numbers $c_1, \ldots, c_s$ are called a system of eutactic coefficients for Λ if the identity

$$(v, v) = \sum_{k=1}^{s} c_k (v, m^{(k)})^2 \tag{5}$$

holds for all $v \in \Lambda$. A lattice is eutactic if it has a system of strictly positive eutactic coefficients. It is semi-eutactic if it has a system of non-negative eutactic coefficients.

We remark that if Λ has a system of eutactic coefficients (or positive eutactic coefficients) then it has another such set, obtained by averaging, in which the c_k are invariant under the automorphism group Aut (Λ). In §6 we give all such invariant systems of eutactic coefficients for each lattice considered.

The dual lattice Λ^* has associated quadratic form

$$f^*(\eta) = \eta A^{-1} \eta^{\mathrm{tr}}.$$

In the η-coordinates (5) states that Λ is eutactic if and only if

$$f^*(\eta) = \sum_{k=1}^{s} c_k \left[\sum_{i=1}^{n} \eta_i \zeta_i^{(k)} \right]^2 \tag{6}$$

holds identically in $\eta = (\eta_1, \ldots, \eta_n)$.

Equations (5) and (6) imply that in a eutactic lattice the minimal vectors $m^{(1)}, \ldots, m^{(s)}$ are the projections of a set of mutually orthogonal vectors in $\mathbb{R}^s$. (When all the c_k are equal, $\pm m^{(1)}, \ldots, \pm m^{(s)}$ are the projections of a coordinate frame in $\mathbb{R}^s$, and form a eutactic star (Coxeter 1973; Hadwiger 1940; Seidel 1978).)

It is easy to prove that for the root lattices $A_n (n \geqslant 1)$, $D_n (n \geqslant 3)$, $E_n (n = 6, 7, 8)$ (cf. Conway & Sloane 1988*a*, *b*) the eutactic coefficients are equal to $1/h$, where h is the Coxeter number.

The preceding concepts are related by the following theorem.

THEOREM 1 (Voronoi 1907). *A lattice is extreme if and only if it is perfect and eutactic.*

Several proofs are available: Barnes (1957*c*), Coxeter (1951), Kneser (1955), Lekkerkerker (1969) and Ryshkov & Baranovskii (1979). Barnes's proof makes use of Stiemke's theorem, which is a corollary to one of the fundamental theorems of linear programming (Stiemke 1915; Schrijver 1986).

To make our proofs of theorems 5 and 6 self-contained we establish part of this theorem, by showing that an extreme lattice is perfect. Let

$$f(\zeta) = \sum_{i,j=1}^{n} \zeta_i a_{ij} \zeta_j$$

be an extreme positive definite quadratic form, of determinant d and minimal norm μ, with minimal vectors $\pm m^{(1)}, \ldots, \pm m^{(s)}$ and corresponding n-tuples $\zeta^{(1)}, \ldots, \zeta^{(s)}$, so that (2) holds.

If f is not perfect then by (3) there exists a non-zero quadratic form

$$g(\zeta) = \sum_{i,j=1}^{n} \zeta_i b_{ij} \zeta_j$$

such that $g(\pm\zeta^{(i)}) = 0, i = 1, \ldots, s$. Consider

$$f'(\zeta) = f(\zeta) + \epsilon g(\zeta),$$

with determinant d' and minimal norm μ' (say). For all ϵ in a sufficiently small interval $-\epsilon_0 < \epsilon < \epsilon_0$, (*a*) f' is positive definite and (*b*) f' attains its minimum at a subset of $\zeta^{(1)}, \ldots, \zeta^{(s)}$. Because f is extreme, there is an interval $-\epsilon_1 < \epsilon < \epsilon_1$ in which (see (1))

$$(\mu')^n/d' \leqslant \mu^n/d.$$

Thus for one of the $\zeta^{(k)}$, ζ^* say, we have

$$f'(\zeta^*)^n/d' \leqslant f(\zeta^*)^n/d.$$

However, $g(\zeta^*) = 0, f'(\zeta^*) = f(\zeta^*)$, and we conclude that $d' \geqslant d$ for $-\epsilon_1 < \epsilon < \epsilon_1$. This is impossible, for as we now show d' is a strictly concave ($\cap$) function of ϵ on this interval. Indeed, let P, Q be arbitrary distinct positive definite symmetric matrices. By simultaneously diagonalizing P and Q it is easy to show that $\lambda P + (1-\lambda) Q$ for $0 \leqslant \lambda \leqslant 1$ is a positive definite matrix whose determinant is a strictly concave function of λ. This completes the proof.

4. Perfect lattices in dimensions 1 to 4

The goal of this section is to prove theorems 5 and 6, characterizing the perfect lattices and densest lattice sphere packings in dimensions 1 to 4.

An n-dimensional lattice Λ is partitioned into 2^n congruence classes modulo 2Λ, and $\Lambda/2\Lambda$ has the structure of an elementary abelian group of type $(2, 2, \ldots, 2)$. In this section we assume that all lattices are scaled so as to have minimal norm $\mu = 1$.

Proposition 2. *In any lattice, if congruence classes $\alpha_1, \ldots, \alpha_k$ contain vectors $v_1, \ldots, v_k$ of norms $n_1, \ldots, n_k$, then the congruence class $\alpha_1 + \ldots + \alpha_k$ contains a vector $v_1 \pm v_2 \pm \ldots \pm v_k$ of norm at most $n_1 + \ldots + n_k$.*

Proof. Inductively we may suppose that

$$N(v_1 \pm \ldots \pm v_{k-1}) \leqslant n_1 + \ldots + n_{k-1}.$$

The proposition follows by choosing the next sign so that $\pm v_k$ and $v_1 \pm \ldots \pm v_{k-1}$ have inner product $\leqslant 0$.

COROLLARY 3. *If two minimal vectors v_1, v_2 are in the same congruence class then $v_2 = \pm v_1$.*

Proof. For some choice of the sign we have $N(v_1 \pm v_2) \leqslant 2$, by proposition 2. However, $v_1 \pm v_2$ is twice a lattice vector, and so if it were not zero then $N(v_1 \pm v_2) \geqslant 4$, a contradiction.

COROLLARY 4. *If the congruence classes α_1, α_2, $\alpha_1 + \alpha_2$ contain minimal vectors v_1, v_2, v_3 respectively, then $v_3 = \pm v_1 \pm v_2$.*

Proof. For some choice of signs we have

$$N(v_1 \pm v_2 \pm v_3) \leqslant 3,$$

by proposition 2, but if not zero this vector must have norm at least 4, a contradiction.

We can now state the main theorems of this section. As mentioned in the introduction, the three-dimensional results are originally due to Gauss (1831) and the four-dimensional results to Korkine & Zolotareff (1877). Other proofs in three dimensions have been given by Cassels (1971, chap. II, theorem III), Coxeter (1961, §18.4), Dempster (1957) and Mordell (1948). However, the proofs given here seem to be new.

THEOREM 5. *The lattices $\mathbb{Z} \cong A_1$, A_2 and $A_3 \cong D_3$ are the unique perfect lattices (or forms) in dimensions 1, 2 and 3 respectively. There are precisely two four-dimensional perfect lattices, A_4 and D_4.*

THEOREM 6. *The lattices $\mathbb{Z} \cong A_1$, A_2, $A_3 \cong D_3$ and D_4 are the unique absolutely extreme lattices (or forms) in dimensions 1, 2, 3 and 4, respectively.*

Because D_4 has density $\Delta = \frac{1}{16}\pi^2 = 0.6169\ldots$, and A_4 has density

$$\Delta = \pi^2/(8\sqrt{5}) = 0.5517\ldots$$

(see, for example, Leech & Sloane 1971 or Conway & Sloane 1988*a*), theorem 6 is an immediate consequence of theorems 1 and 5.

Proof of theorem 5.

The one-dimensional case. There is a unique one-dimensional lattice $\mathbb{Z} \cong A_1$.

The two-dimensional case. Suppose Λ is a two-dimensional perfect lattice with minimal norm 1. By (4) there are at least three pairs of minimal vectors. Corollaries 3 and 4 show that the three non-trivial congruence classes α, β, $\alpha+\beta$ must without loss of generality contain minimal vectors v_1, v_2, $v_1 - v_2$ respectively. The Gram matrix of v_1 and v_2 is therefore

$$\begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & 1 \end{bmatrix},$$

and so v_1, v_2 generate a copy of the hexagonal lattice A_2 (see figure 1). Is this all of Λ? Yes! From the figure we see that any larger lattice must contain a vector of length at most $1/\sqrt{3}$, contradicting the assumption that the minimal norm is 1. For such a lattice must contain a point in the triangle 0AB. The farthest such point from the hexagonal lattice is the centre of the triangle, at distance $1/\sqrt{3}$. (In other words, the hexagonal lattice has covering radius $1/\sqrt{3}$.)

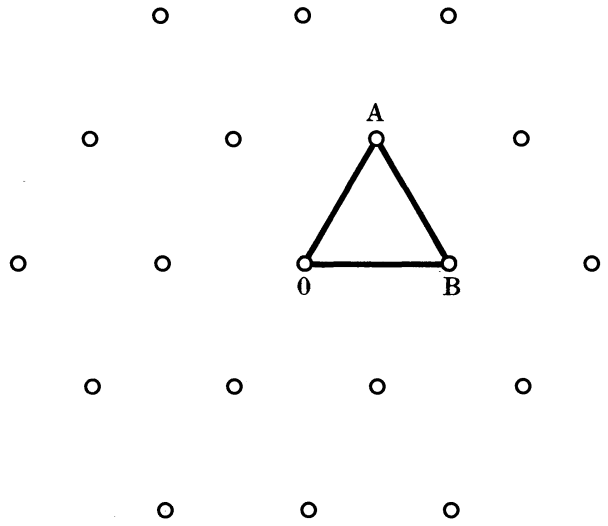

FIGURE 1. The hexagonal lattice A_2.

The three-dimensional case. Now there are seven non-zero congruence classes, which abstractly form a projective plane of order 2 (see figure 2). By (4) there are at least six pairs of minimal vectors, so from corollary 3 at least six of the congruence classes must contain pairs of minimal vectors. Without loss of generality we may suppose that $\alpha+\beta+\gamma$ is the only congruence class that does not contain a minimal vector, if there is such a class.

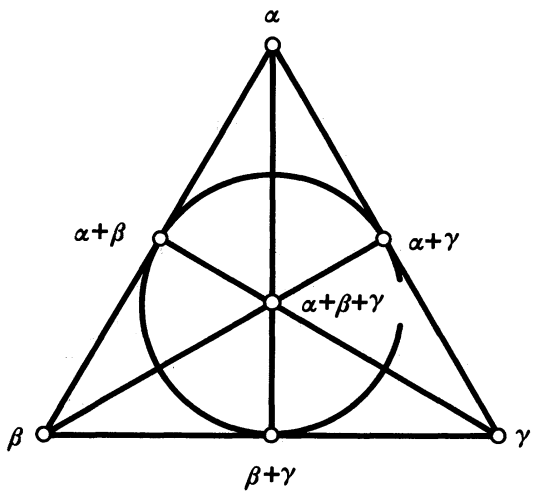

FIGURE 2. The seven non-zero congruence classes of $\Lambda/2\Lambda$, when Λ is a three-dimensional lattice.

