Inequalities for Covering Codes

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Abstract — **Any code C with covering radius R must satisfy a set of linear inequalities that involve the Lloyd polynomial $L_R(x)$; these generalize the sphere bound. The "syndrome graphs" associated with a linear code C help to keep track of low weight vectors in the same coset of C (if there are too many such vectors C cannot exist). As illustrations it is shown that $t[17,10] = 3$ and $t[23,15] = 3$, where $t[n,k]$ is the smallest covering radius of any $[n,k]$ code.**

I. Introduction

THE Delsarte–MacWilliams inequalities have enabled linear programming to be applied successfully to error-correcting codes (see for example [4]). The inequalities presented here (in Theorems 1, 2 and Corollaries 3, 4) are a partial analog for covering codes. Although several other recent papers have given bounds for covering codes [1], [5], [7], the results given here appear to be new. The technique of obtaining bounds by counting syndromes (used in the second proof of Theorem 2) was also used by Brualdi *et al.* [1]. The underlying principle is the fact that if C is a linear $[n,k]$ code with covering radius R, then every $(n-k)$-tuple can be written as a sum of at most R columns of H [2, sec. I-A]. The bookkeeping involved in checking this condition is facilitated by the "syndrome graphs" introduced in Section III. In Section IV we show $t[17,10] = 3$, and the final section contains a list of the known improvements to the table of $t[n,k]$ given in [3]. (Recall that $t[n,k]$ is the minimal covering radius of any binary code of length n and dimension k.)

We restrict ourselves to binary codes; there is a straightforward generalization to other fields. Our notation throughout is that a linear code C has weight distribution $\{A_i\}$ and the dual $C^\perp$ has weight distribution $\{B_i\}$.

II. Linear Inequalities for Covering Codes

Suppose C is a (not necessarily linear) binary code of length n and covering radius R. As in [4, ch. 5], we represent vectors $u = (u_1, \cdots, u_n) \in \boldsymbol{F}_2^n$ by their images $z^u = z_1^{u_1} \cdots z_n^{u_n}$ in the group algebra $\boldsymbol{Q}\boldsymbol{F}_2^n$, and define

$$\chi_v(z^u) = (-1)^{u\cdot v},$$

$$\chi_v(C) = \sum_{c\in C} \chi_v(z^c).$$

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Let

$$K_j(x;n) = \sum_{i=0}^{j} (-1)^i \binom{x}{i}\binom{n-x}{j-i}, \tag{1}$$

$$L_j(x) = \sum_{i=0}^{j} K_i(x;n) = K_j(x-1;n-1), \tag{2}$$

$j = 0,1,\cdots,n$, be Krawtchouk and Lloyd polynomials, respectively [4].

Theorem 1: For any subspace $U \subseteq \boldsymbol{F}_2^n$ and any vector $v \in \boldsymbol{F}_2^n$ we have

$$\sum_{u\in U} \chi_v(z^u)\chi_u(C) L_R(\mathrm{wt}(u)) \geq 2^n. \tag{3}$$

Proof: Since C has covering radius R, in the group algebra we have

$$\Big(\sum_{c\in C} z^c\Big)\Big(\sum_{i=0}^{R} Y_i\Big) = \sum_{w\in \boldsymbol{F}_2^n} \alpha_w z^w, \tag{4}$$

where

$$Y_i = \sum_{\mathrm{wt}(w)=i} z^w$$

and the coefficients α_w are positive integers. We apply χ_u to both sides of (4) and use

$$\chi_u(Y_i) = \sum_{\mathrm{wt}(w)=i} (-1)^{u\cdot w} = K_i(\mathrm{wt}(u);n) \tag{5}$$

[4, p. 135] and (2) to obtain

$$\chi_u(C) L_R(\mathrm{wt}(u)) = \sum_{w\in \boldsymbol{F}_2^n} \alpha_w (-1)^{u\cdot w}.$$

