A New Table of Constant Weight Codes

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Abstract —**A table of binary constant weight codes of length $n \leq 28$ is presented. Explicit constructions are given for most of the 600 codes in the table; the majority of these codes are new. The known techniques for constructing constant weight codes are surveyed, and also a table is given of (unrestricted) binary codes of length $n \leq 28$.**

I. Introduction

THE MAIN GOAL of this paper is to give an extensive table of lower bounds on $A(n,d,w)$, the maximal possible number of binary vectors of length n, Hamming distance at least d apart, and constant weight w. We also give a table of lower bounds on $A(n,d)$, the maximal possible number of binary vectors of length n and Hamming distance at least d apart (with no restriction on weight).

These functions have been studied by many authors, and were tabulated for $n \leq 24$ in [13], [45], [72], [132]. In the present paper we extend the tables to length $n \leq 28$.

Our main concern is with Table I, the table of constant weight codes. The majority of the 600 codes in this table are new, either because we have discovered nicer versions of existing codes, or (more frequently) because we have found better codes than were known before.

Our goal has been to give either an explicit construction or a reference for every code in the table. With some exceptions a reader should be able to reconstruct any of these codes from the information given here. (This is in contrast to [13], where several codes are simply described as being found by an unstated "miscellaneous construction".) However, because of space limitations, we have not included explicit listings for the codes constructed in Section XII (indicated by "y" in Table I) when they contain more than 1500 codewords.

Although [13] gives both upper and lower bounds on $A(n,d,w)$ and $A(n,d)$, in the present paper we give only lower bounds, i.e. tables of actual codes. We have not given upper bounds for several reasons: 1) their calculation is a separate investigation, requiring analytic as opposed to combinatorial methods, 2) all the upper bounds in [13] for $d=10$ should be rechecked (see the Errata section), 3) it is very difficult to check upper bounds found by others,[1] and 4) the paper is long enough already. However, we do mention the cases where we know that our lower bound is actually the exact value.

$A(n,d,w)$ and $A(n,d)$ are fundamental combinatorial quantities. They are also used in the construction of codes for asymmetric channels [16], [39], [49]–[51], [90], [180], DC-free codes [15], [64], [175], and spherical codes [167].

We would appreciate hearing of any improvements to the tables. Please send them to N. J. A. Sloane, Room 2C-376, AT&T Bell Labs, Murray Hill, NJ 07974, USA; electronic mail address user@mhuxo.att.com.

Notation: The following notation will be used throughout. $\mathbb{F}_q$ denotes the Galois field of order q, while $\mathbb{Z}_m$ denotes the integers modulo m. An $[n,k,d]$ code is a linear code with length n, dimension k and minimal distance d [132]. Bars indicate complements of sets or binary vectors.

To save space we have sometimes written vectors in *hexadecimal*, using $0=0000,\cdots,\ 9=1001,\ A=1010,\cdots,$ $F=1111$, usually omitting leading zeros (so the vectors are right-justified). Superscripts (for example in Table XV) indicate the number of vectors in an orbit. Parentheses inside a vector (for example in Tables XII–XIV) indicate that all simultaneous cyclic shifts of the parenthesized sections are to be used. For example (110)(10) is an abbreviation for the six vectors 11010, 01101, 10110, 11001, 01110, 10101.

A *design* $(X,\mathcal{B})$ is a set X (of "points") together with a collection $\mathcal{B}$ of subsets of X (called "blocks"). A $t-(v,k,\lambda)$ design is a design in which $|X|=v$, all blocks contain exactly k points, and any t distinct points of X belong to exactly λ blocks ([14], [27], [31], [77], [96], [163], [176]). A *Steiner system* $S(t,k,v)$ is a $t-(v,k,1)$ design. A *balanced incomplete block design* is a $2-(v,k,\lambda)$ design. More generally an (r,λ)-*design* is a design in which each point belongs to exactly r blocks and each pair of points belong to exactly λ blocks (but the blocks need not all contain the same number of points). A *symmetric* design

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[1] An extreme case is the recent theorem of Lam *et al.* that there is no projective plane of order 10 [120]. This result, based on *thousands* of hours of computer time and as yet unchecked [141], implies $A(111,20,11) \leq 110$ (see [132, p. 528]).

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TABLE I-A
LOWER BOUNDS ON $A(n,4,w)$

