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A [45, 13] CODE WITH MINIMAL DISTANCE 16

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A record binary code of length 45, dimension 13 and minimal distance 16 is constructed in several ways: as an Abelian code (an ideal in the regular representation of $C_3 \times C_{15}$), from a [15, 6, 8] cyclic code over GF(4), and from the $|a + x \mid b + x \mid c + x|$ construction. Its automorphism group has order 360, and its even subcode is a [45, 12, 16] code with only four nonzero weights.

1. Introduction

We describe a remarkable [45, 13, 16] binary linear code $\mathscr{C}$ (compare [7]). Its minimal distance is larger than the estimates provided by the Hartmann–Tzeng and other bounds (see for example [4]), and it therefore appears to have been overlooked by Jensen [3] in his study of Abelian codes of length up to 129.

We give three constructions for $\mathscr{C}$, which has weight distribution:

i	0	16	17	20	21	24	25	28	29	45
A_i	1	405	540	1260	1890	1890	1260	540	405	1

The even subcode $\mathscr{D}$ is a [45, 12, 16] code with only four nonzero weights. Once $\mathscr{D}$ is constructed, we have

$$\mathscr{C} = \mathscr{D} \cup (1 + \mathscr{D}). \tag{1}$$

When discussing codes over GF(4) we follow the notation and conventions of [1].

2. The construction from a cyclic code over GF(4)

We construct GF(16) by adjoining to GF(4) $= \{0, 1, \omega, \bar{\omega}\}$ a root ϵ of $x^2 + \omega x + 1$. Then $\zeta = \epsilon\omega$ is a primitive element of GF(16), with $\zeta^{15} = 1$, $\zeta^2 + \bar{\omega}\zeta + \bar{\omega} = 0$, $\zeta^{10} = \omega$, $\zeta^6 = \epsilon$, $\epsilon^5 = 1$, $\epsilon^4 + \epsilon^3 + \epsilon^2 + \epsilon + 1 = 0$ (cf. [1]).

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Let $\mathscr{C}$ be the [15, 6] cyclic code over GF(4) with generator polynomial

$$\begin{aligned} g(x) &= (x+1)(x+\omega)(x+\bar{\omega})(x^2+\bar{\omega}x+\bar{\omega})(x^2+\bar{\omega}x+1)(x^2+x+\omega) \\ &= 1+x+\omega x^2+\bar{\omega}x^4+\bar{\omega}x^5+\omega x^7+x^8+x^9, \end{aligned} \tag{2}$$

a divisor of $x^{15}+1$. The zeros of the six factors of $g(x)$ are respectively 1; $\omega=\zeta^{10}$; $\bar{\omega}=\zeta^5$; ζ, ζ^4; ζ^3, ζ^{12}; ζ^{11}, ζ^{14}; or in other words g has zeros ζ^i for

$$i \in \{-5, -4, -3, \quad -1, 0, 1, \quad 3, 4, 5\}.$$

Although the Hartmann–Tzeng bound for example gives $d \geqslant 6$, the minimal distance of this code is in fact 8 (see Section 3), and its weight distribution (found by computer) is

i	0	8	10	12	14
A_i	1	405	1260	1890	540

When we map

$$0 \to \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \quad 1 \to \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix}, \quad \omega \to \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \quad \bar{\omega} \to \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}, \tag{3}$$

$\mathscr{C}$ becomes the [45, 12, 16] binary code $\mathscr{D}$ (see Section 4).

Remark. An [18, 9, 8] self-dual code over GF(4) is described in [1]. By taking the subcode that vanishes on coordinates ∞, 0 and Ω, we obtain a [15, 6, 8] code which we have verified is equivalent to $\mathscr{C}$.

3. From the $|a+x \mid b+x \mid c+x|$ construction

By shortening the well-known [6, 3, 4] hexacode over GF(4) ([2, p.82]), we obtain a [5, 2, 4] code $\mathscr{P}$ over GF(4), which we take to have generator matrix

$$\begin{bmatrix} \omega & \bar{\omega} & \bar{\omega} & \omega & 0 \\ 0 & \omega & \bar{\omega} & \bar{\omega} & \omega \end{bmatrix}.$$

Let $\mathscr{F}$ be the [15, 16] code over GF(4) consisting all 3×5 arrays

$$\begin{bmatrix} a+x \\ b+x \\ c+x \end{bmatrix}, \tag{4}$$

where $a, b, c \in \bar{\mathscr{P}}$ (the conjugate of $\mathscr{P}$), $a+b+c=0$, and $x \in \mathscr{P}$ (cf. [2, p. 317], [6, p. 587]). The mapping

$$\begin{bmatrix} c_0 & c_6 & c_{12} & c_3 & c_9 \\ c_{10} & c_1 & c_7 & c_{13} & c_4 \\ c_5 & c_{11} & c_2 & c_8 & c_{14} \end{bmatrix} \leftrightarrow (c_0, c_1, \ldots, c_{14}), \tag{5}$$

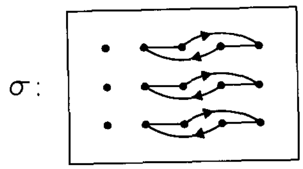

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Fig. 1. The semilinear automorphism σ of order 4.

identifies $\mathscr{F}$ with $\mathscr{E}$. It is now easy to check by hand that $\mathscr{F}$ and therefore $\mathscr{E}$ has minimal distance 8 (compare [6, p. 588]).