By multiplying both sides by $(-1)^{u\cdot v}$ and summing on $u \in U$, we have

$$\sum_{u\in U} \chi_v(z^u)\chi_u(C) L_R(\mathrm{wt}(u)) = \sum_{w\in \boldsymbol{F}_2^n} \alpha_w \sum_{u\in U} (-1)^{u\cdot(v+w)}. \tag{6}$$

But

$$\sum_{u\in U} (-1)^{u\cdot x} = \begin{cases} |U|, & \text{if } x \in U^\perp \\ 0, & \text{otherwise,} \end{cases} \tag{7}$$

so the right side of (6) is at least

$$|U| \sum_{\substack{w\in \boldsymbol{F}_2^n \\ w+v\in U^\perp}} \alpha_w \geq |U||U^\perp| = 2^n,$$

which is the desired result.

From now on we restrict our attention to linear codes.

Theorem 2: Let C be a linear $[n,k]R$ code, i.e. one with covering radius R. Choose any m linearly independent vectors $u_1,\cdots,u_m$ in $C^\perp$ $(1\le m\le n-k)$, and m arbitrary signs $\epsilon_1,\cdots,\epsilon_m$ (equal to ± 1). If $u\in C^\perp$ is a linear combination of the u_i, say

$$u=u_{i_1}+\cdots+u_{i_a},$$

set $\epsilon(u)=\epsilon(i_1)\cdots\epsilon(i_a)$; otherwise set $\epsilon(u)=0$. Then

$$\sum_{u\in C^\perp}\epsilon(u)L_R(\mathrm{wt}(u))\ge 2^{n-k}. \tag{8}$$

Remarks:

i) Since $L_R(0)=\sum_{i=0}^{R}\binom{n}{i}$, we can write (8) as

$$\sum_{i=0}^{R}\binom{n}{i}-2^{n-k}+\sum_{\substack{u\in C^\perp\\ u\neq 0}}\epsilon(u)L_R(\mathrm{wt}(u))\ge 0. \tag{9}$$

The sphere bound

$$\sum_{i=0}^{R}\binom{n}{i}-2^{n-k}\ge 0 \tag{10}$$

is the case $m=0$.

ii) Theorem 2 can be stated more succinctly as follows: if U is a subspace of $C^\perp$ and ϵ is any map from $C^\perp$ to $\{-1,0,+1\}$ satisfying $\epsilon(u)=0$ for $u\in C^\perp\backslash U$ and $\epsilon(u+v)=\epsilon(u)\epsilon(v)$ for $u,v\in U$, then (8) holds.

iii) Although Theorem 2 follows from Theorem 1, we also give a direct proof involving the choice of a parity check matrix, since this approach motivates the rest of the paper.

First proof of Theorem 2: In Theorem 1 let C be linear, $U\subseteq C^\perp$, and choose v so that $\chi_v(z^u)=\epsilon(u)$ holds for all $u\in U$. Then (8) follows from (3) and (7).

Second proof: Let $u_1,\cdots,u_m$ be the first m rows of a parity check matrix H for C. Define $\pi_i=0$ or 1 by $\epsilon_i=(-1)^{\pi_i}$ $(i=1,\cdots,m)$. Then for any $a\in F_2^n$,

$$\prod_{i=1}^{m}\frac{1+\epsilon_i(-1)^{a\cdot u_i}}{2}$$

is equal to 1 if $a\cdot u_i=\pi_i$ for all $i=1,\cdots,m$, and is otherwise 0. Therefore

$$\sum_{\mathrm{wt}(a)\le R}\prod_{i=1}^{m}\frac{1+\epsilon_i(-1)^{a\cdot u_i}}{2}\ge 2^{n-k-m}, \tag{11}$$

since the left side is the number of linear combinations of at most R columns of H whose sum begins $\pi_1\pi_2\cdots\pi_m$, and this must be at least as large as the number of $(n-k)$-tuples with the same beginning. Equation (8) follows by expanding (11), and applying (5) and (2).