n,w	3^{z3}	4^{z5}	5	6	7	8	9	10	11	12	13	14
6	4.^{a}	3	1	1	0	0	0	0	0	0	0	0
7	7.^{ss}	7	3	1	1	0	0	0	0	0	0	0
8	8.^{c}	14.^{ss}	8	4	1	1	0	0	0	0	0	0
9	12.^{s}	18.^{s}	18	12	4	1	1	0	0	0	0	0
10	13.^{s}	30.^{c}	36.^{s}	30	13	5	1	1	0	0	0	0
11	17.^{s}	35.^{pc}	66.^{s}	66	35	17	5	1	1	0	0	0
12	20.^{s}	51.^{p0}	80^{y}	132.^{ss}	80	51	20	6	1	1	0	0
13	26.^{c}	65.^{s}	123^{y}	166^{y}	166	123	65	26	6	1	1	0
14	28.^{s}	91.^{ss}	169^{g}	278^{ya}	325^{yd}	278	169	91	28	7	1	1
15	35.^{s}	105.^{s}	237^{yd}	389^{y}	585^{s}	585	389	237	105	35	7	1
16	37.^{s}	140.^{ss}	312^{yd}	615^{y}	836^{sd}	1170^{m}	836	615	312	140	37	8
17	44.^{s}	156^{ec}	424^{p0}	854^{p1}	1416^{s}	1770^{s}	1770	1416	854	424	156	44
18	48.^{s}	198.^{c}	518^{y}	1260^{p1}	2041^{p3}	3186^{s}	3540^{s}	3186	2041	1260	518	198
19	57.^{c}	228.^{s}	684^{s}	1596^{s}	3172^{p1}	4667^{p0}	6726^{s}	6726	4667	3172	1596	684
20	60.^{s}	285.^{p0}	874^{y}	2280^{g}	4213^{p0}	7730^{p0}	10039^{p0}	13452^{g}	10039	7730	4213	2280
21	70.^{s}	315.^{s}	1071^{s}	2856^{s}	6120^{p0}	10726^{p1}	16856^{p0}	20188^{p2}	20188	16856	10726	6120
22	73.^{s}	385.^{s}	1386.^{s}	3927^{s}	8211^{p0}	16354^{p1}	25570^{p3}	36381^{p2}	39688^{p3}	36381	25570	16354
23	83.^{s}	416^{p0}	1771.^{s}	5313.^{s}	11638^{s}	23276^{s}	40786^{p1}	57436^{p0}	73794^{p1}	73794	57436	40786
24	88.^{s}	498.^{p0}	1895$^{p\bullet}$	7084.^{ss}	15554^{p0}	34914^{g}	59262^{p0}	96496^{p0}	116914^{p0}	146552^{p0}	116914	96496
25	100.^{c}	550.^{s}	2334$^{p\bullet}$	7772^{p1}	21094^{p0}	46390^{p1}	88411^{p0}	140320^{p0}	194756^{p0}	227168^{p0}	227168	194756
26	104.^{c}	650.^{ss}	2670^{s}	10010^{p1}	26920^{p0}	65260^{p1}	128024^{p0}	218853^{p1}	315648^{p0}	394874^{p1}	424868^{p0}	394874
27	117.^{s}	702.^{s}	3276^{s}	12012^{s}	35510^{p1}	87709^{p0}	184420^{p1}	329076^{p0}	506444^{p1}	672148^{p0}	774565^{p1}	774565
28	121.^{z3}	819.^{ss}	3718^{p0}	15288^{g}	44747^{p0}	121403^{p0}	259703^{p0}	502068^{p0}	806303^{p0}	1154541^{p0}	1399597^{p0}	1520224^{p0}

TABLE I-B
LOWER BOUNDS ON $A(n,6,w)$

n,w	4^{z4}	5	6	7	8	9	10	11	12	13	14
8	2.	2	1	1	1	0	0	0	0	0	0
9	3.^{s}	3	3	1	1	1	0	0	0	0	0
10	5.^{a}	6.^{s}	5	3	1	1	1	0	0	0	0
11	6.^{s}	11.^{c}	11	6	3	1	1	1	0	0	0
12	9.^{s}	12.^{c}	22.^{hm}	12	9	4	1	1	1	0	0
13	13.^{c}	18.^{s}	26.^{c}	26	18	13	4	1	1	1	0
14	14.^{c}	28.^{s}	42.^{c}	42.^{hn}	42	28	14	4	1	1	1
15	15.^{c}	42.^{s}	70.^{s}	69^{h1}	69	70	42	15	5	1	1
16	20.^{s}	48.^{s}	112.^{t1}	109^{h1}	120^{t1}	109	112	48	20	5	1
17	20.	68.^{ss}	112	166^{h1}	184^{g}	184	166	112	68	20	5
18	22.^{zh}	68	132^{sb}	243^{h1}	260^{y}	304^{g}	260	243	132	68	22
19	25.^{z2}	76^{c}	172^{sb}	338^{sb}	408^{sb}	504^{g}	504	408	338	172	76
20	30.^{z8}	84^{c}	232^{t2}	462^{t2}	588^{t2}	832^{t2}	944^{t2}	832	588	462	232
21	31.^{s}	105^{c}	269^{h3}	570^{sb}	774^{y}	1184^{y}	1454^{y}	1454	1184	774	570
22	37.^{m}	132^{m}	319^{g}	759^{h2}	1139^{y}	1792^{y}	2182^{y}	2636^{t2}	2182	1792	1139
23	40.^{z4}	147^{t2}	399^{s}	969^{s}	1436^{y}	2271^{y}	2970^{y}	3585^{y}	3585	2970	2271
24	42.^{s}	168^{s}	532^{s}	1368^{s}	1882^{yd}	3041^{ya}	4200^{y}	5267^{y}	5616^{y}	5267	4200
25	50.^{s}	210.^{s}	700^{s}	1900^{s}	2590^{yd}	4127^{ya}	6036^{ya}	7960^{y}	9031^{ya}	9031	7960
26	52.^{c}	260.^{s}	910.^{s}	2600^{s}	3532^{yd}	5703^{y}	8695^{ya}	12037^{ya}	14836^{gs}	15977^{gs}	14836
27	54.^{c}	260	1170.^{s}	3510.^{s}	4786^{yd}	7727^{ya}	12368^{ya}	18096^{ya}	23879^{gs}	27553^{gs}	27553
28	63.^{ss}	272^{y}	1170	4680.^{ss}	6315^{yd}	10313^{ya}	17447^{ya}	28368^{gs}	40188^{gs}	49462^{gs}	52995^{gs}

(or square 2-design) is a 2-design with as many blocks as points.

We have included some codes that are close to the best presently known when they are easy to construct and the best code is not. These nonrecord codes are indicated by ‡.

II. THE TABLES OF $A(n,d,w)$ AND $A(n,d)$

We begin with the main tables, Tables I and II, which give lower bounds on $A(n,d,w)$ and $A(n,d)$. The rest of the paper is devoted to describing the codes in Table I (Table II being largely self-explanatory).

Most of the entries in Table I are new, either because we have improved the lower bound, or because we have found a more symmetric code or more compact definition than was known before. The notes to Table I describe the simplest construction we know for a code with the given parameters. We have usually not attempted to indicate the original discoverer of a code. For as mentioned in Section I our chief concern is to describe these codes explicitly. Those interested in the history of these codes may consult the extensive bibliography (see for example [11], [13], [23], [69], [103], [132], [177], [178]) and the Acknowledgment at the end of the paper. Table II contains one new entry, $A(25,10) \geq 151$.