If we let v_1, v_2, v_3 be minimal vectors in α, β, γ then by corollary 4 there are minimal vectors $v_1 \pm v_2$, $v_1 \pm v_3$, $v_2 \pm v_3$, and so the Gram matrix of v_1, v_2, v_3 has the form

$$\begin{bmatrix} 1 & \frac{1}{2}\epsilon_3 & \frac{1}{2}\epsilon_2 \\ \frac{1}{2}\epsilon_3 & 1 & \frac{1}{2}\epsilon_1 \\ \frac{1}{2}\epsilon_2 & \frac{1}{2}\epsilon_1 & 1 \end{bmatrix},$$

where ϵ_1, ϵ_2, ϵ_3 are $+1$ or -1. Then the vectors $w_1 = \epsilon_1 v_1$, $w_2 = \epsilon_2 v_2$, $w_3 = \epsilon_3 v_3$ have Gram matrix

$$\begin{bmatrix} 1 & \frac{1}{2}\epsilon & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & 1 & \frac{1}{2}\epsilon \\ \frac{1}{2}\epsilon & \frac{1}{2}\epsilon & 1 \end{bmatrix},$$

where $\epsilon = \epsilon_1 \epsilon_2 \epsilon_3$. If $\epsilon = -1$ the three rows add to zero, implying that the vectors w_1, w_2, w_3 belong to a two-dimensional lattice, which is impossible. Hence $\epsilon = +1$, and the vectors w_1, w_2, w_3 generate a copy of the face-centred cubic lattice A_3. Again this must be all of Λ, because A_3 (on this scale) has covering radius $1/\sqrt{2}$, which is less than 1 (see, for example, Coxeter 1973; Conway & Sloane 1982*a*, 1988*a*).

Note that we have now established the following result.

PROPOSITION 7. *In a perfect lattice the following are true.*

(*i*) *Seven congruence classes such as* α, β, γ, $\alpha+\beta$, $\alpha+\gamma$, $\beta+\gamma$, $\alpha+\beta+\gamma$ *cannot all contain minimal vectors.*

(*ii*) *If we have minimal vectors*

$$v_i \in \alpha, v_j \in \beta, v_k \in \gamma,$$

$$v_i - v_j \in \alpha+\beta, v_i - v_k \in \alpha+\gamma, v_j \pm v_k \in \beta+\gamma,$$

then in fact $v_j - v_k \in \beta+\gamma$ *is a minimal vector.*

The four-dimensional case. In the four-dimensional case there are 15 non-zero congruence classes α, β, γ, δ, $\alpha+\beta, \ldots, \alpha+\beta+\gamma+\delta$, at least 10 of which contain pairs of minimal vectors (by (4) and corollary 3). We may suppose that the classes that do not contain minimal vectors are included in one of:

(*a*) α, β, γ, δ, $\alpha+\beta+\gamma+\delta$ (if no four are dependent);

(*b*) α, β, γ, $\alpha+\beta+\gamma$, δ (if some four are dependent but no three are dependent);

(*c*) α, β, $\alpha+\beta$, γ, $\alpha+\beta+\gamma$ (if some three are dependent and there is a further dependence);

(*d*) α, β, $\alpha+\beta$, γ, δ (if some three are dependent but there is no further dependence).

We discuss these cases in rather telegraphic language. The vectors v_i, $v_i - v_j$ we mention are implicitly asserted to be minimal vectors.

(*a*) We may choose minimal vectors as follows:

(i) $v_1 \in \gamma+\delta$;

(ii) $v_2 \in \alpha+\gamma$ so that $v_1 - v_2 \in \alpha+\delta$;

(iii) $v_3 \in \beta+\gamma$ so that $v_1 - v_3 \in \beta+\delta$.

Then $v_2 - v_3 \in \alpha+\beta$, by proposition 7.

(iv) $v_4 \in \alpha+\beta+\delta$ so that $v_1 - v_4 \in \alpha+\beta+\gamma$.

Then $v_2-v_4\in\beta+\gamma+\delta$, $v_3-v_4\in\alpha+\gamma+\delta$ by proposition 7. The vectors $v_1,\ldots,v_4$ now have Gram matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & 1 \end{bmatrix}$$

and so generate a copy of the lattice A_4. Because A_4 has covering radius $\sqrt{\frac{3}{5}}<1$, this is all of Λ.

(*b*) In this case the seven classes shown in figure 3 all contain minimal vectors. This is prohibited by proposition 7.

(*c*) We choose minimal vectors so that

(i) $v_1\in\alpha+\gamma$;

(ii) $v_2\in\delta$ so that $v_1-v_2\in\alpha+\gamma+\delta$;

(iii) $v_3\in\beta+\gamma+\delta$ so that $v_1-v_3\in\alpha+\beta+\delta$;

Then $v_2-v_3\in\beta+\gamma$ by proposition 7.

(iv) $v_4\in\gamma+\delta$ so that $v_1-v_4\in\alpha+\delta$.

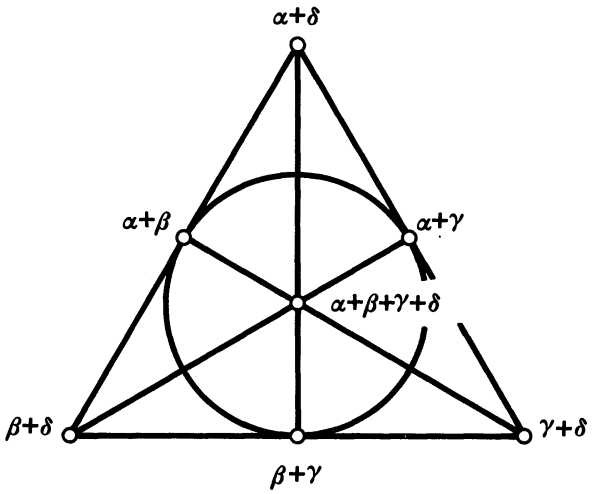

FIGURE 3. The proof for case (*b*).

We now have minimal vectors

$$v_2-v_3\pm v_4\in\beta+\delta,$$

$$v_2-v_3\pm(v_1-v_4)\in\alpha+\beta+\gamma+\delta,$$

and we have identified minimal vectors in ten non-zero congruence classes. The vectors $v_1,\ldots,v_4$ have the Gram matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & a \\ \frac{1}{2} & \frac{1}{2} & 1 & b \\ \frac{1}{2} & a & b & 1 \end{bmatrix}$$

The only condition on a and b given by the minimal vectors identified so far is that $a-b=\pm\frac{1}{2}$. So we do not yet have a perfect lattice, unless some other congruence class contains a minimal vector. But then this reduces to case (*b*) or (*d*).

(*d*) We choose minimal vectors so that

(i) $v_1\in\alpha+\gamma$;

(ii) $v_2\in\alpha+\delta$ so that $v_1-v_2\in\gamma+\delta$;

(iii) $v_3\in\beta+\gamma+\delta$ so that $v_1-v_3\in\alpha+\beta+\delta$.

Then $v_2-v_3\in\alpha+\beta+\gamma$ by proposition 7.

(iv) $v_4\in\alpha+\beta+\gamma+\delta$ so that $v_1-v_4\in\beta+\delta$.

Then $v_2-v_4\in\beta+\gamma$ by proposition 7.

It remains to discuss the class $\alpha+\gamma+\delta$, which must contain minimal vectors of both forms $\pm(v_1-v_4)\pm(v_2-v_3)$ and $\pm(v_2-v_4)\pm(v_1-v_3)$. By corollary 3 we deduce that in fact this class contains the minimal vector $v_1+v_2-v_3-v_4$. Therefore $v_1,\ldots,v_4$ have the Gram matrix

$$\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & \frac{1}{2} & 1 & 0 \\ \frac{1}{2} & \frac{1}{2} & 0 & 1 \end{bmatrix},$$

and generate a copy of the lattice D_4. Because D_4 has covering radius $1/\sqrt{2}<1$, this is all of Λ.

This completes the proof of theorem 5.

5. Dimensions 5 and 6; also the lattices A_n^{+r}, $A_n^{(r)}$ and $A_{n,r}$

Korkine & Zolotareff (1877) showed that there are three perfect lattices in five dimensions, and Barnes (1957, 1957*a*) that there are seven perfect lattices in six dimensions. These lattices are summarized in table 1 below and are described in more detail in the following section. Besides the root lattices A_n, D_n and E_n, the five- and six-dimensional lists involve certain generalizations of the A_n lattices that we shall describe in this section.

The lattice A_n^{+r} (for $n\geqslant 1$ and r a divisor of $n+1$) is the unique sublattice of the dual lattice A_n^* that contains A_n to index r. This is also called $A_n[q]$ (where $qr=n+1$). It was called A_n^r in Coxeter (1951).

The lattice $A_n^{(1)}=A_n$ ($n\geqslant 1$) is generated by all cyclic permutations of the $(n+1)$-tuple $(1,-1,0,\ldots,0)$. Similarly $A_n^{(2)}$ ($n\geqslant 2$) is generated by all cyclic permutations of the $(n+1)$-tuple $(1,-2,1,0,\ldots,0)$, $A_n^{(3)}$ ($n\geqslant 3$) by those of $(1,-3,3,-1,0,\ldots,0)$, and so on. The lattices $P_6^5=A_6^{(2)}$ and $p_7^5=A_7^{(2)}$ belong to this family. $A_n^{(2)}$ is called P_n by Barnes and Stacey. Craig (1978, 1984) has studied these lattices from the point of view of the sphere packing problem. It is not difficult to show that $A_n^{(r)}$ has determinant $(n+1)^{2r-1}$. Furthermore if $n+1$ is a prime and $r<\frac{1}{2}n$, the minimal norm of $A_n^{(r)}$ is at least $2r$ (Craig 1978, 1984; Conway & Sloane 1988*a*, chap. 8, §6).

The lattices $A_{n,r}$ were introduced (under different names) by Barnes (1957*a*, *b*, 1959). We give a slightly different definition using complex coordinates. We take

a set $\mathscr{V} = \{v_1, \ldots, v_N\}$ of vectors, each of which has a single non-zero coordinate taking values from $\{1, \omega, \omega^2\}$, where $\omega = e^{\frac{2}{3}\pi i}$; and consider the lattice consisting of all integer combinations of the differences $v_i - v_j$, equipped with the inner product $(x, y) = \mathrm{Re} \sum x_i \bar{y}_i$. It is fairly easy to see that the structure of such a lattice depends only on its dimension n and the number r of coordinates that are non-zero for precisely *one* of the v_i. We therefore call this lattice $A_{n,r}$.

There are really two cases. (i) If $r = n - 2t$ we may assume without loss of generality that the set $\mathscr{V}$ consists of $3t$ vectors of the form

$$(\ldots, \omega^\nu, \ldots; 0, \ldots, 0), \quad \nu = 0, 1 \text{ or } 2$$

(where the non-zero coordinate is one of the first t); together with r further vectors

$$(0, \ldots, 0; \ldots, 1, \ldots)$$

(where it is one of the last r). Then $A_{n,r}$ is spanned by the differences of these vectors, and contains

$$9\binom{t}{2} + 3tr + \binom{r}{2}$$

pairs of minimal vectors, namely

$$9\binom{t}{2} \text{ pairs} \qquad \pm(\ldots, \omega^\mu, \ldots, -\omega^\nu, \ldots; 0, \ldots, 0),$$

$$3tr \text{ pairs} \qquad \pm(\ldots, \omega^\nu, \ldots; \ldots, -1, \ldots),$$

$$\binom{r}{2} \text{ pairs} \qquad \pm(0, \ldots, 0; \ldots, 1, \ldots, -1, \ldots).$$

This is the lattice called L_n^t by Barnes.

(ii) If $r = n + 1 - 2t$ we may assume that $\mathscr{V}$ consists of $2t$ vectors

$$(\ldots, \omega^{\pm 1}, \ldots; 0, \ldots, 0),$$

(where the non-zero coordinate is one of the first t); together with r further vectors

$$(0, \ldots, 0; \ldots, 1, \ldots)$$

(where it is one of the last r). Again $A_{n,r}$ is spanned by the differences of these vectors, and contains

$$5\binom{t}{2} + 2tr + \binom{r}{2}$$

pairs of minimal vectors, namely

$$4\binom{t}{2} \text{ pairs} \qquad \pm(\ldots, \omega^{\pm 1}, \ldots, -\omega^{\pm 1}, \ldots; 0, \ldots, 0),$$

$$\binom{t}{2} \text{ pairs} \qquad \pm(\ldots, 1, \ldots, -1, \ldots; 0, \ldots, 0),$$

$$2tr \text{ pairs} \qquad \pm(\ldots, \omega^{\pm 1}, \ldots; \ldots, -1, \ldots),$$

$$\binom{r}{2} \text{ pairs} \qquad \pm(0, \ldots, 0; \ldots, 1, \ldots, -1, \ldots).$$

This is the lattice called M_n^t by Barnes.