Corollary 3: If w is any nonzero weight in $C^\perp$ then

a) $$\sum_{i=0}^{R}\binom{n}{i}-2^{n-k}\ge|L_R(w)|, \tag{12}$$

and

b) $$M(w)\ge 0, \tag{13}$$

where

$$M(x)=L_R(0)-2^{n-k}-L_R(x). \tag{14}$$

Proof: a) is the case $m=1$ of Theorem 2; b) is an immediate consequence.

Corollary 4: If $u_1,u_2,u_3=u_1+u_2$ are nonzero vectors in $C^\perp$ of weights w_1,w_2,w_3 respectively, then

a)
$$\sum_{i=0}^{R}\binom{n}{i}-2^{n-k}+\epsilon_1L_R(w_1)+\epsilon_2L_R(w_2)+\epsilon_1\epsilon_2L_R(w_3)\ge 0 \tag{15}$$

holds for all choices of signs $\epsilon_1=\pm 1$, $\epsilon_2=\pm 1$; and

b) $$M(w_1)+M(w_2)-M(w_3)\ge 0. \tag{16}$$

Proof: a) is the case $m=2$ of Theorem 2; b) follows by choosing $\epsilon_1=\epsilon_2=-1$.

Remarks:

i) Equation (16) states that the three numbers $M(w_1)$, $M(w_2)$, $M(w_3)$ are a possible set of edge lengths of a triangle (which may have zero area—for example if $M(w_1)+M(w_2)=M(w_3)$).

ii) Although assertions (13) and (16) are generally the most useful, there are many applications of the more general results (8), (12), (15).

iii) Lloyd's theorem for linear codes. Let C be a perfect $[n,k]R$ code. Consider the graph Γ with vertices the cosets of C and where two cosets x,y are joined if and only if $\mathrm{wt}(x+y)=1$. If we choose a parity check matrix H, then we may label the vertices of Γ with syndromes with respect to H. Vertices labeled s,t are joined in Γ if and only if $s+t$ is a column of H. The graph Γ is regular with valency n and has eigenvalues $n-2w_j$ with multiplicity B_{w_j}, where B_{w_j} is the number of codewords in $C^\perp$ with weight w_j. Since the diameter of Γ is R, there are at least R nonzero weights in $C^\perp$. Since C is perfect we have

$$L_R(0)=2^{n-k},$$

and for every nonzero weight w in $C^\perp$ we have

$$M(w)=L_R(0)-2^{n-k}-L_R(w)=0.$$

We conclude that the polynomial $L_R(x)$ of degree R has exactly R distinct integer roots in $[0,n]$.

Corollary 5: Suppose there is an integer β $(0\le\beta\le n)$ such that for all integers w $(0\le w\le n)$ either

$$M(w)\le\beta \quad\text{or}\quad M(w)\ge 2\beta+1.$$

Then

$$\{u\in C^\perp:\ M(\mathrm{wt}(u))\le\beta\}$$

together with the zero vector is a linear subcode of $C^\perp$.

Proof: Let $u_1,u_2\in C^\perp$ be linearly independent vectors with $M(\mathrm{wt}(u_i))\le\beta$ $(i=1,2)$. We must show $M(\mathrm{wt}(u_1+u_2))\le\beta$. Let $w_i=\mathrm{wt}(u_i)$, $w_3=\mathrm{wt}(u_1+u_2)$. Then

$M(w_1)+M(w_2)\geq M(w_3)\geq 0$, so $M(w_3)\leq 2\beta$. Therefore $M(w_3)\leq\beta$, by hypothesis.

Corollary 6: The zero vector and the vectors $u\in C^{\perp}$ with $M(\mathrm{wt}(u))=0$ form a linear subcode of $C^{\perp}$.

Proof: This is the case $\beta=0$.

