Graphs and Combinatorics 5, 315–325 (1989)

Graphs and Combinatorics

Gray Codes for Reflection Groups

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Abstract. Let G be a finite group generated by reflections. It is shown that the elements of G can be arranged in a cycle (a "Gray code") such that each element is obtained from the previous one by applying one of the generators. The case $G = \mathscr{A}_1^n$ yields a conventional binary Gray code. These generalized Gray codes provide an efficient way to run through the elements of any finite reflection group.

1. Introduction

The classical version of a Gray code is a Hamiltonian circuit through the 2^n vertices of the n-cube, or equivalently an ordering of the 2^n binary vectors of length n such that each pair of adjacent vectors (including the first and last) differ in a single position. For the extensive literature see the bibliography. The first appearance of the "Gray code" that we have located is in 1872 [29].

As we will show, the classical version is the special case $G = \mathscr{A}_1^n$ of the following.

Theorem. *Let G be a finite group generated by reflections $R_1, \ldots, R_n$. Then there is a Hamiltonian circuit in the Cayley diagram for G corresponding to these generators. In other words the $g = |G|$ elements of G can be arranged in order*

$$\{a_0, a_1, \ldots, a_{g-1}\} \tag{1}$$

so that for each i $(0 \le i \le g-1)$ there is a j so that $a_{i+1} = a_i R_j$ (where $a_g = a_0$).

We call (1) a Gray code for G.

It is well-known that any group generated by reflections can be described by a Coxeter diagram [7, 14, 15, 31]. The finite reflection groups for which the Coxeter diagram is a connected graph are ([7], p. 193, Theorem 1) the groups $\mathscr{A}_n$ $(n \ge 1)$, $\mathscr{B}_n$ $(n \ge 2)$, $\mathscr{D}_n$ $(n \ge 4)$, $\mathscr{E}_6$, $\mathscr{E}_7$, $\mathscr{E}_8$, $\mathscr{F}_4$, $\mathscr{G}_2$, $\mathscr{H}_3$, $\mathscr{H}_4$ and $\mathscr{I}_2(m)$, $(m = 5$ or $m > 7)$.* These are the *irreducible* reflection groups. Figure 1 shows their Coxeter diagrams,

* We follow Grove and Benson [31] in using script letters for these groups, to distinguish them from the Lie groups and Euclidean lattices with similar names.

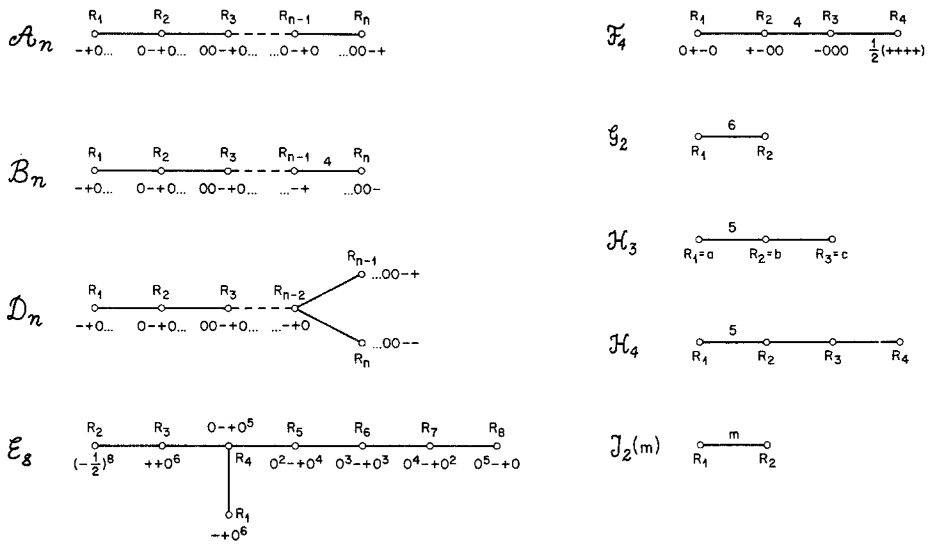

Fig. 1. Coxeter diagrams for the irreducible finite reflection groups, with nodes labeled by generating reflections (+ stands for +1, − for −1). For $\mathscr{E}_7$, omit the node labeled R_8 from $\mathscr{E}_8$; for $\mathscr{E}_6$, omit nodes R_7 and R_8.

$\mathscr{A}_n$	$(n \geq 1)$	$(n+1)!$
$\mathscr{B}_n$	$(n \geq 2)$	$2^n\, n!$
$\mathscr{D}_n$	$(n \geq 4)$	$2^{n-1}\, n!$
$\mathscr{E}_6$		51840
$\mathscr{E}_7$		2903040
$\mathscr{E}_8$		696729600
$\mathscr{F}_4$		1152
$\mathscr{G}_2$		12
$\mathscr{H}_3$		120
$\mathscr{H}_4$		14400
$\mathscr{I}_2(m)$	$(m=5$ or $m \geq 7)$	$2m$

Fig. 2. Orders of irreducible reflection groups.

labeled with the generating reflection R_i. (The notation is explained in §3.) The orders of these groups are given in Fig. 2. An arbitrary finite reflection group is then a direct product of irreducible ones.