We remark that A_n^{+r}, $A_n^{(r)}$ and $A_{n,r}$ are indeed generalizations of A_n, because $A_n \cong A_n^{+1} \cong A_n^{(1)} \cong A_{n,n+1}$.

We showed in §4 that the perfect lattices in up to four dimensions are all root lattices:

$$P_1^1 \cong A_1, \quad P_2^1 \cong A_2, \quad P_3^1 \cong A_3 \cong D_3, \quad P_4^1 \cong D_4, \quad P_4^2 \cong A_4.$$

The perfect lattices in five and six dimensions can all be described either as root lattices or as these generalizations of A_n. The perfect five-dimensional lattices are

$$P_5^1 \cong D_5, \quad P_5^2 \cong A_{5,0} \cong A_{5,1} \cong A_5[2] \cong A_5^{+3}, \quad P_5^3 \cong A_5,$$

and in six dimensions $P_6^1, \ldots, P_6^7$ are respectively

$$E_6, \quad A_{6,0} \cong E_6^*, \quad D_6, \quad A_{6,2}, \quad A_6^{(2)}, \quad A_{6,1}, \quad A_6.$$

6. Properties of the known perfect lattices in dimensions $n \leqslant 7$

This section contains a detailed description of the known perfect lattices in dimensions $n \leqslant 7$.

Table 1 summarizes their main properties. These lattices have been known by many names in the past. In this paper we denote the n-dimensional perfect lattices by $P_n^1, P_n^2, \ldots$, in decreasing order of density, beginning with the absolutely extreme forms, except that we use lower-case letters $p_7^1, p_7^2, \ldots$ as provisional names for the known seven-dimensional perfect lattices. The first column of the table gives our symbol for the typical lattice and the second column gives other names for it. Successive columns then give d_2 (rounded to four decimal places), which is the determinant on the scale at which the minimal norm is 2; d and μ, which are the determinant and minimal norm on the finest scale at which all inner products are integers; s, the number of pairs $\pm v$ of minimal vectors; and g, the order of the automorphism group. Note that $d_2 = (2/\mu)^n d$. The density (see (1)) is given by

$$\Delta = V_n 2^{-\frac{1}{2}n} d_2^{-1},$$

and the kissing number of this lattice is $2s$.

The meaning of the symbols in the final column is as follows:

A, absolutely extreme; perfect, eutactic;

E, extreme (but not absolutely so); perfect, eutactic;

S, perfect (but not extreme); semi-eutactic (but not eutactic);

P, perfect (but not extreme); neither semi-eutactic nor eutactic.

The remainder of this section gives further information about each perfect lattice Λ. We first give the values of s, d and g. Then we define Λ to be the set of integer linear combinations of N vectors $A, B, C, \ldots$ (where $N \geqslant n$) whose inner product matrix is given. The n vectors indicated by arrows form an integral base for Λ. The remaining $N-n$ vectors can be eliminated using the relations (if any) given just below the inner product matrix. In cases when simple coordinates are available for these vectors we give them to the right of the inner product matrix ($+1$ and -1 are abbreviated to $+$ and $-$, respectively).

Then generators for the automorphism group of Λ are enclosed in angled

TABLE 1. SUMMARY OF PERFECT LATTICES IN DIMENSIONS $n \leqslant 7$.

(Key: d_2, determinant on scale at which minimal norm is 2; d and μ, determinant and minimal norm on finest scale at which lattice is integral; s, number of pairs of minimal vectors; g, order of group; A, absolutely extreme, E, extreme, S, semi-eutactic, P, perfect.)

name	other names	d_2	d	μ	s	g	type
P_1^1	$I_1 \cong A_1 \cong \phi_0$	2	1	1	1	2	A
P_2^1	$A_2 \cong \phi_0$	3	3	2	3	12	A
P_3^1	$A_3 \cong D_3 \cong \phi_0$	4	4	2	6	48	A
P_4^1	D_4	4	4	2	12	1152	A
P_4^2	$A_4 \cong \phi_0$	5	5	2	10	240	E
P_5^1	D_5	4	4	2	20	3840	A
P_5^2	$A_5^{+3} \cong A_{5,0} \cong A_{5,1}$	5.0625	162	4	15	1440	E
P_5^3	$A_5 \cong \phi_0$	6	6	4	15	1440	E
P_6^1	$E_6 \cong \phi_2$	3	3	2	36	103680	A
P_6^2	$E_6^* \cong A_{6,0} \cong \phi_4$	3.7969	243	4	27	103680	E
P_6^3	$D_6 \cong \phi_1$	4	4	2	30	46080	E
P_6^4	$A_{6,2} \cong \phi_5$	5.0625	324	4	22	288	S
P_6^5	$A_6^{(2)} \cong \Lambda_3(\lambda) \cong \phi_6$	5.3594	343	4	21	672	E
P_6^6	$A_{6,1} \cong \phi_3$	5.4844	351	4	21	96	E
P_6^7	$A_6 \cong \phi_0$	7	7	2	21	10080	E
p_7^1	$E_7 \cong A_7^{+2} \cong \phi_2$	2	2	2	63	$2^{10}3^45 \cdot 7$	A
p_7^2	$E_7^* \cong A_7^{+4} \cong \phi_{16}$	3.7458	64	3	28	$2^{10}3^45 \cdot 7$	E
p_7^3	$A_{7,1} \cong L_7^3 \cong \phi_4$	3.7969	486	4	36	2592	E
p_7^4	$D_7 \cong \phi_1$	4	4	2	42	645120	E
p_7^5	$A_7^{(2)} \cong P_7 \cong \phi_6$	4	512	4	36	256	E
p_7^6	ϕ_{20}	4.2140	9216	6	32	192	E
p_7^7	ϕ_{19}	4.2187	540	4	34	48	E
p_7^8	ϕ_9	4.5	576	4	32	128	E
p_7^9	ϕ_{13}	4.5660	356720	10	28	192	E
p_7^{10}	K_7	4.5937	588	4	32	64	E
p_7^{11}	K_5	4.6091	10080	6	30	8	E
p_7^{12}	ϕ_{15}	4.6822	10240	6	28	2880	E
p_7^{13}	K_9	4.6924	76880	8	28	40	E
p_7^{14}	ϕ_{21}	4.7261	10336	6	29	8	E
p_7^{15}	K_{10}	4.7374	77618	8	28	28	E
p_7^{16}	K_6	4.7407	10368	6	30	48	E
p_7^{17}	ϕ_{18}	4.8139	10528	6	28	8	E
p_7^{18}	K_2	4.8145	78880	8	28	8	P
p_7^{19}	ϕ_{11}	4.8395	10584	6	28	192	E
p_7^{20}	K_{11}	4.8733	10658	6	28	36	E
p_7^{21}	ϕ_{14}	4.9163	10752	6	28	32	E
p_7^{22}	ϕ_{10}	4.9219	630	4	30	32	E
p_7^{23}	K_3	4.9291	10780	6	28	16	E
p_7^{24}	K_1	4.9419	10808	6	28	4	E
p_7^{25}	K_4	5.0334	11008	6	28	8	E
p_7^{26}	$A_{7,3} \cong L_7^2 \cong \phi_5$	5.0625	648	4	30	864	S
p_7^{27}	$A_{7,0} \cong \phi_{12}$	5.0625	648	4	30	768	E
p_7^{28}	K_8	5.0625	648	4	30	24	E
p_7^{29}	ϕ_7	5.3437	684	4	29	32	P
p_7^{30}	ϕ_{17}	5.3594	686	4	28	28	E
p_7^{31}	ϕ_8	5.625	720	4	28	16	E
p_7^{32}	$A_{7,2} \cong M_7^3 \cong \phi_3$	5.9062	756	4	28	192	E
p_7^{33}	$A_7 \cong \phi_0$	8	8	2	28	80640	E

brackets. The vectors $A, B, C, \ldots$ have been carefully chosen so that the automorphisms are usually permutations of them (together with the operation -1 that negates every vector). Sometimes we were only able to achieve such an effect by adjoining other vectors $A', B', C', \ldots$. In these cases it is understood that $(X')' = X$ and that if a permutation takes X to Y then it takes X' to Y'. The letters R and S are reserved for root vectors, i.e. vectors $R \in \Lambda$ such that the reflection in R,

$$\mathrm{ref}_R : v \to v - \frac{2(v,R)}{(R,R)} R, \quad (v \in \Lambda),$$

is in the automorphism group.

We define the spectrum of a minimal vector $v \in \Lambda$ to be the system of numbers $n_0, n_1, \ldots, n_\nu$, $\nu = [\frac{1}{2}\mu]$, where n_k is the number of pairs of minimal vectors $\pm\omega \in \Lambda$ for which $(v, w) = \pm k$.

The rest of the text for a given lattice describes the orbits of the automorphism group on pairs of minimal vectors.

There is one line for each orbit. The first column gives the invariant eutactic coefficients c_k (see §3) multiplied by their common denominator. (For the lattices p_7^5, p_7^7, p_7^8, p_7^{10}, p_7^{11} the invariant eutactic coefficients are not uniquely determined, and in fact form a one-dimensional set which we have parametrized by a real variable a. If a is set to zero a particular positive set of eutactic coefficients is produced.)

The next entry (in parentheses) is the number of pairs of minimal vectors in the orbit. (If the orbit contains only l out of a larger number m pairs of vectors with a given spectrum we write (l/m) in the second column rather than (l).) We then give the spectrum of these vectors, and a representative vector for the orbit.

When $s = \frac{1}{2}n(n+1)$, the Voronoi neighbours of the lattice are in one-to-one correspondence with the pairs of minimal vectors, and we display these neighbours in the final column, headed 'nbr'.

Thus the entry

$78c_k$	number	spectrum			representative	nbr
5	(12)	2	10	8	$A-C$	P_6^1

for the lattice P_6^6 indicates that there are 12 pairs of minimal vectors like $\pm(A-C)$, each of which has eutactic coefficient $c_k = \frac{5}{78}$, is orthogonal (inner product 0) to two other pairs, has inner product ± 1 with 10 other pairs, and inner product ± 2 with the remaining eight pairs. The Voronoi neighbour corresponding to the wall obtained by removing such a vector is of type P_6^1.

The perfect lattices in dimensions one to four

$P_1^1 \cong I_1 \cong A_1 \cong \phi_0$. Absolutely extreme. ($s = 1$, $d = 1$, $g = 2$.)

	A	basis
$\to A$	$\boxed{1}$	$(+)$

Group: $\langle -1 \rangle$.

$2c_k$	number	spectrum	representative	nbr
1	(1)	0	A	P_1^1

$P_2^1 \cong A_2 \cong \phi_0$. Absolutely extreme. ($s = 3$, $d = 3$, $g = 12$.)

	A	B	basis
$\to A$	2	1	$(+\ 0\ -)$
$\to B$	1	2	$(0\ +\ -)$

Group: $\langle S_3$ on coordinates, $-1\rangle$.

$3c_k$	number	spectrum	representative	nbr
1	(3)	0 2	A	P_2^1

$P_3^1 \cong A_3 \cong D_3 \cong \phi_0$. Absolutely extreme. ($s = 6$, $d = 4$, $g = 48$.)

	A	B	C	basis
$\to A$	2	1	1	$(0\ +\ +)$
$\to B$	1	2	1	$(+\ 0\ +)$
$\to C$	1	1	2	$(+\ +\ 0)$

Group: ⟨all permutations and sign-changes of coordinates⟩.