TABLE I-C
LOWER BOUNDS ON $A(n,8,w)$

n,w	5	6	7	8	9	10	11	12	13	14
10	2.	2	1	1	1	1	0	0	0	0
11	2.	2	2	1	1	1	1	0	0	0
12	3.[j]	4.[d]	3	3	1	1	1	1	0	0
13	3.	4.	4	3	3	1	1	1	1	0
14	4.[s]	7.[d]	8.[s]	7	4	3	1	1	1	1
15	6.[a]	10.[s]	15.[c]	15	10	6	3	1	1	1
16	6.	16.[q2]	16.[c]	30.[hm]	16	16	6	4	1	1
17	7.[s]	17.[pc]	24.[ec]	34.[c]	34	24	17	7	4	1
18	9.[s]	21.[q3]	33[s]	46[s]	48[s]	46	33	21	9	4
19	12.[s]	28.[s]	52[s]	78[s]	88[s]	88	78	52	28	12
20	16.[s]	40.[s]	80.[s]	130[s]	160[s]	176[s]	160	130	80	40
21	21.[c]	56.[s]	120.[s]	210.[s]	280[s]	336[s]	336	280	210	120
22	21.	77.[s]	176.[s]	330.[s]	280	616[s]	672[s]	616	280	330
23	23.[c]	77	253.[s]	506.[s]	400[s]	616	1288[s]	1288	616	4[illegible]0
24	24.[c]	78[x2]	253	759.[ss]	640[t4]	960[t4]	1288	2576.[t4]	1288	96[illegible]
25	30.[s]	100.[s]	254[x2]	759	829[t5]	1248[ya]	1662[t5]	2576	2576	1662
26	30.	130.[ss]	257[y]	760[x2]	883[y]	1519[t5]	1988[ya]	3070[y]	3328[ya]	3070
27	30	130	278[y]	766[xy]	970[y]	1597[y]	2295[y]	3335[ya]	3923[y]	3923
28	33.[m]	130	296[y]	833[y]	1107[y]	1806[y•]	2756[ya]	4114[ya]	4805[ya]	5280[t6]

TABLE I-D
LOWER BOUNDS ON $A(n,10,w)$

n,w	6	7	8	9	10	11	12	13	14
12	2.	2	1	1	1	1	1	0	0
13	2.	2	2	1	1	1	1	1	0
14	2.	2.	2	2	1	1	1	1	1
15	3.[j]	3.[j]	3	3	3	1	1	1	1
16	3.	4.[j]	4.[j]	4	3	3	1	1	1
17	3.	5.[j]	6.[j]	6	5	3	3	1	1
18	4.[j]	6.[j]	9.[q2]	10.[s]	9	6	4	3	1
19	4.	8.[x]	12.[sb]	19.[c]	19	12	8	4	3
20	5.[s]	10.[q2]	17.[m]	20[c]	38.[hm]	20	17	10	5
21	7.[a]	13.[xh]	21.[c]	27[pc]	38	38	27	21	13
22	7.	16.[pc]	24[sd]	35[pc]	42[ec]	46[c]	42	35	24
23	8.[x2]	20[y]	33[pc]	45[pc]	54[pc]	63[pc]	63	54	45
24	9.[x2]	24[c]	38[pc]	56[c]	72[c]	90[pc]	96[c]	90	72
25	10.[s]	28[ec]	48[ec]	72[ec]	100[c]	125[c]	130[ec]	130	125
26	13.[q2]	28	54[pc]	84[pc]	130[c]	168[pc]	185[y]	191[y]	185
27	14.[q9]	36[q3]	66[pc]	111[c]	159[pc]	213[ya]	257[y]	283[ya]	283
28	16.[m]	37[q4]	78[pc]	132[pc]	195[yd]	280[ya]	356[ya]	414[ya]	435[yd]

Key to Table I

An entry followed by a period is known to be exact.

- *a* = From a trivial design or its dual (Section III).
- *c* = Cyclic code (Table XI).
- *cm* = Conference matrix code ((19) of Section III).
- *d* = Doubling ((2) of Section III).
- *d*1 = 2-(25, 9, 3) design (Section III).
- *d*2 = From (r,λ)-design (Section III).
- *ec* = Extended cyclic (or "cyclic with fixed point") code (Table XII).
- *g* = Group code—orbits under a group with more than one generator (Table XV).
- *gf* = Group code plus extra vectors (Table XV).
- *gp* = Group code followed by polishing—group code need not be subcode (Table XV).
- *gs* = From S_2-sets (Theorem 16 and Table V).
- *hm* = Hadamard matrix code (Theorem 10).
- *hn* = From Hadamard matrix of order 12 (Section III).
- *h*1 = From Theorem 20.
- *h*2 = From Theorem 21.
- *h*3 = Hämäläinen (see remark following Theorem 21).
- *j* = Juxtaposing ((1) of Section III). Details are left to the reader.
- *m* = Miscellaneous construction (Section XI).
- *p*0 = Partitioning construction with $n_1=[n/2]$, $\epsilon=0$ (Section VI).
- *p*1 = Partitioning construction with $n_1=[n/2]$, $\epsilon=1$ (Section VI).
- *p*2 = Partitioning construction with $n_1=[n/2]-1$, $\epsilon=0$ (Section VI).
- *p*3 = Partitioning construction with $n_1=[n/2]-1$, $\epsilon=1$ (Section VI).
- *pc* = Orbits under a single permutation (Table XIV).
- *qi* = Quasi-cyclic code, for $2\le i\le 9$—fixed by a permutation containing i cycles of length n/i (Table XIII).
- *s* = Section of code below or diagonally down to right, obtained from (5) of Section III.
- *sb* = Section of code below, obtained by direct examination of the code (Section III).