The codes $\mathscr{C}, \mathscr{D}, \mathscr{E}, \mathscr{F}$ share the same automorphism group G (using the definition in [1]). Described in terms of $\mathscr{F}$, the group G is generated by scalar multiplication by ω, bodily permutations of the three rows of (4), bodily cyclic shifts of the five columns of (4), and the semilinear automorphism σ of order 4 defined in *Fig. 1.* The permutation σ^2 and the cyclic shifts of the columns of (4) generate a dihedral permutation group of order 10 on the columns, and G has order $3 \cdot 3! \cdot 10 \cdot 2 = 360$.

The group G is (imprimitively) transitive on the 15 coordinates of $\mathscr{F}$. We checked that G is the full automorphism group of $\mathscr{F}$ (and $\mathscr{E}$), and (when acting on the 45 coordinates in the obvious way) of $\mathscr{C}$ and $\mathscr{D}$.

The dual code $\mathscr{D}^\perp$ is a [45, 33, 3] code with weight distribution (in part) $A_0 = 1$, $A_3 = 15$, $A_5 = 189$, $A_6 = 1995$, $A_7 = 11475, \ldots$ $(A_{45-i} = A_i)$, and the 15 words of weight 3 show that the division of the 45 coordinates into 15 blocks of 3 is intrinsic. The dual code $\mathscr{C}^\perp$ is a [45, 32, 6] code, and is the even subcode of $\mathscr{D}^\perp$.

There are four orbits of minimal weight codewords of $\mathscr{F}$ (and therefore of $\mathscr{C}, \mathscr{D}, \mathscr{E}$) under G, namely

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & \bar\omega & \omega & \omega & \bar\omega \\ 0 & \bar\omega & \omega & \omega & \bar\omega \end{bmatrix},$$

$$\begin{bmatrix} 0 & \omega & \bar\omega & \bar\omega & \omega \\ 0 & 0 & \omega & \omega & 0 \\ 0 & 0 & \omega & \omega & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 0 & \bar\omega & 0 & 0 & \bar\omega \\ 0 & 0 & \omega & \omega & 0 \end{bmatrix},$$

$$\begin{bmatrix} 0 & 0 & \omega & \omega & 0 \\ \bar\omega & 0 & 1 & 0 & \omega \\ \bar\omega & \omega & 0 & 1 & 0 \end{bmatrix},$$

with respectively 45, 90, 90 and 180 images, for a total of 405.

4. As an Abelian code

The map (3), when interpreted literally, sends the codewords of $\mathscr{C}$ onto 3×15 binary arrays. The image set may therefore be regarded as an ideal in the group ring $\mathrm{GF}(2) \cdot H$, where $H = C_3 \times C_{15}$ is a product of cyclic groups. In this form $\mathscr{D}$ is an Abelian code, generated as a cyclic submodule of $\mathrm{GF}(2) \cdot H$ by the single vector

$$v = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}, \tag{6}$$

which is the image of $g(x)$ (see (2)) under the map (3), and similarly $\mathscr{C}$ is generated either by $\mathbf{1}$ and v, or by the single vector $\mathbf{1} + v$.The code $\mathscr{C}$ was originally obtained in this form, while carrying out a computer search for Abelian codes (cf. [5], [6, pp. 677, 733]). A generator matrix for $\mathscr{C}$ is given in *Fig. 2*.

$$\left[\begin{array}{l}
100000000000011100100110010011111011001101111 \\
010000000000010010110101011010000110101011000 \\
001000000000001001011010101101000011010101100 \\
000100000000011000001011000101111010100111001 \\
000010000000010000100011110001000110011110011 \\
000001000000001100010001111000100011001111001 \\
000000100000011110101110101111001010101010011 \\
000000010000010011110001000100011110011000110 \\
000000001000001001111000100010001111001100011 \\
000000000100011100011010000010011100101011110 \\
000000000010010110101011010010110101011000000 \\
000000000001001011010101101001011010101100000 \\
000000000000111001001100100111110110011011111
\end{array}\right]$$

Fig. 2. Generator matrix for [45, 13, 16] binary code $\mathscr{C}$.

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