$4c_k$	number	spectrum	representative	nbr
1	(6)	1 4	A	P_3^1

$P_4^1 \cong D_4$. Absolutely extreme. ($s = 12$, $d = 4$, $g = 1152$.)

	A	B	C	R	basis
$\to A$	2	1	1	2	$(+\ 0\ 0\ +)$
$\to B$	1	2	1	2	$(0\ +\ 0\ +)$
$\to C$	1	1	2	2	$(0\ 0\ +\ +)$
$\to R$	2	2	2	4	$(0\ 0\ 0\ 2)$

Group: ⟨all permutations and sign-changes of coordinates, also right-multiplication of vectors by

$$\frac{1}{2}\begin{pmatrix} + & + & + & + \\ + & - & + & - \\ + & + & - & - \\ + & - & - & + \end{pmatrix}\rangle.$$

$6c_k$	number	spectrum	representative
1	(12)	3 8	A

$P_4^2 \cong A_4 \cong \phi_0$. Extreme. ($s = 10$, $d = 5$, $g = 2 \cdot 5! = 240$.)

	A	B	C	D	basis
$\to A$	2	1	1	1	$(+\ 0\ 0\ 0\ -)$
$\to B$	1	2	1	1	$(0\ +\ 0\ 0\ -)$
$\to C$	1	1	2	1	$(0\ 0\ +\ 0\ -)$
$\to D$	1	1	1	2	$(0\ 0\ 0\ +\ -)$

Group: $\langle S_5$ on coordinates, $-1\rangle$.

$5c_k$	number	spectrum	representative	nbr
1	(10)	3 6	A	P_4^1

The five-dimensional perfect lattices

$P_5^1 \cong D_5$. Absolutely extreme. ($s = 20$, $d = 4$, $g = 2^5 \cdot 5! = 3840$.)

	A	B	C	D	R	basis
$\rightarrow A$	2	1	1	1	2	(+ 0 0 0 +)
$\rightarrow B$	1	2	1	1	2	(0 + 0 0 +)
$\rightarrow C$	1	1	2	1	2	(0 0 + 0 +)
$\rightarrow D$	1	1	1	2	2	(0 0 0 + +)
$\rightarrow R$	2	2	2	2	4	(0 0 0 0 2)

Group: ⟨all permutations and sign-changes of coordinates⟩.

$8c_k$	number	spectrum	representative
1	(20)	7 12	A

$P_5^2 \cong A_5^{+3} \cong A_5[2] \cong A_{5,1} \cong A_{5,0}$. Extreme. ($s = 15$, $d = 162$, $g = 1440$.)

	A	B	C	D	E	F	basis
$\rightarrow A$	4	1	1	−2	−2	−2	$(-1\ +2\ +2\ -1\ -1\ -1)/\sqrt{3}$
$\rightarrow B$	1	4	1	−2	−2	−2	$(+2\ -1\ +2\ -1\ -1\ -1)/\sqrt{3}$
$\rightarrow C$	1	1	4	−2	−2	−2	$(+2\ +2\ -1\ -1\ -1\ -1)/\sqrt{3}$
$\rightarrow D$	−2	−2	−2	4	1	1	$(-1\ -1\ -1\ -1\ +2\ +2)/\sqrt{3}$
$\rightarrow E$	−2	−2	−2	1	4	1	$(-1\ -1\ -1\ +2\ -1\ +2)/\sqrt{3}$
F	−2	−2	−2	1	1	4	$(-1\ -1\ -1\ +2\ +2\ -1)/\sqrt{3}$

Relations: $A+B+C+D+E+F = 0$.
Group: $\langle S_6$ on coordinates, $-1\rangle$.

$12c_k$	number	spectrum	representative	nbr
1	(15)	0 8 6	A	P_5^1

$P_5^3 \cong A_5 \cong \phi_0$. Extreme. ($s = 15$, $d = 6$, $g = 2 \cdot 6! = 1440$.)

	A	B	C	D	E	basis
$\rightarrow A$	2	1	1	1	1	(+ 0 0 0 0 −)
$\rightarrow B$	1	2	1	1	1	(0 + 0 0 0 −)
$\rightarrow C$	1	1	2	1	1	(0 0 + 0 0 −)
$\rightarrow D$	1	1	1	2	1	(0 0 0 + 0 −)
$\rightarrow E$	1	1	1	1	2	(0 0 0 0 + −)

Group: $\langle S_6$ on coordinates, $-1\rangle$.

$6c_k$	number	spectrum	representative	nbr
1	(15)	6 8	A	P_5^1

The six-dimensional perfect lattices

$P_6^1 \cong E_6 \cong \phi_2$. Absolutely extreme. ($s = 36$, $d = 3$, $g = 2^8 \cdot 3^4 \cdot 5 = 103680$.)

	A	B	C	D	E	F	basis
$\rightarrow A$	2	1	1	1	1	0	$(0+0\ 0\ 0\ 0-0)$
$\rightarrow B$	1	2	1	1	1	0	$(0\ 0+0\ 0\ 0-0)$
$\rightarrow C$	1	1	2	1	1	1	$(0\ 0\ 0+0\ 0-0)$
$\rightarrow D$	1	1	1	2	1	1	$(0\ 0\ 0\ 0+0-0)$
$\rightarrow E$	1	1	1	1	2	1	$(0\ 0\ 0\ 0\ 0+-0)$
$\rightarrow F$	0	0	1	1	1	2	$(+--+++--)/2$

Group: $\langle$Weyl group of type E_6, $-1\rangle$.

$12c_k$	number	spectrum	representative
1	(36)	15 20	A

$P_6^2 \cong E_6^* \cong A_{6,0} \cong \phi_4$. Extreme. ($s = 27$, $d = 243$, $g = 2^8 \cdot 3^4 \cdot 5 = 103680$.)

	A	B	C	D	E	F	G	basis
A	4	1	1	1	1	2	2	$(\ 0+2+2-1-1-1-1\ \ 0)/\sqrt{3}$
$\rightarrow B$	1	4	1	1	1	2	2	$(\ 0+2-1+2-1-1-1\ \ 0)/\sqrt{3}$
$\rightarrow C$	1	1	4	1	1	2	2	$(\ 0+2-1-1+2-1-1\ \ 0)/\sqrt{3}$
$\rightarrow D$	1	1	1	4	1	2	2	$(\ 0+2-1-1-1+2-1\ \ 0)/\sqrt{3}$
$\rightarrow E$	1	1	1	1	4	2	2	$(\ 0+2-1-1-1-1+2\ \ 0)/\sqrt{3}$
$\rightarrow F$	2	2	2	2	2	4	1	$(+3+5-1-1-1-1-1-3)/2\sqrt{3}$
$\rightarrow G$	2	2	2	2	2	1	4	$(-3+5-1-1-1-1-1+3)/2\sqrt{3}$

Relation: $A+B+C+D+E = 2F+2G$.
Group: $\langle$Weyl group of type E_6, $-1\rangle$.

$18c_k$	number	spectrum	representative
1	(27)	0 16 10	A

$P_6^3 \cong D_6 \cong \phi_1$. Extreme. ($s = 30$, $d = 4$, $g = 2^6 \cdot 6! = 46080$.)

	A	B	C	D	E	R	basis
$\rightarrow A$	2	1	1	1	1	2	$(+\ 0\ 0\ 0\ 0+)$
$\rightarrow B$	1	2	1	1	1	2	$(0+0\ 0\ 0+)$
$\rightarrow C$	1	1	2	1	1	2	$(0\ 0+0\ 0+)$
$\rightarrow D$	1	1	1	2	1	2	$(0\ 0\ 0+0+)$
$\rightarrow E$	1	1	1	1	2	2	$(0\ 0\ 0\ 0++)$
$\rightarrow R$	2	2	2	2	2	4	$(0\ 0\ 0\ 0\ 0\ 2)$

Group: $\langle$all permutations and sign-changes of coordinates$\rangle$.

$10c_k$	number	spectrum	representative
1	(30)	13 16	A

$P_6^4 \cong A_{6,2} \cong \phi_5$. Semi-eutactic. ($s = 22$, $d = 324$, $g = 288$.)

	A	B	C	D	E	F	R	basis
A	4	1	1	2	2	2	2	$\sqrt{2}(1\ 0; 0\ -1)$
$\to B$	1	4	1	2	2	2	2	$\sqrt{2}(\omega\ 0; 0\ -1)$
$\to C$	1	1	4	2	2	2	2	$\sqrt{2}(\bar{\omega}\ 0; 0\ -1)$
$\to D$	2	2	2	4	1	1	2	$\sqrt{2}(0\ 1; 0\ -1)$
$\to E$	2	2	2	1	4	1	2	$\sqrt{2}(0\ \omega; 0\ -1)$
$\to F$	2	2	2	1	1	4	2	$\sqrt{2}(0\ \bar{\omega}; 0\ -1)$
$\to R$	2	2	2	2	2	2	4	$\sqrt{2}(0\ 0; 1\ -1)$

Relations: $A+B+C = D+E+F$, $A+A' = \ldots = F+F' = R$.
Group: $\langle S_3(A,B,C),\ S_3(D,E,F),\ (AD)(BE)(CF),\ (XX'),\ -1\rangle$.

$36c_k$	number	spectrum			representative
3	(12)	3	10	8	A
2	(9)	1	12	8	$A-D$
0	(1)	9	0	12	R

$P_6^5 \cong A_6^{(2)} \cong \Lambda_3(\lambda) \cong P_6 \cong \phi_6$. Extreme. ($s = 21$, $d = 343$, $g = 672$.)

	A	B	C	D	E	F	G	basis
$\to A$	4	−2	1	−1	−1	1	−2	$v_{0,3} = (0\ +\ -\ 0\ 0\ -\ +)$
$\to B$	−2	4	−2	1	−1	−1	1	$v_{1,4} = (+\ 0\ +\ -\ 0\ 0\ -)$
$\to C$	1	−2	4	−2	1	−1	−1	$v_{2,5} = (-\ +\ 0\ +\ -\ 0\ 0)$
$\to D$	−1	1	−2	4	−2	1	−1	$v_{3,6} = (0\ -\ +\ 0\ +\ -\ 0)$
$\to E$	−1	−1	1	−2	4	−2	1	$v_{4,0} = (0\ 0\ -\ +\ 0\ +\ -)$
$\to F$	1	−1	−1	1	−2	4	−2	$v_{5,1} = (-\ 0\ 0\ -\ +\ 0\ +)$
G	−2	1	−1	−1	1	−2	4	$v_{6,2} = (+\ -\ 0\ 0\ -\ +\ 0)$

Relations: $A+B+C+D+E+F+G = 0$.
Group: $\langle v_{i,j} \to v_{i+1,j+1}, v_{i,j} \to v_{2i,2j}, v_{i,j} \to v_{-j,-i}, v_{i,j} \to \pm v_{\pi(i),\bar{\pi}(j)}, -1\rangle$, where the $\pm$ sign is $+$ if and only if $ij \equiv 0, 1, 2, 4 \pmod 7$ and

$$v_{n+1,n} = v_{n,n+3} + v_{n+2,n+5} + v_{n+3,n+6} + v_{n+6,n+2},$$

$$v_{n+2,n} = v_{n+5,n+1} + v_{n+6,n+2},$$

with subscripts read modulo 7, and $\pi = (1\ 2)\ (3\ 6)$, $\bar{\pi} = (1\ 4)\ (5\ 6)$.

$14c_k$	number	spectrum			representative	nbr
1	(21)	4	8	8	A	P_6^1

$P_6^6 \cong A_{6,1} \cong \phi_3$. Extreme. ($s = 21$, $d = 351$, $g = 96$.)