The group $\mathscr{A}_{n-1}$ is isomorphic to the symmetric group S_n (see §3). Thus a Gray code for $\mathscr{A}_{n-1}$ is an ordering of the $n!$ permutations of n letters such that (if we use the generators indicated in Fig. 1) each permutation is obtained from the previous one by following it by one of the transpositions

$$(1,2), (2,3), \ldots, (n-1, n). \tag{2}$$

Such arrangements of permutations were given by Johnson [32] and Trotter [53] in the early 1960's. Other Hamiltonian circuits through all $n!$ permutations (satisfying different constraints) arise in bell-ringing [49, 57–60]. Although several other generalizations of Gray codes have appeared [3, 22, 33–37, 42, 48, 52], we believe our version is new.

The theorem is proved in §2, and §3 gives some examples. In particular we give specific Gray codes for all the examples $(\mathscr{A}_1, \mathscr{A}_2, \mathscr{A}_3, \mathscr{B}_2, \mathscr{B}_3, \mathscr{G}_2, \mathscr{H}_3, \mathscr{I}_2(m))$ in dimensions $n \leq 3$.

It is worth remarking that not all Cayley diagrams for groups contain Hamiltonian circuits. Rankin [49] (see also [60]) gives a necessary condition that a certain class of groups must satisfy, and uses it to deduce for example that the Cayley diagram for the alternating group A_6 with generators $(2, 4, 6, 5, 3)$ and $(1, 6, 3)(2, 4, 5)$ does not contain a Hamiltonian circuit.

The problem of finding Gray codes for reflection groups arose in the following context (see Dobkin et al. [20] and Levy and Wilks [39]). Suppose Euclidean n-space $\mathbf{R}^n$ is tesselated by simplices, and suppose that $f: \mathbf{R}^n \to \mathbf{R}^{n-1}$ is affine (linear) on each simplex. The 0-contour of f, $f^{-1}(0)$, is generically a piecewise-linear curve in $\mathbf{R}^n$. An effective method for finding this contour is to follow it as it passes through successive simplices in the tesselation.

Normally the contour passes into and out of simplices through their facets – their $(n-1)$-faces. It can happen, though, that the contour passes out of a simplex through a face of lower dimension. When this happens, it is not clear which simplex the contour enters next, and a search must be made among adjacent simplices sharing the exiting face.

An excellent choice of tesselation that facilitates these computations is a tesselation by reflection. The group of isometries generated by reflections in the facets of any given simplex of such a tesselation generates the entire, infinite symmetry group of the tesselation. The reason computations are simpler with this class of tesselations is that it is easy to reflect a simplex in one of its facets. Generically this is the only step necessary in finding the contour of f, as the curve passes through facets between adjacent simplices.

Now when the curve passes through a lower dimensional face τ of a simplex σ it is necessary to find all simplices sharing τ. Each such simplex is the image of σ under some isometry of $\mathbf{R}^n$. Together these isometries form a finite reflection group, generated by reflections in the facets of σ that leave τ fixed. Since these generating reflections are computationally easy to apply, a reasonable scheme for enumerating simplices that share τ as a face is to begin with σ and apply generating reflections repeatedly until the group is enumerated. Clearly this will happen most efficiently if each simplex is visited only once; such a scheme is precisely a Gray code for the finite reflection group.

2. Proof of Theorem

We begin by reducing the problem to the case when the Coxeter diagram is connected. Suppose the Coxeter diagram for a reflection group G is the disjoint union of two Coxeter diagrams corresponding to reflection groups G_1 and G_2, so that $G \cong G_1 \times G_2$. Let $|G_1| = r \geq 2, |G_2| = s \geq 2$, where (see Fig. 2) r and s are even. Let $\{a_0, a_1, \ldots, a_{r-1}\}$ and $\{b_0, b_1, \ldots, b_{s-1}\}$ be Gray codes for G_1 and G_2 respectively. Then

$$\begin{aligned} \{(a_0, b_0),\ (a_0, b_1),\ &\ldots, (a_0, b_{s-1}), \\ (a_1, b_{s-1}), (a_1, b_{s-2}), &\ldots, (a_1, b_0), \\ (a_2, b_0),\ (a_2, b_1),\ &\ldots, (a_2, b_{s-1}), \\ (a_3, b_{s-1}), (a_3, b_{s-2}), &\ldots\ldots\} \end{aligned} \tag{3}$$

(reading from left to right) is a Gray code for $G_1 \times G_2$.

We now suppose the Coxeter diagram for G is connected and hence is a tree (see Fig. 1). The proof is by induction on n, the number of nodes of the Coxeter diagram. For $n = 1$ and 2 the result is trivial (see §3), so we assume $n \geq 3$.