Examples:

a) It was shown in [3] that $t[23,15]=2$ or 3. Suppose C is a [23,15]2 code. The corresponding values of $L_2(x)$ and $M(x)$ for $0\leq x\leq 23$ are as follows:

x:	0	1	2	3	4	5	6	7
$L_2(x)$:	277	231	189	151	117	87	61	39
$M(x)$:	−256	−210	−168	−130	−96	−66	−40	−18
x:	8	9	10	11	12	13	14	15
$L_2(x)$:	21	7	−3	−9	−11	−9	−3	7
$M(x)$:	0	14	24	30	32	30	24	14
x:	16	17	18	19	20	21	22	23
$L_2(x)$:	21	39	61	87	117	151	189	231
$M(x)$:	0	−18	−40	−66	−96	−130	−168	−210

From (13), the nonzero codewords in $C^{\perp}$ have weights in the range $\{8,9,\cdots,16\}$, and from Corollary 6 the codewords of weights 0, 8, 16 form a linear subcode.

We shall continue this example in Section III, and show that C does not exist.

b) From [3], $t[16,7]=3$ or 4. Suppose C is a [16,7]3 code. The corresponding values of $L_3(x)$ and $M(x)$ are as follows:

x:	0	1	2	3	4	5	6	7	8
$L_3(x)$:	697	455	273	143	57	7	−15	−17	−7
$M(x)$:	−512	−270	−88	42	128	178	200	202	192
x:	9	10	11	12	13	14	15	16	
$L_3(x)$:	7	17	15	−7	−57	−143	−273	−455	
$M(x)$:	178	168	170	192	242	328	458	640	

From (13), the nonzero codewords in $C^{\perp}$ have weights $w\geq 3$. In this case (12) yields the additional result that $w\leq 14$.

At present it is not known if C exists.

III. Syndrome Graphs

Every nonzero codeword $c\in C$ specifies a dependence among the columns of H. In particular, the existence of a codeword in C of weight $w\leq 2R$ implies that two sets of at most R columns of H have equal sums (usually in several ways). For example, if $c=111100\cdots 0\in C$, we have the identities

$$h_1+h_2=h_3+h_4,$$
$$h_1+h_3=h_2+h_4,$$
$$h_1+h_4=h_2+h_3,$$

where h_i is the ith column of H. Each such identity reduces the number of *distinct* sums of at most R columns of H. If there are enough identities it may be possible to show that the code cannot exist. (Such identities indicate that there are low weight vectors in the same coset of C.)

We now define certain "syndrome graphs" $G_{\pi_1\cdots\pi_m}$, which make it possible to record and utilize even *partial* information of this type.

As in Section II, let $u_1,\cdots,u_m\in C^{\perp}$ be the first m rows of a parity check matrix H $(1\leq m\leq n-k)$. For each binary vector $\pi_1\cdots\pi_m$ we define a graph $G_{\pi_1\cdots\pi_m}$ as follows. The vertices are the subsets $\{i_1,\cdots,i_r\}\subseteq\{1,\cdots,n\}$ $(0\leq r\leq R)$ for which the binary vector a supported on that subset satisfies $a\cdot u_j=\pi_j$ for all $j=1,\cdots,m$. Two vertices $\{i_1,\cdots,i_r\},\{j_1,\cdots,j_s\}$ are joined by an edge if and only if

$$\sum_{\rho=1}^{r}h_{i_\rho}=\sum_{\sigma=1}^{s}h_{j_\sigma}. \tag{17}$$

Proposition 7: The syndrome graph $G_{\pi_1\cdots\pi_m}$ contains exactly 2^{n-k-m} connected components, each of which is a complete graph.

Proof: It follows from the definition that the number of components in this graph is equal to the number of distinct sums $f=(f_1,\cdots,f_{n-k})^{tr}$ of at most R columns of H for which $f_1=\pi_1,\cdots,f_m=\pi_m$. Since C has covering radius R, this number is exactly 2^{n-k-m}. It also follows from the definition that if node V_1 is joined to node V_2, and V_2 to V_3, then V_1 is joined to V_3; thus every component is a complete subgraph.