	A	B	C	D	E	F	basis
$\to A$	4	1	2	2	2	2	$\sqrt{2}(\omega\ 0\ 0;\ -1)$
$\to B$	1	4	2	2	2	2	$\sqrt{2}(\bar{\omega}\ 0\ 0;\ -1)$
$\to C$	2	2	4	1	2	2	$\sqrt{2}(0\ \omega\ 0;\ -1)$
$\to D$	2	2	1	4	2	2	$\sqrt{2}(0\ \bar{\omega}\ 0;\ -1)$
$\to E$	2	2	2	2	4	1	$\sqrt{2}(0\ 0\ \omega;\ -1)$
$\to F$	2	2	2	2	1	4	$\sqrt{2}(0\ 0\ \bar{\omega};\ -1)$

Group: $\langle (AB), (CD), (EF), (ACE)(BDF), (AF)(BE)(CD), -1 \rangle$.

$78k_c$	number	spectrum			representative	nbr
5	(12)	2	10	8	$A-C$	P_6^1
6	(6)	5	7	8	A	P_6^4
7	(3)	2	12	6	$A+B-C-D$	P_6^3

$P_6^7 \cong A_6 \cong \phi_0$. Extreme. ($s = 21$, $d = 7$, $g = 2 \cdot 7! = 10080$.)

	A	B	C	D	E	F	basis
$\to A$	2	1	1	1	1	1	$(+\ 0\ 0\ 0\ 0\ 0\ -)$
$\to B$	1	2	1	1	1	1	$(0\ +\ 0\ 0\ 0\ 0\ -)$
$\to C$	1	1	2	1	1	1	$(0\ 0\ +\ 0\ 0\ 0\ -)$
$\to D$	1	1	1	2	1	1	$(0\ 0\ 0\ +\ 0\ 0\ -)$
$\to E$	1	1	1	1	2	1	$(0\ 0\ 0\ 0\ +\ 0\ -)$
$\to F$	1	1	1	1	1	2	$(0\ 0\ 0\ 0\ 0\ +\ -)$

Group: $\langle S_7$ on coordinates, $-1 \rangle$.

$7c_k$	number	spectrum		representative	nbr
1	(21)	10	10	A	P_6^3

The known seven-dimensional perfect lattices

$p_7^1 \cong E_7 \cong A_7^{+2} \cong A_7[4] \cong \phi_2$. Absolutely extreme. ($s = 63$, $d = 2$, $g = 2^{10}3^45 \cdot 7 = 2903040$.)

	A	B	C	D	E	F	G	H	basis
A	2	1	1	1	1	1	1	0	$(+\ 0\ 0\ 0\ 0\ 0\ 0\ -)$
$\to B$	1	2	1	1	1	1	1	0	$(0\ +\ 0\ 0\ 0\ 0\ 0\ -)$
$\to C$	1	1	2	1	1	1	1	0	$(0\ 0\ +\ 0\ 0\ 0\ 0\ -)$
$\to D$	1	1	1	2	1	1	1	1	$(0\ 0\ 0\ +\ 0\ 0\ 0\ -)$
$\to E$	1	1	1	1	2	1	1	1	$(0\ 0\ 0\ 0\ +\ 0\ 0\ -)$
$\to F$	1	1	1	1	1	2	1	1	$(0\ 0\ 0\ 0\ 0\ +\ 0\ -)$
$\to G$	1	1	1	1	1	1	2	1	$(0\ 0\ 0\ 0\ 0\ 0\ +\ -)$
$\to H$	0	0	0	1	1	1	1	2	$(-\ -\ -\ +\ +\ +\ +\ -)/2$

Relations: $A+B+C+2H = D+E+F+G$.
Group: Weyl group of type E_7.

$18c_k$	number	spectrum		representative
1	(63)	30	32	A

$p_7^2 \cong E_7^* \cong A_7^{+4} \cong A_7[2] \cong \phi_{16}$. Extreme. ($s = 28$, $d = 64$, $g = 2^{10}3^45 \cdot 7 = 2903040$.)

	A	B	C	D	E	F	G	R	basis
A	3	1	1	1	1	1	1	2	$(+3\ -1\ -1\ -1\ -1\ -1\ -1\ +3)/\sqrt{8}$
$\to B$	1	3	1	1	1	1	1	2	$(-1\ +3\ -1\ -1\ -1\ -1\ -1\ +3)/\sqrt{8}$
$\to C$	1	1	3	1	1	1	1	2	$(-1\ -1\ -3\ -1\ -1\ -1\ -1\ +3)/\sqrt{8}$
$\to D$	1	1	1	3	1	1	1	2	$(-1\ -1\ -1\ +3\ -1\ -1\ -1\ +3)/\sqrt{8}$
$\to E$	1	1	1	1	3	1	1	2	$(-1\ -1\ -1\ -1\ +3\ -1\ -1\ +3)/\sqrt{8}$
$\to F$	1	1	1	1	1	3	1	2	$(-1\ -1\ -1\ -1\ -1\ +3\ -1\ +3)/\sqrt{8}$
$\to G$	1	1	1	1	1	1	3	0	$(-1\ -1\ -1\ -1\ -1\ -1\ +3\ +3)/\sqrt{8}$
$\to R$	2	2	2	2	2	2	0	4	$(\ 0\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ -4\ +4)/\sqrt{8}$

Relations: $A+B+C+D+E+F = 2G+3R$.
Group: Weyl group of type E_7.

$12c_k$	number	spectrum	representative	nbr
1	(28)	0 27	A	p_7^1

$p_7^3 \cong A_{7,1} \cong L_7^3 \cong \phi_4$. Extreme. ($s = 36$, $d = 486$, $g = 2592$.)

	A	B	C	D	E	F	G	H	I	basis
A	4	1	1	2	2	2	2	2	2	$\sqrt{2}(1\ 0\ 0;\ -1)$
$\to B$	1	4	1	2	2	2	2	2	2	$\sqrt{2}(\omega\ 0\ 0;\ -1)$
$\to C$	1	1	4	2	2	2	2	2	2	$\sqrt{2}(\bar{\omega}\ 0\ 0;\ -1)$
$\to D$	2	2	2	4	1	1	2	2	2	$\sqrt{2}(0\ 1\ 0;\ -1)$
$\to E$	2	2	2	1	4	1	2	2	2	$\sqrt{2}(0\ \omega\ 0;\ -1)$
$\to F$	2	2	2	1	1	4	2	2	2	$\sqrt{2}(0\ \bar{\omega}\ 0;\ -1)$
$\to G$	2	2	2	2	2	2	4	1	1	$\sqrt{2}(0\ 0\ 1;\ -1)$
$\to H$	2	2	2	2	2	2	1	4	1	$\sqrt{2}(0\ 0\ \omega;\ -1)$
I	2	2	2	2	2	2	1	1	4	$\sqrt{2}(0\ 0\ \bar{\omega};\ -1)$

Relations: $A+B+C = D+E+F = G+H+I$.
Group: $\langle S_3(A,B,C),\ S_3(D,E,F),\ S_3(G,H,I),\ S_3$ (blocks), $-1\rangle$.

$108c_k$	number	spectrum	representative
5	(27)	3 20 12	$A-D$
6	(9)	9 14 12	A

$p_7^4 \cong D_7 \cong \phi_1$. Extreme. ($s = 42$, $d = 4$, $g = 2^7 \cdot 7! = 645120$.)

	A	B	C	D	E	F	R	basis
$\to A$	2	1	1	1	1	1	2	$(+\ 0\ 0\ 0\ 0\ 0\ +)$
$\to B$	1	2	1	1	1	1	2	$(\ 0\ +\ 0\ 0\ 0\ 0\ +)$
$\to C$	1	1	2	1	1	1	2	$(\ 0\ 0\ +\ 0\ 0\ 0\ +)$
$\to D$	1	1	1	2	1	1	2	$(\ 0\ 0\ 0\ +\ 0\ 0\ +)$
$\to E$	1	1	1	1	2	1	2	$(\ 0\ 0\ 0\ 0\ +\ 0\ +)$
$\to F$	1	1	1	1	1	2	2	$(\ 0\ 0\ 0\ 0\ 0\ +\ +)$
$\to R$	2	2	2	2	2	2	4	$(\ 0\ 0\ 0\ 0\ 0\ 0\ 2)$

Group: ⟨all permutations and sign-changes of coordinates⟩.

$12c_k$	number	spectrum		representative
1	(42)	21	20	A

$p_7^5 \cong A_7^{(2)} \cong P_7 \cong \phi_6$. Extreme. ($s = 36$, $d = 512$, $g = 256$.)

	A	B	C	D	E	F	G	H	basis
$\to A$	4	1	2	1	2	2	2	2	(+ − 0 − + 0 0 0)
$\to B$	1	4	1	2	2	2	2	2	(0 0 + − 0 − + 0)
$\to C$	2	1	4	1	2	2	2	2	(+ 0 0 0 + − 0 −)
$\to D$	1	2	1	4	2	2	2	2	(0 − + 0 0 0 + −)
$\to E$	2	2	2	2	4	1	2	1	(0 − + 0 + − 0 0)
$\to F$	2	2	2	2	1	4	1	2	(0 0 0 − + 0 + −)
$\to G$	2	2	2	2	2	1	4	1	(+ − 0 0 0 − + 0)
H	2	2	2	2	1	2	1	4	(+ 0 + − 0 0 0 −)

Relations: $A+B+C+D = E+F+G+H$.
Group: ⟨$(ABCD)$, $(AD)(BC)$, $(EFGH)$, $(EH)(FG)$, $(AE)(BF)(CG)(DH)$, -1⟩.

$64c_k$	number	spectrum			representative
3	(16)	9	12	14	$A-E$
4	(8)	7	18	10	A
$3-a$	(8)	11	8	16	$A+B-E-F$
$2+2a$	(4)	17	0	18	$A-C$

$p_7^6 \cong \phi_{20}$. Extreme. ($s = 32$, $d = 9216$, $g = 192$.)

	A	B	C	D	E	F	G	H
$\to A$	6	2	2	2	3	3	3	3
$\to B$	2	6	2	2	3	3	3	3
$\to C$	2	2	6	2	3	3	3	3
$\to D$	2	2	2	6	3	3	3	3
$\to E$	3	3	3	3	6	2	3	1
$\to F$	3	3	3	3	2	6	1	3
$\to G$	3	3	3	3	3	1	6	2
H	3	3	3	3	1	3	2	6

Relations: $A+B+C+D = E+F+G+H$.
Group: ⟨$S_4(A,B,C,D)$, $V_4(E,F,G,H)$, -1⟩.

$96c_k$	number	spectrum				representative
3	(16)	1	12	8	10	$A-E$
4	(6)	0	12	11	8	$A+B-F-G$
4	(4)	2	12	9	8	A
4	(4)	4	12	5	10	E
4	(2)	4	10	7	10	$E-G$

$p_7^7 \cong \phi_{19}$. Extreme. ($s = 34$, $d = 540$, $g = 48$.)

	A	B	C	D	E	F	G	H	I	R
$\rightarrow A$	4	2	2	0	1	1	1	2	2	0
$\rightarrow B$	2	4	2	1	0	1	2	1	2	0
$\rightarrow C$	2	2	4	1	1	0	2	2	1	0
$\rightarrow D$	0	1	1	4	2	2	1	2	2	0
$\rightarrow E$	1	0	1	2	4	2	2	1	2	0
$\rightarrow F$	1	1	0	2	2	4	2	2	1	0
$\rightarrow G$	1	2	2	1	2	2	4	2	2	2
H	2	1	2	2	1	2	2	4	2	2
I	2	2	1	2	2	1	2	2	4	2
R	0	0	0	0	0	0	2	2	2	4

Relations: $A+D+G = B+E+H = C+F+I = G+H+I-R$, $G-G' = H-H' = I-I' = R$.
Group: $\langle (ABC)(DEF)(GHI),\ (AC)(DF)(GI),\ (AD)(BE)(CF),\ (GG')(HH')(II'),\ -1\rangle$.