TABLE I-E
LOWER BOUNDS ON $A(n,12,w)$

n,w	7	8	9	10	11	12	13	14
14	2.	2	1	1	1	1	1	1
15	2.	2	2	1	1	1	1	1
16	2.	2.	2	2	1	1	1	1
17	2.	2.	2	2	2	1	1	1
18	3.^{j}	3.^{d}	4.^{j}	3	3	3	1	1
19	3.	3.	4.	4	3	3	3	1
20	3.	5.^{d}	5.^{j}	6.^{a}	5	5	3	3
21	3.	5.	7.^{j}	7.^{j}	7	7	5	3
22	4.^{j}	6.^{d}	8.^{pc}	11.^{d}	12.^{s}	11	8	6
23	4.	6.	10.^{s}	16.^{xh}	23.^{c}	23	16	10
24	4.	9.^{d}	16.^{s}	24.^{c}	24^{c}	46.^{hm}	24	24
25	5.^{j}	10.^{q5}	25.^{d1}	28^{pc}	36^{ec}	50^{cm}	50	36
26	5.	13.^{d}	26^{c}	30^{pc}	39^{q2}	54^{z9}	58^{z9}	54
27	6.^{s}	15.^{q9}	39.^{z9}	39^{ec}	54^{c}	82^{t7}	81^{t7}	81
28	8.^{a}	19.^{pc}	39	48^{g}	63^{pc}	84^{c}	96^{pc}	106^{m}

TABLE I-F
LOWER BOUNDS ON $A(n,14,w)$

n,w	8	9	10	11	12	13	14
16	2.	2	1	1	1	1	1
17	2.	2	2	1	1	1	1
18	2.	2.	2	2	1	1	1
19	2.	2.	2	2	2	1	1
20	2.	2.	2.	2	2	2	1
21	3.^{j}	3.^{j}	3.^{j}	3	3	3	3
22	3.	3.	4.^{j}	4.^{j}	4	3	3
23	3.	3.	4.	4.	4	4	3
24	3.	4.^{j}	5.^{j}	6.^{j}	6.^{j}	6	5
25	3.	5.^{j}	6.^{j}	7.^{j}	8.^{j}	8	7
26	4.^{j}	6.^{j}	8.^{d2}	10.^{j}	13.^{q2}	14.^{s}	13
27	4.	6.	9.^{j}	13.^{s}	19^{q9}	27.^{s}	27
28	4.	7.^{j}	11.^{x}	21.^{q4}	28^{c}	28^{c}	54.^{hm}

sd = Section of code diagonally down to right, obtained by direct examination of the code (Section III).

sf = Section of code below or diagonally down to right, followed by addition of extra vectors (Section XI).

sp = Section of code below or diagonally down to right, followed by polishing (Section XI).

ss = From Steiner systems $S(2,3,7)$, $S(3,4,8)$, $S(5,6,12)$, $S(3,4,14)$, $S(3,4,16)$, $S(5,6,24)$, $S(3,4,26)$, $S(3,4,28)$ for $d=4$; $S(3,5,17)$, $S(2,4,28)$, $S(5,7,28)$ for $d=6$; $S(5,8,24)$, $S(3,6,26)$ for $d=8$ (Table IV).

*t*1 = From translate of Nordstrom–Robinson code (Section IX).

*t*2 = Adding tails to translates of Nordstrom–Robinson code (Section IX).

*t*4 = From translate of Golay code (Section IX).

*t*5 = Adding tails to translates of Golay code (Section IX).

*t*6 = From translate of Karlin's [27, 14, 7] code (Section IX).

*t*7 = From translate of [26, 7, 11] code (Section IX).

x = Lexicographic code (Section VIII).

xh = Lexicode with seed (Table VIII).

xy = Lexicode with seed (Table XVI).

*x*2 = Complement of lexicode with sum constraint (Table VII).

y = No known structure (Table XVI).

TABLE I-G
VALUES OF $A(n,16,w)$

n,w	9	10	11	12	13	14
18	2.	2	1	1	1	1
19	2.	2	2	1	1	1
20	2.	2.	2	2	1	1
21	2.	2.	2	2	2	1
22	2.	2.	2.	2	2	2
23	2.	2.	2.	2	2	2
24	3.^{j}	3.^{d}	3.^{j}	4.^{d}	3	3
25	3.	3.	3.	4.	4	3
26	3.	3.	4.^{j}	4.	4.	4
27	3.	3.	5.^{j}	5.^{j}	6.^{j}	6
28	3.	4.^{d}	5.	7.^{d}	7.^{j}	8.^{d}

TABLE I-H
VALUES OF $A(n,18,w)$

n,w	10	11	12	13	14
20	2.	2	1	1	1
21	2.	2	2	1	1
22	2.	2.	2	2	1
23	2.	2.	2	2	2
24	2.	2.	2.	2	2
25	2.	2.	2.	2	2
26	2.	2.	2.	2.	2
27	3.^{j}	3.^{j}	3.^{j}	3.^{j}	3
28	3.	3.	3.	4.^{j}	4.^{j}

ya = Obtained by extending the code above it in the table; no other known structure.

yd = Obtained by extending the code diagonally above it to left; no other known structure.

*z*2 = [170].

*z*3 = Theorem 4.

*z*4 = Theorem 6.

*z*5 = Theorem 5.

*z*8 = [22].

*z*9 = [95] (see Table XVI).

Key to Table II

Unmarked entries are either trivial or are obtained by shortening the code below.

An entry followed by a period is known to be exact.

1 = Extended Hamming code ([132], p. 23).

2 = Conference matrix code ([132], p. 585, [165]).

3 = Best ([12], [45], p. 140).

4 = From $S(5,6,12)$, 6 disjoint words of weight 2 and complements ([45], p. 139, [132], p. 585).

5 = Romanov [155]—see Section VI.

6 = From Hamming code over GF(5) [79].

7 = From the $u|u+v$ construction ([132], p. 76, [166]).

8 = Hadamard matrix code ([132], p. 49).

8*a* = "Nadler" code ([13], [129], [132], pp. 75, 79).