The merit of this graphical approach is that each time one can deduce—for example from the Delsarte–MacWilliams identities—that a codeword of weight at most $2R$ exists in C, the number of known edges in one or more of the graphs $G_{\pi_1\cdots\pi_m}$ increases. And of course a graph with many edges cannot have too many connected components.

This leads to the following proposition.

Proposition 8: Suppose the ith connected component of the syndrome graph $G_{\pi_1\cdots\pi_m}$ contains v_i vertices and $e_i=v_i(v_i-1)/2$ edges $(1\leq i\leq K=2^{n-k-m})$, and let $v=\sum v_i$, $e=\sum e_i$ be the total number of vertices and edges, respectively. Then the nonnegative integers $\delta_i=v_i-1$ $(1\leq i\leq K)$ satisfy

$$\sum_{i=1}^{K}\delta_i=v-2^{n-k-m}, \tag{18}$$

$$\sum_{i=1}^{K}\delta_i^2=2e-v+2^{n-k-m}. \tag{19}$$

Proof: This follows immediately from the definition of δ_i.

Remarks:

i) If it is known only that the syndrome graph $G_{\pi_1,\cdots,\pi_m}$ contains *at least* η edges, then Proposition 8 implies that there exist nonnegative integers $\{\delta_i\}$ satisfying

$$\sum_{i=1}^{K}\delta_i=v-2^{n-k-m}, \tag{20}$$

$$\sum_{i=1}^{K}\delta_i^2\geq 2\eta-v+2^{n-k-m}. \tag{21}$$

ii) It is worth emphasizing that $G_{\pi_1 \cdots \pi_m}$ depends both on the choice of $u_1, \cdots, u_m \in C^\perp$ and on the binary vector $\pi_1 \cdots \pi_m$.

iii) It is sometimes useful to consider how the different syndrome graphs are related. There is a "parent" graph $\mathcal{G}$ with $L_R(0)$ vertices labeled by all subsets $\{i_1, \cdots, i_r\} \subseteq \{1, \cdots, n\}$, $0 \le r \le R$, with edges defined by (17). Each $G_{\pi_1 \cdots \pi_m}$ is an induced subgraph[1] of $\mathcal{G}$. Furthermore if $u_{i_1}, \cdots, u_{i_s}$ is a subset of $u_1, \cdots, u_m$, then $G_{\pi_1 \cdots \pi_m}$ is an induced subgraph of $G_{\pi_{i_1} \cdots \pi_{i_s}}$.

iv) Consider the case $m = 1$, $\pi_1 = 1$. Then $M(\mathrm{wt}(u_1)) = 0$ if and only if G_1 contains no edges (for then equality must hold in (11)). More generally, if $u_1, \cdots, u_m$ are such that $M(\mathrm{wt}(u_i)) = 0$ $(1 \le i \le m)$, then all the syndrome graphs $G_{\pi_1 \cdots \pi_m}$ for which at least one π_i is 1 contain no edges (this follows from the last sentence of the previous remark).

Example a) (Continued):

Let C be a [23,15]2 code, and let $\{A_i\}, \{B_i\}$ be the weight distributions of $C, C^\perp$ respectively. From Section II and the tables in [3], [6] we have $A_1 = A_2 = 0$, $B_1 = \cdots = B_7 = B_{17} = \cdots = B_{23} = 0$, $B_8 \ge 1$. We now use linear programming to minimize $A_3 + A_4$ subject to the Delsarte–MacWilliams constraints; the result is $A_3 + A_4 \ge 12.33\cdots$, i.e. $A_3 + A_4 \ge 13$.