$180c_k$	number	spectrum			representative
$10+a$	(12)	7	14	12	$A-H$
$10-2a$	(6)	9	12	12	A
$8+2a$	(6)	9	10	14	G
10	(6)	5	18	10	$A-B$
$8-2a$	(3)	5	16	12	$G-H$
$3-6a$	(1)	15	0	18	R

$p_7^8 \cong \phi_9$. Extreme. ($s = 32$, $d = 576$, $g = 128$.)

	A	B	C	D	E	F	G	H	R	S
A	4	1	0	0	−1	2	−1	−1	2	0
B	1	4	0	0	2	−1	−1	−1	2	0
C	0	0	4	1	−1	−1	−1	2	2	0
$\rightarrow D$	0	0	1	4	−1	−1	2	−1	2	0
$\rightarrow E$	−1	2	−1	−1	4	1	0	0	0	2
$\rightarrow F$	2	−1	−1	−1	1	4	0	0	0	2
$\rightarrow G$	−1	−1	−1	2	0	0	4	1	0	2
$\rightarrow H$	−1	−1	2	−1	0	0	1	4	0	2
$\rightarrow R$	2	2	2	2	0	0	0	0	4	0
$\rightarrow S$	0	0	0	0	2	2	2	2	0	4

Relations: $A+A' = \ldots = D+D' = R$, $E+E' = \ldots = H+H' = S$, $A+E = B+F = Y$ (say), $C+G = D+H = Z$ (say), $Y+Z = R+S$.

Group: $\langle (AB)(EF),\ (CD)(GH),\ (AC)(BD)(EG)(FH),\ (AE)(BF)(CG)(DH)(RS),$ $\mathrm{ref}_R,\ \mathrm{ref}_S,\ -1\rangle$.

$48c_k$	number	spectrum			representative
3	(16)	5	16	10	A
$2+a$	(8)	7	12	12	$A-F$
2	(4)	3	18	10	$A+C-R$
$2-2a$	(2)	15	0	16	R
$3-2a$	(1)	3	16	12	$A+B+C+D-2R$
$5-2a$	(1)	7	16	8	$C+D+E+F-R-S$

$p_7^9 \cong \phi_{13}$. Extreme. ($s = 28$, $d = 356720$, $g = 192$.)

	A	B	C	D	E	F	R
$\to A$	10	3	3	3	3	3	8
$\to B$	3	10	3	3	3	3	8
$\to C$	3	3	10	3	3	3	8
$\to D$	3	3	3	10	3	3	8
$\to E$	3	3	3	3	10	5	8
$\to F$	3	3	3	3	5	10	8
$\to R$	8	8	8	8	8	8	16

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle S_4(A,B,C,D),\ (EF),\ (XX'),\ -1\rangle$.

$7280c_k$	number	spectrum						representative	nbr
142	(8)	1	1	10	5	0	10	A	p_7^1
109	(8)	2	0	9	6	4	6	$A-E'$	p_7^3
269	(6)	1	0	9	8	5	4	$A-B'$	p_7^8
171	(4)	4	0	7	6	0	10	E	p_7^6
339	(1)	15	0	0	0	0	12	$E-F$	p_7^{12}
451	(1)	1	8	6	4	8	0	$A+B+C+D+E+F-3R$	p_7^4

$p_7^{10} \cong K_7$. Extreme. ($s = 32$, $d = 588$, $g = 64$.)

	A	B	C	D	E	F	G	H	R
A	4	1	2	2	0	2	2	1	2
B	1	4	1	2	2	0	2	2	2
$\to C$	2	1	4	1	2	2	0	2	2
$\to D$	2	2	1	4	1	2	2	0	2
$\to E$	0	2	2	1	4	1	2	2	2
$\to F$	2	0	2	2	1	4	1	2	2
$\to G$	2	2	0	2	2	1	4	1	2
$\to H$	1	2	2	0	2	2	1	4	2
$\to R$	2	2	2	2	2	2	2	2	4

Relations: $A+E = C+G$, $B+F = D+H$, $A+A' = \ldots = H+H' = R$.

Group: $\langle (ABCDEFGH),\ (AH)(BG)(CF)(DE),\ (XX'),\ -1\rangle$.

$84c_k$	number	spectrum			representative
$4+a$	(16)	8	11	12	A
6	(8)	7	14	10	$A-D$
$4-2a$	(4)	9	10	12	$A-C$
$6-2a$	(2/3)	5	16	10	$A-E'$
$5-4a$	(1)	15	0	16	R
2	(1/3)	5	16	10	$A+E-B-F$

$p_7^{11} \cong K_5$. Extreme. ($s = 30$, $d = 10080$, $g = 8$.)

	A	B	C	D	E	F	G	H
A	6	0	3	1	3	3	2	2
$\rightarrow B$	0	6	1	3	3	3	2	2
$\rightarrow C$	3	1	6	2	2	2	−1	−1
$\rightarrow D$	1	3	2	6	2	2	−1	−1
$\rightarrow E$	3	3	2	2	6	0	3	1
$\rightarrow F$	3	3	2	2	0	6	1	3
$\rightarrow G$	2	2	−1	−1	3	1	6	3
$\rightarrow H$	2	2	−1	−1	1	3	3	6

Relations: $A+B = E+F$.
Group: $\langle (AB)(CD),\ (EF)(GH),\ -1\rangle$.

$2520c_k$	number	spectrum				representative
$75+a$	(4)	8	5	4	12	$B-D-G$
99	(4)	2	11	8	8	$D-E+G$
129	(4)	4	9	8	8	$D-E-F+H$
117	(2)	6	8	5	10	A
48	(2/4)	3	10	6	10	C
45	(2/4)	3	10	6	10	E
$78+a$	(2)	3	9	7	10	G
$102-2a$	(2)	5	8	6	10	$A-C$
135	(2)	4	10	7	8	$A-E$
162	(2)	2	11	10	6	$E-G$
$100+a$	(2)	7	7	3	12	$C+D-E-F+H$
$67-2a$	(1)	7	6	4	12	$G-H$
$87-2a$	(1)	5	6	6	12	$C+D-E-F+G+H$

$p_7^{12} \cong \phi_{15}$. Extreme. ($s = 28$, $d = 10240$, $g = 2880$.)

	A	B	C	D	E	F	R
$\to A$	6	2	2	2	2	2	5
$\to B$	2	6	2	2	2	2	5
$\to C$	2	2	6	2	2	2	5
$\to D$	2	2	2	6	2	2	5
$\to E$	2	2	2	2	6	2	5
$\to F$	2	2	2	2	2	6	5
$\to R$	5	5	5	5	5	5	10

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle S_6(A,B,C,D,E,F),\ (XX'),\ -1\rangle$.

$240c_k$	number	spectrum				representative	nbr
11	(15)	0	8	15	4	$A-B'$	p_7^9
8	(12)	0	12	5	10	A	p_7^1
19	(1)	0	12	15	0	$A+B+C+D+E+F-3R$	p_7^{28}

$p_7^{13} \cong K_9$. Extreme. ($s = 28$, $d = 76880$, $g = 40$.)

	A	B	C	D	E	F	G
$\to A$	8	1	3	3	1	4	4
$\to B$	1	8	1	3	3	4	4
$\to C$	3	1	8	1	3	4	4
$\to D$	3	3	1	8	1	4	4
$\to E$	1	3	3	1	8	4	4
$\to F$	4	4	4	4	4	8	3
$\to G$	4	4	4	4	4	3	8

Group: $\langle (ABCDE),\ (AE)(BD),\ (FG),\ -1\rangle$.

$2480c_k$	number	spectrum					representative	nbr
87	(10)	0	9	5	7	6	$A-F$	p_7^{11}
66	(5)	1	10	2	6	8	A	p_7^5
43	(5)	2	10	1	4	10	$A+B-F-G$	p_7^{25}
117	(5)	2	8	5	6	6	$A+B+C-F-G$	p_7^{27}
61	(2)	0	10	1	6	10	F	p_7^2
48	(1)	5	0	12	0	10	$A+B+C+D+E-2F-2G$	p_7^{28}

$p_7^{14} \cong \phi_{21}$. Extreme. ($s = 29$, $d = 10336$, $g = 8$.)

	A	B	C	D	E	F	G	H
$\rightarrow A$	6	−1	0	−3	2	1	2	2
$\rightarrow B$	−1	6	−3	0	1	2	2	2
$\rightarrow C$	0	−3	6	3	0	1	1	1
$\rightarrow D$	−3	0	3	6	1	0	1	1
$\rightarrow E$	2	1	0	1	6	−1	1	1
$\rightarrow F$	1	2	1	0	−1	6	1	1
$\rightarrow G$	2	2	1	1	1	1	6	2
H	2	2	1	1	1	1	2	6

Relations: $2A+2B+C+D = E+F+G+H$.
Group: $\langle (AB)(CD)(EF), (GH), -1 \rangle$.

$7752c_k$	number	spectrum				representative
359	(4/6)	2	10	8	8	$B+C-G$
287	(4)	5	7	6	10	$A+B+C-G$
301	(2/4)	3	9	8	8	A
348	(2/4)	3	9	8	8	C
313	(2)	4	9	7	8	E
275	(2)	1	10	9	8	G
133	(2)	2	10	6	10	$B+C$
444	(2)	7	7	4	10	$B+C-D$
599	(2)	3	11	8	6	$B+C-F$
85	(2)	3	10	5	10	$B+C-D-F$
277	(2)	2	9	9	8	$A+B+C-E-F$
371	(2/6)	2	10	8	8	$A+B+C-F-G$
168	(1)	8	4	4	12	$C-D$

$p_7^{15} \cong K_{10}$. Extreme. ($s = 28$, $d = 77618$, $g = 28$.)

	A	B	C	D	E	F	G
$\rightarrow A$	8	−4	0	1	1	0	−4
$\rightarrow B$	−4	8	−4	0	1	1	0
$\rightarrow C$	0	−4	8	−4	0	1	1
$\rightarrow D$	1	0	−4	8	−4	0	1
$\rightarrow E$	1	1	0	−4	8	−4	0
$\rightarrow F$	0	1	1	0	−4	8	−4
$\rightarrow G$	−4	0	1	1	0	−4	8

Group: $\langle (ABCDEFG), (AG)(BF)(CE), -1 \rangle$.

$1576c_k$	number	spectrum					representative	nbr
54	(7)	3	6	6	4	8	A	p_7^7
29	(7)	2	10	1	4	10	$A+B$	p_7^1
63	(7)	1	8	6	6	6	$A+B+C$	p_7^{14}
51	(7)	0	6	9	6	6	$A+B+C-E-F$	p_7^{11}

$p_7^{16} \cong K_6$. Extreme. ($s = 30$, $d = 10368$, $g = 48$.)

	A	B	C	D	E	F	R
$\to A$	6	1	1	2	3	3	4
$\to B$	1	6	1	1	2	3	4
$\to C$	1	1	6	1	1	2	4
$\to D$	2	1	1	6	1	1	4
$\to E$	3	2	1	1	6	1	4
$\to F$	3	3	2	1	1	6	4
$\to R$	4	4	4	4	4	4	8

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle (ABCDEFA'B'C'D'E'F'),\ (AF)(BE)(CD),\ -1 \rangle$.

$72c_k$	number	spectrum				representative
3	(12)	1	12	8	8	A
2	(6)	5	8	4	12	$A-E$
3	(6)	2	12	7	8	$A-F$
3	(6)	4	10	5	10	$A+B-E-F$

$p_7^{17} \cong \phi_{18}$. Extreme. ($s = 28$, $d = 10528$, $g = 8$.)

	A	B	C	D	E	F	G
$\to A$	6	0	2	0	1	1	3
$\to B$	0	6	0	2	1	1	3
$\to C$	2	0	6	3	3	3	1
$\to D$	0	2	3	6	3	3	1
$\to E$	1	1	3	3	6	2	-1
$\to F$	1	1	3	3	2	6	-1
$\to G$	3	3	1	1	-1	-1	6

Group: $\langle (AB)(CD),\ (EF),\ -1 \rangle$.