Let the nodes of the Coxeter diagram be labeled $R_1, \ldots, R_n$ (corresponding to generating reflections), where R_n is a node of degree 1 connected only to node R_{n-1}. Let H be the subgroup generated by $R_1, \ldots, R_{n-1}$.

The Cayley diagrams for G and H will be denoted by Γ and Δ respectively. Γ has a node for each element $g \in G$, with n directed edges labeled $R_1, \ldots, R_n$ leaving each node, the edge labeled R_i being directed from g to g' if $gR_i = g'$ (see for example [30, 56]). Since $R_i^2 = 1$, $g'R_i = g$, and so the edges occur in pairs. We must show that the directed graph Γ contains a Hamiltonian circuit.

Since G is the union of $m = |G|/|H|$ cosets of H, the Cayley diagram Γ contains the disjoint union of m subgraphs $\Delta_1, \ldots, \Delta_m$ (corresponding for example to the left cosets aH) each of which is isomorphic to Δ. Every node of Γ belongs to one of the Δ_i, and the edges joining the Δ_i's all carry the label R_n. By induction there is a Hamiltonian circuit through Δ, with edges labeled $R_1, \ldots, R_{n-1}$. There are therefore Hamiltonian circuits with identical labelings through each Δ_i, and we use these particular circuits in the construction below.

We construct an (undirected) graph Φ from Γ by contracting each Δ_i to a point, discarding all edges except those labeled R_n, replacing these by undirected edges and discarding multiple edges. (The nodes of the contracted graph Φ correspond to left cosets aH, and nodes corresponding to aH and bH are joined by an edge just when aHR_n intersects bH.) Since Γ is connected (all Cayley diagrams are connected [30]) so is Φ, and we can find a spanning tree in Φ. We form the Hamiltonian circuit for Γ by expanding this spanning tree into a circuit.

Consider two subgraphs Δ_i and Δ_j joined by a pair of edges labeled R_n, joining $P \in \Delta_i$ and $Q \in \Delta_j$, as in Fig. 3. Let $\ldots P'PP'' \ldots$ be part of the Hamiltonian circuit for Δ_i, with the edges $P'P$, PP'' labeled R_α, R_β respectively. We cannot have $R_\alpha = R_\beta$, because $R_\alpha^2 = 1$ and we are assuming $n \geq 3$. So at least one of R_α, R_β is different from R_{n-1}, say $R_\beta \neq R_{n-1}$. Then the nodes R_β, R_n are not joined in the Coxeter diagram for G, so R_β commutes with R_n. Consider the pair of edges at P'' labeled R_n (indicated by the dotted lines in Fig. 3), joining P'' to Q'' say. Since R_n and R_β commute, there is an edge labeled R_β from Q to Q'', and (by applying a suitable element of H to the original Hamiltonian circuit for Δ_j) we can find a Hamiltonian circuit for Δ_j containing this edge.

We can now construct a Hamiltonian circuit through the nodes of $\Delta_i \cup \Delta_j$: start

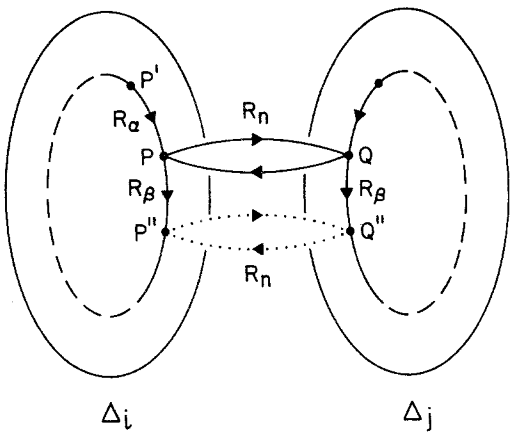

Fig. 3. Construction of Hamiltonian circuit through the nodes of $\Delta_i \cup \Delta_j$.

at P'', follow the circuit for Δ_i forwards to P, go to Q, follow the circuit for Δ_j backwards to Q'', and return to P''.

By using this technique for every edge of the spanning tree joining the Δ_i's, we build up a Hamiltonian circuit for Γ. This completes the proof.

The inductive proof of the theorem also serves as a recursive algorithm for finding a Gray code for a particular group. For a detailed example, see the construction of a Gray code for the icosahedral group $\mathscr{H}_3$ in the following section.

3. Examples

There are three equivalent ways to specify a Gray code for a group G: (i) as a list of group elements $\{a_0, \dots, a_{g-1}\}$, as in Eq. (1); (ii) by specifying the reflection needed to derive each successive group element from the previous one – if $a_\nu = a_{\nu-1} R_{i_\nu}$ in (1) $(1 \le \nu \le g)$, the Gray code can be described by the command

$$\text{apply successively } \{R_{i_1}, R_{i_2}, \dots, R_{i_g}\}; \tag{4}$$

or (iii) by taking a particular vector w in the appropriate space and listing its images under the successive group elements of Eq. (1):

$$\{wa_0, wa_1, \dots, wa_{g-1}\}. \tag{5}$$

We now discuss the individual groups, in alphabetical order. The construction of a Gray code for the icosahedral group $\mathscr{H}_3$ is described in some detail, but our treatment of the other groups is brief. Remember that the cases $n = 1$ and $n = 2$ are special, and the inductive construction applies only for $n \ge 3$.