9 = Nordstrom–Robinson code (Section IX, [5], [132], p. 73).

10 = Nonlinear code from Construction X ([132], p. 583, [164], p. 505).

11 = From Construction X4 ([132], p. 585, Example 7, [164], p. 507).

12 = Wagner [179].

TABLE II
LOWER BOUNDS ON $A(n,d)$

n,d	4	6	8	10	12	14	16	18	20
5	2.	1.	1.	1.	1.	1.	1.	1.	1.
6	4.	2.	1.	1.	1.	1.	1.	1.	1.
7	8.	2.	1.	1.	1.	1.	1.	1.	1.
8	16.[1]	2.	2.	1.	1.	1.	1.	1.	1.
9	20.[2]	4.	2.	1.	1.	1.	1.	1.	1.
10	40.[3]	6.	2.	2.	1.	1.	1.	1.	1.
11	72	12.	2.	2.	1.	1.	1.	1.	1.
12	144[4]	24.[8]	4.	2.	2.	1.	1.	1.	1.
13	256.	32.[8a]	4.	2.	2.	1.	1.	1.	1.
14	512.	64.	8.	2.	2.	2.	1.	1.	1.
15	1024.	128.	16.	4.	2.	2.	1.	1.	1.
16	2048.[1]	256.[9]	32.[14]	4.	2.	2.	2.	1.	1.
17	2720[5]	256	36[2]	6.[17]	2.	2.	2.	1.	1.
18	5248	512	64	10.	4.	2.	2.	2.	1.
19	10496[6]	1024	128	20.	4.	2.	2.	2.	1.
20	20480[7]	2048[10]	256	40.[8]	6.[17]	2.	2.	2.	2.
21	36864	2560[11]	512.	42[18]	8.[17]	4.	2.	2.	2.
22	73728	4096	1024.	48[17]	12.	4.	2.	2.	2.
23	147456	8192	2048.	68[19]	24.	4.	2.	2.	2.
24	294912[7]	16384[12]	4096.[15]	128[20]	48.[8]	6.[17]	4.	2.	2.
25	524288	16384	4096	151[21]	52[2]	8.[17]	4.	2.	2.
26	1048576	32768	4096	256	64	14.	4.	2.	2.
27	2097152	65536	8192	512	128[23]	28.	6.[17]	4.	2.
28	4194304[1]	131072[13]	16384[16]	1024[22]	128	56.[8]	8.[17]	4.	2.

13 = Shortened nonprimitive BCH code of length 32 ([132], p. 586).
14 = Reed–Muller code ([132], Chap. 13).
15 = Golay code ([45], Chaps. 3, 11, [132], Chap. 20).
16 = Self-dual double circulant code ([45], p. 189, [132], p. 509, [148])—see (53).
17 = From Hadamard matrices using Levenshtein's construction ([122], [132], p. 50).
18 = Extended quasi-cyclic code [104].
19 = Extended cyclic code [105] (see Table XI).
20 = Hashim–Pozdniakov linear code [88].
21 = Cyclic code (see Table XI).
22 = Piret [147].
23 = Linear code ((51) of Section IX, [87], [89], [132], p. 593).

We see from Table I that the exact value of $A(n,d,w)$ is now known for all lengths $n \le 11$ (the first undetermined value being $80 \le A(12,4,5) \le 84$). Similarly, from Table II, $A(n,d)$ (for d even) is known exactly for $n \le 10$, the first undetermined value being $72 \le A(11,4) \le 79$ (the upper bound is from [12]).

Of course there is no theoretical difficulty in computing $A(n,d)$ or $A(n,d,w)$. One "simply" forms the graph with 2^n or $\binom{n}{w}$ vertices, corresponding to all possible codewords, joins two vertices by an edge if their Hamming distance is at least d, and finds the largest clique. Although several new algorithms have recently been proposed for clique-finding (see [6], [7], [70], [71], [140]), the unsolved problems in Tables I and II appear to be beyond their range. Similarly, finding the largest code invariant under a given permutation group (see Section X) requires finding the largest clique in a graph with weights attached to the vertices.

III. Johnson and Related Bounds

S. M. Johnson [99]–[102] obtained a number of upper bounds on $A(n,d,w)$, given in Theorems 2, 8 and (7), (16), (18). The cases when equality holds in these bounds (see Conjecture 3 and Theorems 4–7, 9–13) are of particular interest, because of their connection with Steiner systems, block designs and Hadamard matrices. We also mention some related lower bounds obtained from conference matrices.

Theorem 1 (Trivial values):

a) If d is odd, $A(n,d,w) = A(n,d+1,w)$.
b) $A(n,d,w) = A(n,d,n-w)$.
c) $A(n,d,w) = 1$ if $2w < d$.
d) If $d = 2w$ then $A(n,d,w) = \lfloor n/w \rfloor$.
e) $A(n,2,w) = \binom{n}{w}$.

Remark: Let $N(a,d,w)$ denote the inverse of $A(n,d,w)$, that is, the smallest n for which one can find a words of weight w at mutual distance d, where d is even. Then one can show that

$$N(0,d,w) = 0,$$
$$N(1,d,w) = w,$$

and if $w \ge d/2$ then

$$N(2,d,w) = w + d/2,$$
$$N(3,d,w) = \max\{3d/2, w + d/2\},$$
$$N(4,d,w) = \max\{3d - 2w, \lceil d + 2w/3 \rceil, w + d/2\}$$

(if $w < d/2$ these quantities do not exist).

By *juxtaposing* two codes—placing them side by side—we obtain

$$A(n_1+n_2, d_1+d_2, w_1+w_2) \ge \min\{A(n_1,d_1,w_1), A(n_2,d_2,w_2)\}. \quad (1)$$

In particular we have the *doubling construction*:

$$A(2n,2d,2w) \geq A(n,d,w). \quad (2)$$

Lower bounds in Table I obtained from (1), (2) are indicated by "j" and "d" respectively.