$7896c_k$	number	spectrum				representative	nbr
365	(4/5)	2	10	7	8	$D-E$	p_7^7
341	(4)	1	10	10	6	$B+C-E-G$	p_7^{15}
479	(2)	4	8	9	6	B	p_7^8
349	(2)	4	8	7	8	D	p_7^1
142	(2)	1	10	6	10	E	p_7^1
85	(2)	2	10	5	10	$B-G$	p_7^1
420	(2)	2	8	11	6	$A+B-E-G$	p_7^{11}
446	(2)	1	10	8	8	$B+C-E-F-G$	p_7^5
168	(2)	6	6	5	10	$A+B+D-E-F-G$	p_7^6
239	(2)	3	8	8	8	$A+B+D-E-F-2G$	p_7^3
529	(1)	4	10	5	8	G	p_7^1
385	(1)	7	6	4	10	$C-D$	p_7^{13}
361	(1/5)	2	10	7	8	$A+B-G$	p_7^1
457	(1)	5	8	6	8	$A+B+C+D-E-F-2G$	p_7^1

$p_7^{18} \cong K_2$. Perfect. ($s = 28$, $d = 78880$, $g = 8$.)

	A	B	C	D	E	F	G	H
A	8	1	3	−4	0	4	1	1
$\rightarrow B$	1	8	−4	3	4	0	1	1
$\rightarrow C$	3	−4	8	1	0	4	1	1
$\rightarrow D$	−4	3	1	8	4	0	1	1
$\rightarrow E$	0	4	0	4	8	3	0	4
$\rightarrow F$	4	0	4	0	3	8	4	0
$\rightarrow G$	1	1	1	1	0	4	8	−4
$\rightarrow H$	1	1	1	1	4	0	−4	8

Relations: $A+D = B+C$.
Group: $\langle (AB)(CD)(EF)(GH),\ (AC)(BD),\ -1 \rangle$.

$19720c_k$	number	spectrum					representative	nbr
568	(4)	2	7	6	4	8	A	p_7^{11}
496	(4)	1	8	4	6	8	$A-F$	p_7^{7}
908	(4)	0	9	6	6	6	$B-E+F-G$	p_7^{24}
552	(2)	4	8	0	5	10	E	p_7^{1}
738	(2)	3	8	4	4	8	G	p_7^{14}
224	(2)	1	8	5	5	8	$E-H$	p_7^{5}
664	(2)	3	8	3	5	8	$B+C-E$	p_7^{5}
1048	(2)	2	8	6	5	6	$E-G-H$	p_7^{25}
452	(2)	1	8	6	4	8	$A-B+E-F-H$	p_7^{1}
798	(1)	7	0	10	0	10	$B+C$	p_7^{16}
387	(1)	5	0	12	0	10	$G+H$	p_7^{1}
864	(1)	3	10	0	6	8	$A-B+E-F$	p_7^{10}
−38	(1)	3	0	14	0	10	$B+C-G-H$	p_7^{1}

$p_7^{19} \cong \phi_{11}$. Extreme. ($s = 28$, $d = 10584$, $g = 192$.)

	A	B	C	D	E	F	R
$\rightarrow A$	6	3	1	1	1	1	4
$\rightarrow B$	3	6	1	1	1	1	4
$\rightarrow C$	1	1	6	3	1	1	4
$\rightarrow D$	1	1	3	6	1	1	4
$\rightarrow E$	1	1	1	1	6	3	4
$\rightarrow F$	1	1	1	1	3	6	4
$\rightarrow R$	4	4	4	4	4	4	8

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle (ACE)(BDF),\ (AF)(BE)(CD),\ (AB),\ (CD),\ (EF),\ (XX'),\ -1 \rangle$.

$168c_k$	number	spectrum				representative	nbr
7	(12)	6	6	5	10	A	p_7^{5}
5	(12)	6	4	9	8	$A+C-R$	p_7^{1}
11	(3)	15	0	0	12	$A-B$	p_7^{8}
19	(1)	3	12	12	0	$A+B+C+D+E+F-3R$	p_7^{4}

$p_7^{20} \cong K_{11}$. Extreme. ($s = 28$, $d = 10658$, $g = 36$.)

	A	B	C	D	E	F	G	H	I
$\to A$	6	3	0	−2	−1	−1	−2	0	3
$\to B$	3	6	3	0	−2	−1	−1	−2	0
$\to C$	0	3	6	3	0	−2	−1	−1	−2
$\to D$	−2	0	3	6	3	0	−2	−1	−1
$\to E$	−1	−2	0	3	6	3	0	−2	−1
$\to F$	−1	−1	−2	0	3	6	3	0	−2
$\to G$	−2	−1	−1	−2	0	3	6	3	0
H	0	−2	−1	−1	−2	0	3	6	3
I	3	0	−2	−1	−1	−2	0	3	6

Relations: $A+D+G = B+E+H = C+F+I$.
Group: $\langle (ABCDEFGHI),\ (AI)(BH)(CG)(DF),\ -1 \rangle$.

$438c_k$	number	spectrum				representative	nbr
16	(9)	4	8	7	8	A	p_7^5
22	(9)	7	6	4	10	$A-B$	p_7^7
14	(9)	4	6	9	8	$A-B+C$	p_7^1
43	(1)	9	0	18	0	$A+D+G$	p_7^1

$p_7^{21} \cong \phi_{14}$. Extreme. ($s = 28$, $d = 10752$, $g = 32$.)

	A'	B'	A	B	C	D	E	F	R
A'	6	2	−2	2	3	1	1	3	4
B'	2	6	2	−2	3	3	1	1	4
$\to A$	−2	2	6	2	1	3	3	1	4
$\to B$	2	−2	2	6	1	1	3	3	4
$\to C$	3	3	1	1	6	3	3	3	4
$\to D$	1	3	3	1	3	6	3	3	4
$\to E$	1	1	3	3	3	3	6	3	4
$\to F$	3	1	1	3	3	3	3	6	4
$\to R$	4	4	4	4	4	4	4	4	8

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle (A'B'AB)(CDEF),\ (AB)(DF);\ (XX'),\ -1 \rangle$.

$336c_k$	number	spectrum				representative	nbr
21	(8)	6	6	5	10	C	p_7^7
9	(8)	3	8	8	8	$A-D$	p_7^1
12	(4)	2	8	9	8	A	p_7^5
16	(4)	9	4	4	10	$C-D$	p_7^{14}
11	(2)	6	4	9	8	$C-E$	p_7^1
33	(2)	6	8	9	4	$A+B+C-E-R$	p_7^{31}

$p_7^{22} \cong \phi_{10}$. Extreme. ($s = 30$, $d = 630$, $g = 32$.)

	A	B	C	D	E	F	G	H	I	J
A	4	1	−1	2	2	2	1	1	2	0
$\rightarrow B$	1	4	2	−1	2	2	1	1	2	0
$\rightarrow C$	−1	2	4	1	1	1	2	2	0	2
D	2	−1	1	4	1	1	2	2	0	2
E	2	2	1	1	4	1	−1	2	2	0
$\rightarrow F$	2	2	1	1	1	4	2	−1	2	0
$\rightarrow G$	1	1	2	2	−1	2	4	1	0	2
$\rightarrow H$	1	1	2	2	2	−1	1	4	0	2
$\rightarrow I$	2	2	0	0	2	2	0	0	4	−1
$\rightarrow J$	0	0	2	2	0	0	2	2	−1	4

Relations: $A+C = B+D = E+G = F+H$.

Group: $\langle (AB)(CD),\ (EF)(GH),\ (AC)(BD)(EG)(FH)(IJ),\ (AE)(BF)(CG)(DH),\ -1 \rangle$.

$420c_k$	number	spectrum			representative
21	(8)	4	15	10	A
24	(8)	6	13	10	$A-I$
25	(4/5)	3	16	10	$A-E$
36	(4)	5	16	8	$A-D+J$
12	(2)	9	8	12	I
33	(2)	7	12	10	$A-D$
5	(1/5)	3	16	10	$A-D-E+H$
36	(1)	7	14	8	$F-H-I-J$

$p_7^{23} \cong K_3$. Extreme. ($s = 28$, $d = 10780$, $g = 16$.)

	A	B	C	D	E	F	G	H
$\rightarrow A$	6	3	−1	−1	2	2	1	3
$\rightarrow B$	3	6	−1	−1	2	2	3	1
$\rightarrow C$	−1	−1	6	3	1	3	2	2
$\rightarrow D$	−1	−1	3	6	3	1	2	2
$\rightarrow E$	2	2	1	3	6	0	3	3
$\rightarrow F$	2	2	3	1	0	6	3	3
$\rightarrow G$	1	3	2	2	3	3	6	0
H	3	1	2	2	3	3	0	6

Relations: $E+F = G+H$.

Group: $\langle (AB)(GH),\ (CD)(EF),\ (AC)(BD)(EG)(FH),\ -1 \rangle$.

$4620c_k$	number	spectrum				representative	nbr
194	(8/12)	4	8	7	8	$A+D-E$	p_7^7
148	(4)	3	9	7	8	A	p_7^1
157	(4)	4	8	5	10	E	p_7^1
272	(4)	3	9	9	6	$A-H$	p_7^{10}
200	(4/12)	4	8	7	8	$B+D-E-F$	p_7^7
192	(2)	7	6	4	10	$A-B$	p_7^{11}
173	(2)	3	8	8	8	$E-G$	p_7^5

$p_7^{24} \cong K_1$. Extreme. ($s = 28$, $d = 10808$, $g = 4$.)

	A	B	C	D	E	F	G	H
A	6	−2	1	3	1	3	1	1
$\to B$	−2	6	3	1	3	1	1	1
$\to C$	1	3	6	−2	1	3	3	−1
$\to D$	3	1	−2	6	3	1	−1	3
$\to E$	1	3	1	3	6	2	1	0
$\to F$	3	1	3	1	2	6	0	1
$\to G$	1	1	3	−1	1	0	6	−3
$\to H$	1	1	−1	3	0	1	−3	6

Relations: $A+B = C+D$.
Group: $\langle (AB)(CD)(EF)(GH),\ -1 \rangle$.

$32424c_k$	number	spectrum				representative	nbr
979	(2/4)	2	10	7	8	A	p_7^5
878	(2/4)	2	10	7	8	C	p_7^1
1177	(2/4)	3	9	7	8	E	p_7^5
1176	(2/4)	3	9	7	8	G	p_7^{11}
1121	(2)	4	7	6	10	$B-E$	p_7^1
1894	(2)	4	9	6	8	$C-F$	p_7^{10}
1109	(2)	5	7	5	10	$C-G$	p_7^1
1261	(2)	2	9	8	8	$A+B-E$	p_7^7
1579	(2)	7	6	4	10	$C-F-G$	p_7^7
1411	(2/3)	4	8	7	8	$A+C-F-G$	p_7^7
821	(2)	4	7	8	8	$A+B-E-G-H$	p_7^1
1445	(2)	8	5	4	10	$B-C-E+F-H$	p_7^{18}
2734	(1)	4	8	11	4	$B-C$	p_7^{30}
1711	(1/3)	4	8	7	8	$G+H$	p_7^8
2551	(1)	4	10	7	6	$A+B-G-H$	p_7^{28}
1130	(1)	4	6	9	8	$B-C-E+F$	p_7^1

K_4. Extreme. ($s = 28$, $d = 11008$, $g = 8$.)