The Coxeter diagrams are given in Fig. 1. The n nodes are labeled by generating reflections $R_1, \dots, R_n$. A reflection R_i in the hyperplane perpendicular to a vector a,

$$R_i\colon x \to x - 2\frac{x \cdot a}{a \cdot a} a,$$

is specified by giving the vector a.

$\mathscr{A}_n$. The n vectors a for $\mathscr{A}_n$ have the form $(0,\dots,0,1,-1,0,\dots,0)$, with $n+1$ coordinates adding to zero. The corresponding reflections are the transpositions $(1,2), (2,3), \dots, (n, n+1)$ of the $n+1$ coordinates, and so $\mathscr{A}_n$ is isomorphic to the symmetric group S_{n+1}. Several algorithms for obtaining Gray codes are known [18, 22, 32, 53]. The following algorithm (which arises from the proof of the theorem) seems as simple as any. It will also be used when we construct Gray codes for the other groups.

We specify a Gray code C_n for $\mathscr{A}_{n-1} \cong S_n$ in the third form, as in Eq. (5), taking $w = (1, 2, \dots, n)$. We must say how S_n acts on vectors: a typical permutation

$$\pi = \begin{pmatrix} 1 & 2 & \cdots & n \\ \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix} \in S_n$$

sends a vector $x = (x_1, \dots, x_n)$ to $x\pi = y = (y_1, \dots, y_n)$, where $y_{\pi_v} = x_v$ $(v = 1, \dots, n)$.

For $\mathscr{A}_1$, the Gray code is $C_2 = \{12, 21\}$.

For $\mathscr{A}_2$, $C_3 = \{123, 213, 231, 321, 312, 132\}$.

For $n \geq 3$, we observe that C_n contains a pair of adjacent vectors $\alpha_i i$, $\beta_i i$ (both ending in i), for all $i = 1, \dots, n$. To obtain C_{n+1} from C_n, we append $n+1$ to every vector of C_n, and then, for each $i = 1, \dots, n$, insert between $\alpha_i i(n+1)$ and $\beta_i i(n+1)$ the sequence of vectors $D_n^i i$, where D_n^i is obtained from C_n by starting at $\alpha_i i$, proceeding *backwards* through the code to $\beta_i i$, and then replacing each occurrence of i by $n+1$. For example, Fig. 4 shows the construction of C_4 from C_3. The arrows indicate where the sequences $D_3^1 1$, $D_3^2 2$ and $D_3^3 3$ are inserted.

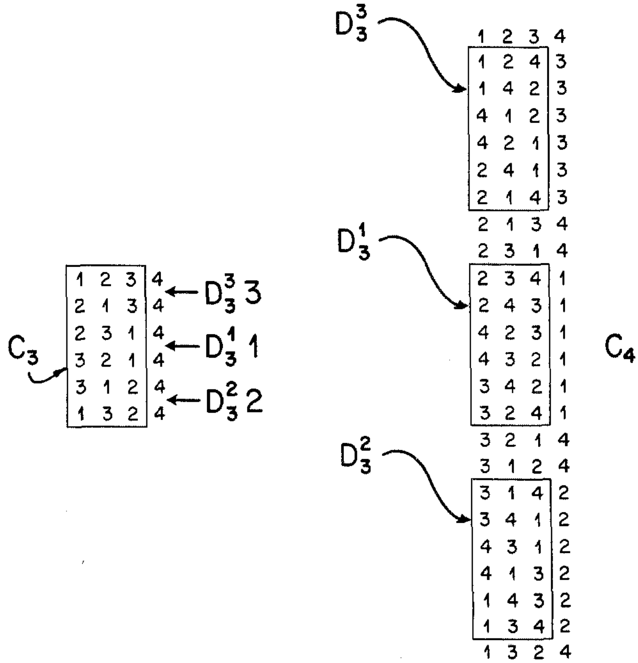

Fig. 4. Construction of Gray code for S_4 from that for S_3.

We omit the straightforward derivation of this algorithm from the proof of the theorem, and just mention that for this group the nodes of the subgraph Δ_i are all vectors ending in i, and the contracted graph Φ is a complete graph.

The Gray code for $\mathscr{A}_1$ has two elements, which could therefore be denoted by 0 and 1. Then the Gray code for $\mathscr{A}_1^n$ obtained by successive applications of Eq. (3) is the usual binary "reflected" Gray code.