Suppose $C^\perp$ contains a codeword u_1 of weight 12. The corresponding syndrome graph G_0 (i.e., taking $m = 1$, $\pi_1 = 0$) has $v = 1 + \binom{12}{2} + 11 + \binom{11}{2} = 133$ vertices. Furthermore G_0 contains at least one edge for every codeword of weight 3 or 4 in C, i.e. $e \ge 13$. (For example, suppose $u_1 = 11\cdots100\cdots0 \in C^\perp$, of weight 12. If $c = 110\cdots010\cdots0 \in C$, of weight 3, then there is an edge in G_0 joining vertex $\{1,2\}$ to vertex $\{13\}$. On the other hand if $c = 0\cdots01110\cdots0 \in C$, the three vertices $\{13,14\}$, $\{13,15\}, \{14,15\}$ in G_0 are joined by three edges to form a triangle (similarly if c has weight 4). The edges determine c, and so distinct c's produce distinct edges.)

From (20) and (21) it follows that $\sum_{i=1}^{128} \delta_i = 5$ and $\sum_{i=1}^{128} \delta_i^2 \ge 21$, which implies $\delta_j = 5$ for some j and $\delta_i = 0$ for $i \ne j$. We take $j = 1$. Thus G_0 contains one component with six vertices (and at least 13 edges) and 127 isolated vertices. It is now easy to check that there is no way to choose 13 vectors of weight 3 or 4 orthogonal to u_1 for which the corresponding edges are restricted to only six vertices. Thus $C^\perp$ contains no vector of weight 12, i.e. $B_{12} = 0$. Similar arguments apply if u_1 has weight 11 or 13, and we obtain $B_{11} = B_{12} = B_{13} = 0$. Linear programming now shows that there is no feasible solution to the Delsarte–MacWilliams constraints. Thus C does not exist, and we have established the following result.

Theorem 9:

$$t[23,15] = 3.$$

Remark: We originally proved Theorem 9 without using a computer. To save space we omit this argument.

[1]A graph H is an *induced subgraph* of G if the vertices of H are a subset of the vertices of G, and the edges of H are just the edges of G that join these vertices.

IV. $t[17,10] = 3$

Theorem 10:

$$t[17,10] = 3.$$

Proof: From [3], $t[17,10] = 2$ or 3. Suppose C is a [17,10]2 code. The corresponding values of $L_2(x)$ and $M(x)$ are as follows:

x:	0	1	2	3	4	5	6	7	8
$L_2(x)$:	154	120	90	64	42	24	10	0	−6
$M(x)$:	−128	−94	−64	−38	−16	2	16	26	32
x:	9	10	11	12	13	14	15	16	17
$L_2(x)$:	−8	−6	0	10	24	42	64	90	120
$M(x)$:	34	32	26	16	2	−16	−38	−64	−94

From (13), the nonzero weights in $C^\perp$ are in the range $5 \le w \le 13$; and from [6], $C^\perp$ has $d_{\min} = 5$ or 6. From (16), there is at most one vector of weight 5 in $C^\perp$, and also if there is such a vector, then there can be no vector of weight 6. Linear programming now shows there is no feasible solution to the Delsarte–MacWilliams inequalities if $B_5 = 1$, $B_6 = 0$. We conclude that $C^\perp$ has $d_{\min} = 6$. Equation (16) also shows that any vector disjoint from a vector of weight 6 in $C^\perp$ must also have weight 6 (and by a dimension argument there must be such a vector).

Let $u_1, u_2 \in C^\perp$ be disjoint vectors of weight 6, say $u_1 = 11111100\cdots0$, $u_2 = 00000011111100000$. Let D be the subcode of $C^\perp$ that vanishes on the last five coordinates, and let D' be the projection of $C^\perp$ onto the last five coordinates, so that $\dim D + \dim D' = \dim C^\perp = 7$, and $\dim D \ge 2$ (since $u_1, u_2 \in D$).