Theorem 2 (Johnson; [132], p. 527, [176], p. 98):

$$A(n,d,w) \leq \left\lfloor \frac{n}{w} A(n-1,d,w-1) \right\rfloor, \quad (3a)$$

$$A(n,d,w) \leq \left\lfloor \frac{n}{n-w} A(n-1,d,w) \right\rfloor. \quad (3b)$$

In particular

$$A(n,d,w) \leq A(n-1,d,w-1) + A(n-1,d,w). \quad (4)$$

We also use the contrapositives to (3a), (3b): if there exists a code showing that $A(n,d,w) \geq M$, then

$$A(n-1,d,w-1) \geq \left\lceil \frac{wM}{n} \right\rceil, \quad A(n-1,d,w) \geq \left\lceil \frac{n-w}{n} M \right\rceil. \quad (5)$$

Lower bounds obtained from (5) are indicated by "s" (for "Section") in Table I. In some cases these inequalities can be improved by examining the code and finding a column with an exceptionally large number of 0's or 1's (and deleting that column). Bounds obtained in this way are indicated by "*sb*" or "*sd*" in Table I.

The *first Johnson bound* $J_1(n,d,w)$ is defined to be the smallest upper bound on $A(n,d,w)$ that is obtained by repeatedly applying (3a) and (3b) until Theorem 1 can be used. For example

$$J_1(n,4,3) = \begin{cases} \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor, & \text{if } n \not\equiv 5 \pmod 6 \\ \left\lfloor \frac{n}{3} \left\lfloor \frac{n-1}{2} \right\rfloor \right\rfloor - 1, & \text{if } n \equiv 5 \pmod 6. \end{cases} \quad (6)$$

Clearly

$$A(n,d,w) \leq J_1(n,d,w), \quad (7)$$

and there are reasons for believing that this bound may be tight when n is sufficiently large.

Conjecture 3: For d,w fixed,

$$A(n,d,w) = J_1(n,d,w) \quad (8)$$

for all sufficiently large n. If true this would be a remarkably strong result (it would imply for example by Theorem 7 that Steiner systems $S(t,k,v)$ exist for all t and k provided v is sufficiently large and satisfies the obvious congruences). Rödl [154] has shown that $A(n,d,w)/J(n,d,w)$ approaches 1 as $n \to \infty$ for any fixed d and w. An asymptotic result for certain values of w and d growing with n was given in [186]. The conjecture is known to be true in a few cases, as we shall now see.

Theorem 4 (Kirkman [109], Schönheim [157]; see also [77], p. 237, [106], [153], [169]):

$A(n,4,3) = J_1(n,4,3)$ (see (6)) holds for all n.

Theorem 5 (Brouwer [21], Hanani [81], Kalbfleisch and Stanton [106], Schönheim [157]): For $n \not\equiv 5 \pmod 6$,

$$A(n,4,4) = J_1(n,4,4),$$

the values of this function being

$$\frac{n(n-1)(n-2)}{24}, \quad \text{if } n \equiv 2 \text{ or } 4 \pmod 6, \quad (9)$$

$$\frac{n(n-1)(n-3)}{24}, \quad \text{if } n \equiv 1 \text{ or } 3 \pmod 6, \quad (10)$$

$$\frac{n(n^2-3n-6)}{24}, \quad \text{if } n \equiv 0 \pmod 6. \quad (11)$$

For $n \equiv 5 \pmod 6$ we have $A(5,4,4) = 1$, $A(11,4,4) = 35$ ([12], [13], [45, p. 141]), but for larger n the values are unknown. See also [86].

Theorem 6 (Brouwer [22], [29]):

$$A(n,6,4) = J_1(n,6,4) \quad (12)$$

holds for all n except 8, 9, 10, 11, 17, 19 (for these values see Table I-B).

One special case of equality in the first Johnson bound is particularly important.

Theorem 7 (Schönheim [157]; see also [132], p. 528, [160], [176], p. 100):

$$A(n,2\delta,w) = \frac{n(n-1)\cdots(n-w+\delta)}{w(w-1)\cdots\delta} \quad (13)$$

if and only if a Steiner system $S(w-\delta+1,w,n)$ exists.

In this case equality also holds in (7); the codewords are the blocks of the corresponding Steiner system. The codes obtained in this way are discussed in Section IV.

Theorem 7 enables us to write down immediately the parameters of the codes corresponding to the blocks of a Steiner system. The blocks of an arbitrary $2-(v,k,\lambda)$ design form a code showing that $A(v,d,k) \geq b$, but in general d is not determined by the other parameters. However, the blocks of the *dual* or *transpose* design (obtained by interchanging points and blocks) form a code showing that

$$A(b,2(r-\lambda),r) \geq v. \quad (14)$$

If the design is symmetric these two codes have the same parameters. Theorem 9 describes a case where equality holds both in (14) and in another of Johnson's bounds.

Theorem 8 (Johnson [99]; [132, p. 526]): Let $A(n,2\delta,w) = M$, and write $wM = an + b$, $0 \leq b < n$. Then (by considering the sum of the inner products of all pairs of codewords)

$$(n-b)a(a-1) + ba(a+1) \leq (w-\delta)M(M-1). \quad (15)$$

Theorem 8 rules out certain combinations of n,d,w,M (see [132, p. 526] for an example). The *second Johnson bound* $J_2(n,d,w)$ is defined to be the largest value of M permitted by Theorem 8 (possibly infinity). Clearly

$$A(n,d,w) \leq J_2(n,d,w). \quad (16)$$

The proof of Theorem 8 shows that if equality holds in (15) then every pair of distinct codewords has inner product $w-\delta$, and also that the $M\times n$ array formed by the codewords contains $n-b$ columns of weight a and b columns of weight $a+1$. This is the dual of an (r,λ)-design with $r=w$, $\lambda=w-\delta$ in the notation of Section I. We mention three examples of such designs.