$\mathscr{B}_n$. The subgroup H of $\mathscr{B}_n$ generated by $R_1, \ldots, R_{n-1}$ (see Fig. 1) consists of all permutations of the n coordinates, while the remaining generator R_n negates the last coordinate. Thus the $2^n n!$ elements of $\mathscr{B}_n$ consist of all permutations and sign changes of the n coordinates. We specify group elements $g \in \mathscr{B}_n$ by displaying wg, where $w = (1, 2, \ldots, n)$. For a typical left coset aH, the set waH is obtained from w by changing the signs of a subset of $\{1, 2, \ldots, n\}$ and then permuting the coordinates in all possible ways. The contracted graph Φ is the 1-skeleton of an n-cube, and therefore a standard binary Gray code provides a spanning tree in Φ. The following are Gray codes for $\mathscr{B}_2$ and $\mathscr{B}_3$ (bars indicate negative numbers):

$$\mathscr{B}_2\colon \{12, 1\bar{2}, \bar{2}1, \bar{2}\bar{1}, \bar{1}\bar{2}, \bar{1}2, 2\bar{1}, 21\},$$

$$\begin{aligned}\mathscr{B}_3\colon \{&123, 12\bar{3}, 1\bar{3}2, 1\bar{3}\bar{2}, 1\bar{2}\bar{3}, 1\bar{2}3, 13\bar{2}, 31\bar{2},\\ &3\bar{2}1, 3\bar{2}\bar{1}, 3\bar{1}\bar{2}, \bar{1}3\bar{2}, \bar{1}\bar{2}3, \bar{1}\bar{2}\bar{3}, \bar{1}\bar{3}\bar{2}, \bar{1}\bar{3}2,\\ &\bar{1}2\bar{3}, \bar{1}23, \bar{1}32, 3\bar{1}2, 32\bar{1}, 23\bar{1}, 2\bar{1}3, 2\bar{1}\bar{3},\\ &2\bar{3}\bar{1}, \bar{3}2\bar{1}, \bar{3}\bar{1}2, \bar{3}\bar{1}\bar{2}, \bar{3}\bar{2}\bar{1}, \bar{2}\bar{3}\bar{1}, \bar{2}\bar{1}\bar{3}, \bar{2}\bar{1}3,\\ &\bar{2}3\bar{1}, \bar{2}31, \bar{2}13, \bar{2}1\bar{3}, \bar{2}\bar{3}1, \bar{3}\bar{2}1, \bar{3}1\bar{2}, \bar{3}12,\\ &\bar{3}21, 2\bar{3}1, 21\bar{3}, 213, 231, 321, 312, 132\}\end{aligned}$$

$\mathscr{D}_n$ has order $2^{n-1}n!$ and is similar to $\mathscr{B}_n$, except that only evenly many minus signs are permitted. Again $H = \langle R_1, \ldots, R_{n-1} \rangle \cong S_n$, but now R_n interchanges and negates the last two coordinates. The contracted graph Φ has 2^{n-1} nodes, corresponding to the binary n-typles with an even number of ones, two nodes being joined by an edge just when they are at Hamming distance 2. (Φ is the 1-skeleton of the half-cube $h\gamma_n$ in Coxeter's notation [14].) Taking alternate elements of a standard binary Gray code yields a spanning tree in Φ. Note that $\mathscr{D}_n$ need only be defined for $n \geq 4$ (the analogous Coxeter diagrams for $n = 1, 2$ and 3 define $\mathscr{A}_1$, $\mathscr{A}_1^2$ and $\mathscr{A}_3$).

$\mathscr{E}_6$, $\mathscr{E}_7$, $\mathscr{E}_8$. Gray codes for these groups can be obtained by taking w (in (5)) to be the Weyl vector [7, 13] for the group (a vector having inner product 1 with every node in the Coxeter diagram). The Weyl vectors corresponding to the diagrams in Fig. 1 are:

$$\mathscr{E}_6\colon w = (0, 1, 2, 3, 4, -4, -4, -4),$$

$$\mathscr{E}_7\colon w = (0, 1, 2, 3, 4, 5, -8\tfrac{1}{2}, -8\tfrac{1}{2}),$$

$$\mathscr{E}_8\colon w = (0, 1, 2, 3, 4, 5, 6, -23).$$

The contracted graphs Φ have $|\mathscr{E}_6|/|\mathscr{D}_5| = 27$, $|\mathscr{E}_7|/|\mathscr{E}_6| = 56$ and $|\mathscr{E}_8|/|\mathscr{E}_7| = 240$ nodes respectively. (Their nodes correspond to the minimal vectors in $E_6 + [1]$,

$E_7 + [1]$ and $E_8 + [1] \cong E_8$, respectively, where E_n denotes the n-dimensional lattice corresponding to $\mathscr{E}_n$ and $[i]$ is the ith coset representative for E_n in E_n^* – see [13], Chap. 4.)