Case i): $\dim D = 2$. Then $\dim D' = 5$, $D' = \boldsymbol{F}_2^5$, so in particular $C^\perp$ contains vectors $u_3, \cdots, u_7$ whose last five coordinates are $10000, \cdots, 00001$ respectively. A computer program considered all ways to choose u_3 and $u_4 \in C^\perp$, ending with 10000 and 01000 respectively, and such that $\{u_1, u_2, u_3, u_4\}$ satisfy (8) for all choices of signs. There is an essentially unique solution (after taking account of obvious symmetries), which is shown in Fig. 1. The same program then found that there is no way to choose the fifth row u_5 to end with 00100 and such that $\{u_1, \cdots, u_5\}$ satisfy (8) for all choices of signs.

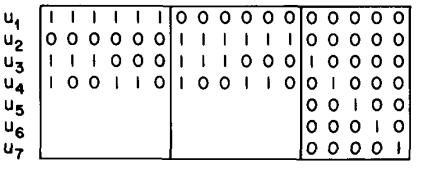

Fig. 1. Partial parity check matrix for [17,10]2 code.

Case ii): $\dim D \ge 3$. The computer program found all ways to adjoin two further vectors $u_3 \in D$, $u_4 \in C^\perp$ to u_1, u_2 so that $\{u_1, u_2, u_3, u_4\}$ satisfy (8) for all choices of signs. There are 27 essentially distinct solutions $\{u_3, u_4\}$ (after allowing for obvious symmetries), but in each solution u_3 has weight six and intersects both u_1 and u_2 in

three coordinates. We may therefore take u_3 as follows:

$$\begin{aligned} u_1 &= 111\ 111\ 000\ 000\ 00000 \\ u_2 &= 000\ 000\ 111\ 111\ 00000 \\ u_3 &= 111\ 000\ 111\ 000\ 00000 \\ u_4 &= \alpha\ \ \beta\ \ \gamma\ \ \delta\ \ \epsilon \end{aligned} \tag{22}$$

Furthermore, in each solution the last five coordinates of u_4 (denoted by ϵ in (22)) have weight at least 2. This shows that $\dim D$ is exactly 3, $\dim D' = 4$, and D' is the [5,4] even weight code. Therefore ϵ has even weight. This reduces the number of inequivalent choices for u_4 (and similarly for the remaining three rows of H) to 12, and we observe that none of these vanishes on any of the first four blocks of three coordinates (i.e., none of $\alpha, \beta, \gamma, \delta$ ever vanishes in (22)). But this is impossible, since there are four rows of H to be completed, and they cannot all be distinct on the first three coordinates. Thus C does not exist.

V. Improvements to the Table of $t[n, k]$

The known improvements to the table of $t[n, k]$ given in [3] are as follows: From [5]: $t[15,6] = 4$. From [7]: $t[23,6] = 7$, $t[25,9] = 6$, $t[26,12] = 5$, $t[28,6] \geq 9$, $t[29,8] \geq 8$, $t[30,7] \geq 9$, $t[32,10] \geq 8$, $t[33,6] \geq 11$. From [1]: $t[27,19] = 2$, $t[41,29] = 3$, $t[42,33] = 2$, $t[58,48] = 2$ (these four values are attained by normal codes); $t[16,9] = 3$, $t[32,23] = 3$, $t[45,35] = 3$. In the present paper: $t[17,10] = 3$, $t[23,15] = 3$. Using $\dot{\oplus}$ to denote an amalgamated direct sum (as in [3]), we also have

$$[27,19]2 \dot{\oplus} [23,12]3 = [49,30]5,$$
$$[42,33]2 \dot{\oplus} [15,11]1 = [56,43]3,$$
$$[42,33]2 \dot{\oplus} [23,12]3 = [64,44]5,$$
$$[49,30]5 \dot{\oplus} [15,11]1 = [63,40]6,$$

which imply $t[49,30] = 5$, $t[56,43] = t[57,44] = 3$, $t[63,40] = 6$, $t[64,44] = 5$.

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