1) A trivial example occurs when

$$n=\binom{k}{a},\ M=k,\qquad b=0,\qquad w=\binom{k-1}{a-1},\quad (17)$$

and the columns of the array consist of all possible binary vectors of weight a. Such codes are denoted by "a" in Table I.

2) In the special case $a=2$ (when $wM<3n$), equality holds in (15) if and only if there is a 1-design consisting of b triples from M points, such that no pair is covered more than $w-\delta$ times. For example $A(26,14,9)=6$ can be obtained in this way; we omit the details.

3) To obtain $A(26,14,10)=8$, by the previous remarks we must find 24 triples and 2 quadruples from an 8-set. If the 8-set is $\{a,b,c,d,A,B,C,D\}$ we may use the triples $\{i,j,K\}$ and $\{i,J,K\}$ and the quadruples $\{a,b,c,d\}$, $\{A,B,C,D\}$.

We shall discuss equality in (16) at the end of this section. We next discuss a weaker version of Theorem 8, also due to Johnson (obtained by ignoring the condition that certain variables must be integers), which states that

$$A(n,2\delta,w)\le\frac{\delta n}{w^2-wn+\delta n},\qquad (18)$$

provided the denominator is positive ([99]; [132], p. 525, [176], p. 97).

Theorem 9 (Semakov and Zinoviev [160]; [176], p. 99): Equality holds in (18) if and only if there exists a 2-design with parameters $b=n$, $r=w$, $\lambda=w-\delta$, $v=\delta n/(w^2-wn+\delta n)$, $k=\delta w/(w^2-wn+\delta n)$.

In Theorem 9 the codewords are the blocks of the dual design; equality then holds in (14). There are lists of small block designs in [77], [133]. The entry $A(25,12,9)=25$ in Table I for example is obtained from Theorem 9 using a symmetric $2-(25,9,3)$ design. There are in fact 78 inequivalent designs with these parameters [58], [133], and so exactly 78 inequivalent codes showing $A(25,12,9)=25$. One example is given in Table XIV.

Hadamard matrices also determine some values of $A(n,d,w)$.

Theorem 10 ([160]; see also [13], Theorem 15, [132], p. 528): A Hadamard matrix H_n of order $n\ge 1$ exists if and only if

$$A\left(n,\frac{n}{2},\frac{n}{2}\right)=2n-2.$$

The code is constructed from the rows of H_n and $-H_n$ by making the entries of the first row of H_n equal to $+1$, then changing $+1$ to 0 and -1 to 1 in every row. The optimality of these codes follows by applying (3) once and then (18).

A modification of this construction shows that $A(14,6,7)\ge 42$. Take a Hadamard matrix of order 12, with first row and column containing $+1$'s, replace $+1$'s by 0's and -1's by 1's, omit the first (zero) row, and label the remaining rows $\rho_0,\cdots,\rho_{10}$. Then the code

$$\begin{array}{lll} \rho_0 & 0\,1 & \\ \overline{\rho_0} & 0\,1 & \\ \rho_i & 1\,0 & (1\le i\le 10)\\ \overline{\rho_i} & 1\,0 & (1\le i\le 10)\\ \rho_0+\rho_i & 0\,1 & (1\le i\le 10)\\ \overline{\rho_0+\rho_i} & 0\,1 & (1\le i\le 10)\end{array}$$

shows that $A(14,6,7)\ge 42$. This idea appears to succeed only for a Hadamard matrix of order 12.[2]

Similarly if a conference matrix ([132], p. 55, [165]) of order $n\equiv 2 \pmod 4$ exists then

$$A\left(n-1,\frac{n-2}{2},\frac{n-2}{2}\right)\ge 2n-2.\qquad (19)$$

Equality holds in (19) for $n=6,10,14,18$; we would like to know if it holds in general.

Honkala *et al.* [95] show that if a conference matrix of order n exists then

$$A(2n,n,n-1)\ge 2n,\qquad (20)$$

if n is a multiple of 4 then

$$A(2n+1,n,n-1)\ge 2n+A\left(n,\frac{n}{2},\frac{n-2}{2}\right),\qquad (21)$$

and if n is a multiple of 4 and a conference matrix of order either $n/2$ or $n+2$ exists then

$$A(2n+1,n,n-1)\ge 3n.\qquad (22)$$

For example (22) gives an alternative proof that $A(25,12,11)\ge 36$ (cf. Table IV).

Finally we return to (16). In 1985 Honkala [92] obtained a considerable generalization of Theorem 10 by showing that equality holds in (16) over a wide range of values. The proof is constructive, by juxtaposing (see (1)) appropriate combinations of Hadamard matrices of orders up to n.

Theorem 11 (Honkala [92]): Provided Hadamard matrices of all orders $4k\le n$ exist,

$$A(n,d,w)=J_2(n,d,w)$$

holds whenever

$$n-d\le w\le d.\qquad (23)$$

In particular, if (23) holds and $n<2d$, then

$$A(n,d,w)=\begin{cases}2u, & \text{if } i-v\le 2j\le i+v,\\ 2u-1, & \text{otherwise,}\end{cases}$$

where $i=2d-n$, $j=d-w$ and $d=ui+v$ with $0\le v<i$.

[2] A fact which ultimately depends on the existence of the outer automorphism of the symmetric group S_6.