$\mathscr{F}_4$. The subgroup H generated by R_1, R_2, R_3 is isomorphic to $\mathscr{B}_3$, while the remaining generator R_4 sends (x_1, x_2, x_3, x_4) to the vector obtained by subtracting $\frac{1}{2}(x_1 + x_2 + x_3 + x_4)$ from each coordinate. As in the case of $\mathscr{E}_n$, particularly simple coordinates for the Gray code are obtained by choosing w in (5) to be the Weyl vector, which for $\mathscr{F}_4$ is

$$w = (1, 2, 3, -8).$$

In fact the 1152 vectors wg, $g \in \mathscr{F}_4$, are found by applying all permutations and sign changes to the components of the vectors

$$w_6 = (1, 4, 5, 6), \qquad w_7 = (2, 3, 4, 7) \qquad \text{and} \qquad w_8 = (1, 2, 3, 8). \tag{6}$$

The contracted graph Φ has 24 nodes, corresponding to the minimal vectors of the lattice D_4 (see [13]), two nodes being joined by an edge just when the difference of the vectors is again a minimal vector. Equivalently, Φ is the 1-skeleton of the polytope $\{3, 4, 3\}$ (see Fig. 8.2A, p. 149 of [14]). The 24 left cosets of H, corresponding to the nodes of Φ, may be labeled $H_{\sigma i}$ ($\sigma = 6$, 7 or 8; $i \in \{\pm 1, \pm 2, \ldots, \pm 8\}$), where $wH_{\sigma i}$ consists of the 48 vectors obtained from w_σ by taking the last coordinate to be i and applying all permutations and sign changes to the other three coordinates. A Gray code for $\mathscr{F}_4$ could now easily be written down, using these vectors and a spanning tree for Φ. We omit the details.

$\mathscr{G}_2$ is isomorphic to the dihedral group of order 12, and Eq. (7) below (with $m = 6$) gives a Gray code.

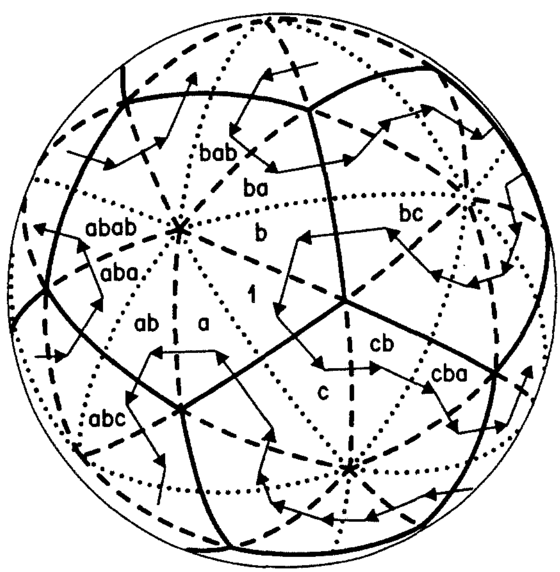

Fig. 5. Surface of 2-sphere divided into 120 spherical triangles corresponding to the elements of the icosahedral group $\mathscr{H}_3$. The arrows show part of a Gray code for $\mathscr{H}_3$.

$\mathscr{H}_3$. The subgroup H of $\mathscr{H}_3$ generated by the reflections $a = R_1$ and $b = R_2$ is a dihedral group $\mathscr{I}_2(5)$ of order 10, for which a Gray code is given by Eq. (7) below. R_1 and R_2 are the reflections in the dotted and dashed lines respectively in Fig. 5, which shows the surface of the 2-sphere tesselated into the 120 spherical triangles corresponding to the elements of $\mathscr{H}_3$. The third generator $c = R_3$ is the reflection in the solid lines. The spherical triangles in the figure have been (partially) labeled by the elements of $\mathscr{H}_3$.

The arrows show (part of) a Gray code for $\mathscr{H}_3$. This was found using the technique used to prove the theorem. The contracted graph Φ has $|\mathscr{H}_3|/|\mathscr{I}_2(5)| = 120/10 = 12$ nodes, and is in fact the 1-skeleton of an icosahedron. We label its nodes 0, 1, ..., 5, 0′, 1′, ..., 5′, nodes i and i' being antipodal, and make use of the spanning tree shown in Fig. 6. (The 12 nodes in Fig. 6 correspond to the 12 spherical pentagons bounded by solid lines in Fig. 5.) The resulting Gray code (in the form of Eq. (4)) is

$$\text{apply successively } \{c, b, a, b, c, (b, a)^2, b, c,$$
$$(b, a)^4, b, c, b, a, b, c, (b, a)^2, b, c,$$
$$(b, c, b, a, b, c, (b, a)^4, b, c, (b, a)^2, b, c)^4, b\}.$$

$\mathscr{H}_4$. The subgroup $H = \langle R_1, R_2, R_3 \rangle$ is isomorphic to $\mathscr{H}_3$, and the contracted graph Φ has $|\mathscr{H}_4|/|\mathscr{H}_3| = 120$ vertices and is isomorphic to the 1-skeleton of the 600-cell $\{3, 3, 5\}$ (see [14], Plate IV).