	A	A'	B	B'	C	C'	D	E	F	R
$\to A$	6	2	2	−2	1	−3	1	3	2	4
A'	2	6	−2	2	−3	1	1	3	2	−4
$\to B$	2	−2	6	2	1	−3	3	1	2	4
B'	−2	2	2	6	−3	1	3	1	2	−4
$\to C$	1	−3	1	−3	6	2	0	0	1	4
C'	−3	1	−3	1	2	6	0	0	1	−4
$\to D$	1	1	3	3	0	0	6	3	3	0
$\to E$	3	3	1	1	0	0	3	6	3	0
$\to F$	2	2	2	2	1	1	3	3	6	0
$\to R$	4	−4	4	−4	4	−4	0	0	0	8

Relations: $A-A' = B-B' = C-C' = R$.

Group: $\langle (AA')(BB')(CC'),\ (AB)(A'B')(DE),\ -1\rangle$.

$2064c_k$	number	spectrum				representative	nbr
71	(4)	1	10	8	8	A	p_7^7
77	(4/7)	2	10	7	8	$B-D$	p_7^7
74	(2/4)	4	8	5	10	C	p_7^7
55	(2/4)	4	8	5	10	E	p_7^1
70	(2)	3	8	8	8	$B+C'$	p_7^{18}
72	(2/4)	3	10	6	8	$E-F$	p_7^{14}
103	(2/4)	3	10	6	8	$A+B'-D$	p_7^1
96	(2)	6	6	5	10	$B+C'-D$	p_7^{16}
46	(2/7)	2	10	7	8	$B+C'-F$	p_7^1
146	(2)	2	10	9	6	$A+B+C'-F$	p_7^{28}
158	(1)	4	10	5	8	F	p_7^{27}
143	(1/2)	6	6	9	6	$D-E$	p_7^{22}
58	(1/7)	2	10	7	8	$A+B'-F$	p_7^1
133	(1/2)	6	6	9	6	$A-B+D-E$	p_7^1

$p_7^{26} \cong A_{7,3} \cong L_7^2 \cong \phi_5$. Semi-eutactic. ($s = 30$, $d = 648$, $g = 864$.)

	A	B	C	D	E	F	R	S	basis
A	4	1	1	2	2	2	2	2	$\sqrt{2}(1\ 0;\ -1\ 0\ 0)$
$\to B$	1	4	1	2	2	2	2	2	$\sqrt{2}(\omega\ 0;\ -1\ 0\ 0)$
$\to C$	1	1	4	2	2	2	2	2	$\sqrt{2}(\bar{\omega}\ 0;\ -1\ 0\ 0)$
$\to D$	2	2	2	4	1	1	2	2	$\sqrt{2}(0\ 1;\ -1\ 0\ 0)$
$\to E$	2	2	2	1	4	1	2	2	$\sqrt{2}(0\ \omega;\ -1\ 0\ 0)$
$\to F$	2	2	2	1	1	4	2	2	$\sqrt{2}(0\ \bar{\omega};\ -1\ 0\ 0)$
$\to R$	2	2	2	2	2	2	4	2	$\sqrt{2}(0\ 0;\ -1\ 1\ 0)$
$\to S$	2	2	2	2	2	2	2	4	$\sqrt{2}(0\ 0;\ -1\ 0\ 1)$

Relations: $A+B+C = D+E+F$.
Group: $\langle S_3(A,B,C),\ S_3(D,E,F),\ (AD)(BE)(CF),\ \mathrm{ref}_R,\ \mathrm{ref}_S,\ -1\rangle$.

$36c_k$	number	spectrum			representative
3	(18)	7	12	10	A
1	(9)	3	16	10	$A-D$
0	(3)	15	0	14	R

$p_7^{27} \cong A_{7,0} \cong \phi_{12}$. Extreme. ($s = 30$, $d = 648$, $g = 768$.)

	A	B	C	D	E	F	R	basis
$\to A$	4	1	2	2	2	2	3	$\sqrt{2}(\omega\ 0\ 0\ -\omega)$
$\to B$	1	4	2	2	2	2	3	$\sqrt{2}(\bar{\omega}\ 0\ 0\ -\omega)$
$\to C$	2	2	4	1	2	2	3	$\sqrt{2}(0\ \omega\ 0\ -\omega)$
$\to D$	2	2	1	4	2	2	3	$\sqrt{2}(0\ \bar{\omega}\ 0\ -\omega)$
$\to E$	2	2	2	2	4	1	3	$\sqrt{2}(0\ 0\ \omega\ -\omega)$
$\to F$	2	2	2	2	1	4	3	$\sqrt{2}(0\ 0\ \bar{\omega}\ -\omega)$
$\to R$	3	3	3	3	3	3	6	$\sqrt{2}(0\ 0\ 0\ \bar{\omega}-\omega)$

Group: $\langle S_4$ on coordinates, conjugations of coordinates, $-1\rangle$.

$72c_k$	number	spectrum			representative
4	(24)	5	14	10	A
5	(6)	5	16	8	$A+B-C-D$

$p_7^{28} \cong K_8$. Extreme. ($s = 30$, $d = 648$, $g = 24$.)

	A	B	C	D	E	F	Z
$\to A$	4	1	2	2	2	1	5
$\to B$	1	4	1	2	2	2	5
$\to C$	2	1	4	1	2	2	5
$\to D$	2	2	1	4	1	2	5
$\to E$	2	2	2	1	4	1	5
$\to F$	1	2	2	2	1	4	5
$\to Z$	5	5	5	5	5	5	14

Group: $\langle (ABCDEF), (AF)(BE)(CD), -1\rangle$.

$216c_k$	number	spectrum			representative
12	(6/9)	7	12	10	A
16	(6)	5	16	8	$A-C$
12	(6)	5	14	10	$A+B-Z$
12	(6)	9	8	12	$A+B+C-Z$
15	(3/9)	7	12	10	$A-D$
7	(3)	5	12	12	$A+B-D-E$

$p_7^{29} \cong \phi_7$. Perfect. ($s = 29$, $d = 684$, $g = 32$.)

	A	B	C	D	E	F	G
$\to A$	4	1	1	2	2	2	2
$\to B$	1	4	2	2	2	2	2
$\to C$	1	2	4	2	2	2	2
$\to D$	2	2	2	4	1	2	2
$\to E$	2	2	2	1	4	2	2
$\to F$	2	2	2	2	2	4	1
$\to G$	2	2	2	2	2	1	4

Group: $\langle (BC), (DE), (FG), (DF)(EG), -1\rangle$.

$228c_k$	number	spectrum			representative
15	(8)	6	12	10	$B-D$
12	(4/6)	9	9	10	D
6	(4)	3	15	10	$A-D$
14	(4)	4	14	10	$D-F$
21	(4)	5	15	8	$A+B-D-E$
18	(2/6)	9	9	10	B
24	(1)	6	14	8	A
−3	(1)	14	0	14	$B-C$
10	(1)	4	16	8	$D+E-F-G$

$p_7^{30} \cong \phi_{17}$. Extreme. ($s = 28$, $d = 686$, $g = 28$.)

	A	B	C	D	E	F	G
$\rightarrow A$	4	1	2	2	2	2	1
$\rightarrow B$	1	4	1	2	2	2	2
$\rightarrow C$	2	1	4	1	2	2	2
$\rightarrow D$	2	2	1	4	1	2	2
$\rightarrow E$	2	2	2	1	4	1	2
$\rightarrow F$	2	2	2	2	1	4	1
$\rightarrow G$	1	2	2	2	2	1	4

Group: $\langle (ABCDEFG),\ (AG)(BF)(CE),\ -1 \rangle$.

$196c_k$	number	spectrum			representative	nbr
14	(7/14)	5	14	8	A	p_7^{10}
11	(7)	7	10	10	$A-C$	p_7^{24}
9	(7)	5	12	10	$A-D$	p_7^{1}
15	(7/14)	5	14	8	$A+B-D-E$	p_7^{1}

$p_7^{31} \cong \phi_8$. Extreme. ($s = 28$, $d = 720$, $g = 16$.)

	A	B	C	D	E	F	R
$\rightarrow A$	4	1	2	2	2	2	3
$\rightarrow B$	1	4	1	2	2	2	3
$\rightarrow C$	2	1	4	1	2	2	3
$\rightarrow D$	2	2	1	4	1	2	3
$\rightarrow E$	2	2	2	1	4	2	3
$\rightarrow F$	2	2	2	2	2	4	3
$\rightarrow R$	3	3	3	3	3	3	6

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle (AE)(BD),\ (BD')(CF),\ (XX'),\ -1 \rangle$.

$240c_k$	number	spectrum			representative	nbr
20	(4/6)	5	12	10	A	p_7^{1}
14	(4)	4	13	10	B	p_7^{1}
6	(4)	7	10	10	C	p_7^{7}
15	(4)	9	8	10	$A-C$	p_7^{22}
10	(4)	4	15	8	$A-D$	p_7^{1}
24	(4)	6	13	8	$B-F$	p_7^{29}
5	(1/6)	5	12	10	$A-E$	p_7^{1}
2	(1/6)	5	12	10	$B-D$	p_7^{21}
27	(1)	7	12	8	$C-F$	p_7^{31}
30	(1)	5	16	6	$A+B-D-E$	p_7^{4}

$p_7^{32} \cong A_{7,2} \cong M_7^3 \cong \phi_3$. Extreme. ($s = 28$, $d = 756$, $g = 192$.)

	A	B	C	D	E	F	R	basis
$\to A$	4	1	2	2	2	2	2	$\sqrt{2}(\omega\ 0\ 0;\ -1\ 0)$
$\to B$	1	4	2	2	2	2	2	$\sqrt{2}(\bar{\omega}\ 0\ 0;\ -1\ 0)$
$\to C$	2	2	4	1	2	2	2	$\sqrt{2}(0\ \omega\ 0;\ -1\ 0)$
$\to D$	2	2	1	4	2	2	2	$\sqrt{2}(0\ \bar{\omega}\ 0;\ -1\ 0)$
$\to E$	2	2	2	2	4	1	2	$\sqrt{2}(0\ 0\ \omega;\ -1\ 0)$
$\to F$	2	2	2	2	1	4	2	$\sqrt{2}(0\ 0\ \bar{\omega};\ -1\ 0)$
$\to R$	2	2	2	2	2	2	4	$\sqrt{2}(0\ 0\ 0;\ -1\ 1)$

Relations: $A+A' = B+B' = \ldots = F+F' = R$.
Group: $\langle (AB), (CD), (EF), (ACE)(BDF), (AF)(BE)(CD), (XX'), -1 \rangle$.

$84c_k$	number	spectrum			representative	nbr
6	(12)	9	8	10	A	p_7^{29}
4	(12)	5	12	10	$A-C$	p_7^{1}
8	(3)	5	16	6	$A+B-C-D$	p_7^{4}
3	(1)	15	0	12	R	p_7^{27}

$p_7^{33} \cong A_7 \cong \phi_0$. Extreme. ($s = 28$, $d = 8$, $g = 2.8! = 80640$.)

	A	B	C	D	E	F	G	basis
$\to A$	2	1	1	1	1	1	1	$(+\ 0\ 0\ 0\ 0\ 0\ 0\ -)$
$\to B$	1	2	1	1	1	1	1	$(0\ +\ 0\ 0\ 0\ 0\ 0\ -)$
$\to C$	1	1	2	1	1	1	1	$(0\ 0\ +\ 0\ 0\ 0\ 0\ -)$
$\to D$	1	1	1	2	1	1	1	$(0\ 0\ 0\ +\ 0\ 0\ 0\ -)$
$\to E$	1	1	1	1	2	1	1	$(0\ 0\ 0\ 0\ +\ 0\ 0\ -)$
$\to F$	1	1	1	1	1	2	1	$(0\ 0\ 0\ 0\ 0\ +\ 0\ -)$
$\to G$	1	1	1	1	1	1	2	$(0\ 0\ 0\ 0\ 0\ 0\ +\ -)$

Group: $\langle S_8$ on coordinates, $-1 \rangle$.

$8c_k$	number	spectrum		representative	nbr
1	(28)	15	12	A	p_7^{4}

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