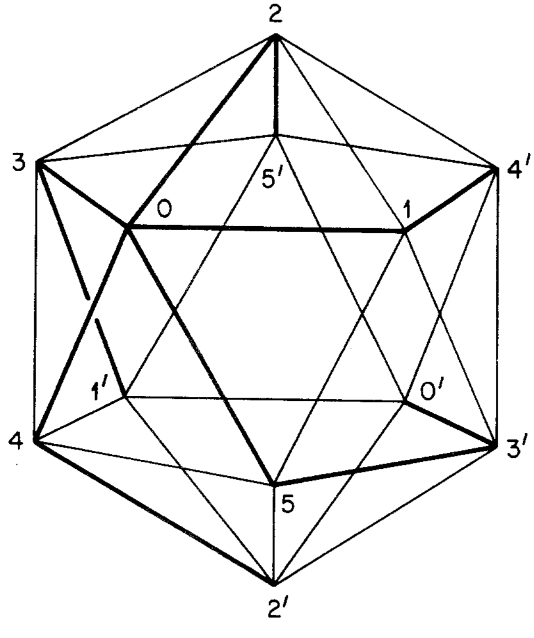

Fig. 6. Contracted graph Φ (the 1-skeleton of an icosahedron) and spanning tree, for icosahedral group $\mathscr{H}_3$.

$\mathscr{I}_2(m)$ (the dihedral group of order $2m$). Since $(R_1 R_2)^m = 1$, a Gray code is

$$\text{apply successively } \{R_1, R_2, R_1, R_2, \ldots, R_2\} \text{ (}2m\text{ elements).} \tag{7}$$

References

1. Abbott, H.L.: Hamiltonian circuits and paths on the n-cube. Canad. Math. Bull. **9**, 557–562 (1966)
2. Afriat, S.N.: The Ring of Linked Rings. London: Duckworth 1982
3. Agrawal, D.P.: Signed modified reflected binary code. IEEE Trans. Comput. **25**, 549–552 (1976)
4. Arazi, B.: An approach to generating different types of Gray codes. Inf. Control **63**, 1–10 (1984)
5. Berlekamp, E.R., Conway, J.H., Guy, R.K.: Winning Ways. NY: Academic Press 1982
6. Bitner, J.R., Ehrlich, G., Reingold, E.M.: Efficient generation of the binary reflected Gray code and its applications. Commun. ACM **19**, 517–521 (1976)
7. Bourbaki, N.: Groupes et Algèbres de Lie. Chapitres 4, 5 et 6. Paris: Hermann 1968
8. Buck, M., Wiedemann, D.: Gray codes with restricted density. Discrete Math. **48**, 163–171 (1984)
9. Caviour, S.R.: An upper bound associated with errors in Gray code. IEEE Trans. Inf. Theory **21**, 596 (1975)
10. Chamberlain, R.M.: Gray codes, Fast Fourier Transforms and hypercubes. Parallel Computing **6**, 225–233 (1988)
11. Cohn, M.: Affine m-ary Gray codes. Inf. Control **6**, 70–78 (1963)
12. Cohn, M., Even, S.: A Gray code counter. IEEE Trans. Comput. **18**, 662–664 (1969)
13. Conway, J.H., Sloane, N.J.A.: Sphere Packings, Lattices and Groups. NY: Springer-Verlag 1988
14. Coxeter, H.S.M.: Regular Polytopes. 3rd ed. NY: Dover 1973
15. Coxeter, H.S.M., Moser, W.O.J.: Generators and Relations for Discrete Groups. 4th ed. NY: Springer-Verlag 1980
16. Crowe, D.W.: The n-dimensional cube and the tower of Hanoi. Amer. Math. Mon. **63**, 29–30 (1956)
17. Darwood, N.: Using the decimal Gray code. Electronic Engineering. 28–29 (Feb. 1972)
18. Dershowitz, N.: A simplified loop-free algorithm for generating permutations. BIT **15**, 158–164 (1975)
19. Dixon, E., Goodman, S.: On the number of Hamiltonian circuits in the n-cube. Proc. Amer. Math. Soc. **50**, 500–504 (1975)
20. Dobkin, D.P., Levy, V.F., Thurston, W.P., Wilks, A.R.: Contour tracing by piecewise linear approximations. Transactions on Graphics **9** (1990), to appear
21. Douglas, R.J.: Bounds on the number of Hamiltonian circuits in the n-cube. Discrete Math. **17**, 143–146 (1977)
22. Ehrlich, G.: Loopless algorithms for generating permutations, combinations, and other combinatorial configurations. J. ACM **20**, 500–513 (1973)
23. Er, M.C.: On generating the n-ary reflected Gray codes. IEEE Trans. Comput. **33**, 739–741 (1984)
24. Er, M.C.: Two recursive algorithms for generating the binary reflected Gray code. J. Inf. Optimization Sci. **6**, 213–216 (1985)
25. Flores, I.: Reflected number systems. IRE Trans. Electronic Computers **5**, 79–82 (1956)
26. Gardner, M.: The curious properties of the Gray code and how it can be used to solve puzzles. Scientific American **227**, 106–109 (No. 2, August 1972)
27. Gilbert, E.N.: Gray codes and paths on the n-cube. Bell Syst. Tech. J. **37**, 815–826 (1958)
28. Gray, F.: Pulse Code Communication. U.S. Patent 2632058, March 17, 1953
29. Gros, L.: Theorie de Baguenodier. Lyons: Aimé Vingtrinier 1872

30. Grossman, I., Magnus, W.: Groups and Their Graphs. NY: Random House 1964
31. Grove, L.C., Benson, C.T.: Finite Reflection Groups. 2nd ed. NY: Springer-Verlag 1985
32. Johnson, S.M.: Generation of permutations by adjacent transpositions. Math. Comput. **17**, 282–285 (1963)
33. Joichi, J.T., White, D.E.: Gray codes in graphs of subsets. Discrete Math. **31**, 29–41 (1980)
34. Joichi, J.T., White, D.E., Williamson, S.G.: Combinatorial Gray codes. SIAM J. Comput. **9**, 130–141 (1980)
35. Kaye, R.: A Gray code for set partitions. Info. Process Lett. **5**, 171–173 (1976)
36. Klee, V.: Long paths and circuits on polytopes. Chap. 17 of B. Grünbaum, Convex Polytopes. NY: Wiley 1967
37. Klingsberg, P.: A Gray code for compositions. J. Algorithms **3**, 41–44 (1982)
38. van Lantschoot, E.J.M.: A systematic method for designing Gray code counters. Comput. J. 43 (1973)
39. Levy, S.V.F., Wilks, A.R.: Computing the contour of a piecewise linear function (to appear)
40. Lucal, H.M.: Arithmetic operations for digital computers using a modified reflected binary code. IRE Trans. Electronic Computers **8**, 449–459 (1959)
41. Ludman, J.E., Sampson, J.L.: A technique for generating Gray codes. J. Stat. Plann. Inference **5**, 171–180 (1981)
42. Lüneburg, H.: Gray-codes. Abh. Math. Semin. Univ. Hamb. **52**, 208–227 (1982)
43. MacWilliams, F.J., Sloane, N.J.A.: The Theory of Error Correcting Codes. Amsterdam: North-Holland 1977
44. Mathialagan, A., Vaidehi, V.: Reduced look-up table for Gray to Binary conversion. J. Inst. Electron. Telecommun. Eng. **32**, 76–77 (1986)
45. Mills, W.H.: Some complete cycles on the n-cube. Proc. Amer. Math. Soc. **14**, 640–643 (1963)
46. Oberman, R.M.M.: A new explanation of the reflected binary code. IEEE Trans. Comput. **23**, 641–642 (1974)
47. Prodinger, H.: Nonrepetitive sequences and Gray code. Discrete Math. **43**, 113–116 (1983)
48. Proskurowski, A., Ruskey, F.: Binary tree Gray codes. J. Algorithms **6**, 225–238 (1985)
49. Rankin, R.A.: A campanological problem in group theory. Proc. Comb. Phil. Soc. **44**, 17–25 (1948)
50. Sharma, B.D., Khanna, R.K.: On m-ary Gray codes. Inf. Sci. **15**, 31–43 (1978)
51. Smith, D.H.: Hamiltonian circuits on the n-cube. Canad. Math. Bull. **17**, 759–761 (1975)
52. Tang, D.T., Liu, C.N.: Distance 2 cyclic chaining of constant weight codes. IEEE Trans. Comput. **22**, 176–180 (1973)
53. Trotter, H.F.: Algorithm 115, PERM. Commun. ACM **5**, 434–435 (1962)
54. Vickers, V.E., Silverman, J.: A technique for generating specialized Gray codes. IEEE Trans. Comput. **29**, 329–331 (1980)
55. Wang, M.C.: An algorithm for Gray-to-binary conversion. IEEE Trans. Comput. **15**, 659–660 (1966)
56. White, A.T.: Graphs, Groups and Surfaces. Amsterdam: North-Holland 1973
57. White, A.T.: Graphs of groups on surfaces. In: Combinatorial Surveys (P.J. Cameron, ed.) pp. 165–197. NY: Academic Press 1977
58. White, A.T.: Ringing the changes. Math. Proc. Comb. Philos. Soc. **94**, 203–215 (1983)
59. White, A.T.: Ringing the changes II. Ars Comb. **A20**, 65–75 (1985)
60. White, A.T.: Ringing the cosets. Amer. Math. Mon. **94**, 721–746 (1987)
61. Yuen, C.K.: The separability of Gray code. IEEE Trans. Inf. Theory **20**, 668 (1974)
62. Yuen, C.K.: Fast analog-to-Gray code converter. Proc. IEEE **65**, 1510–1511 (1977)

Received: October 19, 1988