TABLE III
SOURCES FOR PROOF THAT CERTAIN ENTRIES IN TABLE I MARKED WITH PERIOD ARE EXACT*

Value	Reference
$A(12,6,5)=12$	[132], p. 530.
$A(13,6,5)=18$	[132], p. 531.
$A(14,6,7)=42$	Exhaustive search (see Section XII).
$A(17,6,4)=20$	[20].
$A(19,6,4)=25$	[170].
$A(16,8,7)=16$	Exhaustive search (see Section XII).
$A(17,8,7)=24$	Exhaustive search (see Section XII).
$A(17,8,8)=34$	[149].
$A(22,8,5)=21$	By the Bose-Connor theorem ([18], [96]) a square divisible design $GD(5,1,2;11\times 2)$ does not exist.
$A(24,8,12)=2576$	Linear programming [13].
$A(26,8,5)=30$	[25].
$A(20,10,8)=17$	Exhaustive search (see Section XII).
$A(21,10,7)=13$	From Theorem 8, if $A(21,10,7)\geq 14$ then in fact $A(21,10,7)=15$; but (see [77]) no 2-(15,5,2) design exists.
$A(22,10,7)=16$	Exhaustive search (see Section XII).
$A(23,10,7)=21$	Exhaustive search (see Section XII).
$A(28,12,8)=19$	From Theorem 8, if $A(28,12,8)\geq 20$ then in fact $A(28,12,8)=21$; but (see [77]) no 2-(21,6,2) design exists.

*Exactness of entries not listed here follows from the Johnson bounds (3), (7), (16).

The example $A(26,14,10)=8$ previously mentioned shows that equality may also hold in (16) outside the "Honkala region" (23).

Most proofs that the entries in Table I followed by a period are exact can be obtained from the Johnson bounds ((3), (7), (16)). Sources for proofs that the other entries are exact are given in Table III.

IV. STEINER SYSTEMS

In this section we give a highly compressed survey of Steiner systems (defined in Section I). Because some constructions are available only in obscure sources, and because in Section VI we shall require not just a single Steiner system but as many disjoint ones as possible, Table IV contains a number of explicit constructions for small Steiner systems.

As we saw in the previous section (Theorem 7), if a Steiner system $S(t,k,v)$ exists, it contains $b=\binom{v}{t}/\binom{k}{t}$ blocks, and then $A(v,2k-2t+2,k)=b$. If $S(t,k,v)$ exists then so does the *contracted* (or *derived*) design $S(t-1,k-1,v-1)$, formed from all the blocks containing (say) the last point.

Steiner triple systems $S(2,3,v)$ exist if and only if $v\equiv 1$ or 3 (mod 6) (Kirkman [109]; see Theorem 4). *Steiner quadruple systems* $S(3,4,v)$ exist if and only if $v\equiv 2$ or 4 (mod 6) (Hanani [81]; see Theorem 5).

Theorem 12 (Hanani [82], [83]): $S(2,4,v)$ exists if and only if $v\geq 4$ and $v\equiv 1$ or 4 (mod 12); in these cases

$$A(v,6,4)=v(v-1)/12. \tag{24}$$

$S(2,5,v)$ exists if and only if $v\geq 5$ and $v\equiv 1$ or 5 (mod 20); in these cases

$$A(v,8,5)=v(v-1)/20. \tag{25}$$

Theorem 13 (See [53], [52], Chap. 6; [84]): For any prime power q and any nonnegative integer d there exist Steiner systems $S(2,q,q^d)$ (from the lines in the affine geometry $AG(d,q)$), $S(2,q+1,q^d+q^{d-1}+\cdots+q+1)$ (from the lines in the projective geometry $PG(d,q)$), $S(2,q+1,q^3+1)$ (from a unital in $PG(2,q^2)$), $S(2,2^a,2^{a+b}+2^a-2^b)$ for $a\leq b$ (from a complete 2^a-arc in $PG(2,2^b)$), $S(3,q+1,q^2+1)$ (from an elliptic quadric in $PG(3,q)$), or more generally $S(3,q+1,q^d+1)$ (from subfield sublines of a projective line). Thus

$$A(q^d,2q-2,q)=\frac{q^{d-1}(q^d-1)}{q-1}, \tag{26}$$

$$A(q^d+q^{d-1}+\cdots+q+1,2q,q+1)=\frac{(q^{d+1}-1)(q^d-1)}{(q^2-1)(q-1)}, \tag{27}$$

$$A(q^3+1,2q,q+1)=q^2(q^2-q+1), \tag{28}$$

$$A(2^{a+b}+2^a-2^b,2^{a+1}-2,2^a)=(2^b+1)(2^b-2^{b-a}+1), \tag{29}$$

$$A(q^d+1,2q-2,q+1)=\frac{q^{d-1}(q^{2d}-1)}{q^2-1}. \tag{30}$$

The only Steiner systems presently known with $t\geq 4$ are $S(5,6,12)$, $S(5,8,24)$ (Mathieu, Witt [182], [183], [45]), $S(5,6,24)$, $S(5,7,28)$, $S(5,6,48)$, $S(5,6,84)$ (Denniston [56]), and $S(5,6,72)$ (Mills [138]); and their contractions with $t=4$. All these 5-designs are invariant under the group $PSL(2,v-1)$ for the appropriate value of v.

Table IV lists the known Steiner systems (or $t-(v,k,1)$ designs) with $v\leq 28$ and $2\leq t<k$; b denotes the number of blocks. For each design we give either a construction or a reference, as well as information about the number of inequivalent designs (up to permutation of coordinates) and the number of disjoint designs. Some designs which are contractions of others have been omitted.

The Steiner triple systems on at most 21 points are treated in great detail in [156]. For further information about Steiner systems see [1], [3], [45], [52], [61], [62], [84], [96], [117], [118], [121], [127], [128], [139], [152], [168], [182], [183].

V. GRAHAM–SLOANE TYPE BOUNDS

We next discuss lower bounds on $A(n,d,w)$, roughly in order of increasing complexity. The bounds described in this section require very little computation (and correspondingly produce the weakest results). For $d\geq 6$ we give several different bounds, all roughly comparable (although only codes from Theorem 16 are needed for the present version of Table I).

TABLE IV
STEINER SYSTEMS

$S(2,3,7) = PG(2,2)$; $b = 7$. Cyclic: {(1011000)}; unique. Exactly two disjoint designs exist, the other being {(1101000)}.

$S(3,4,8) = AG(3,2)$; $b = 14$. Extended cyclic: {(1011000)1, (0100111)0}; unique. Exactly two disjoint designs exist, the other being {(1101000)1, (0010111)0} [32].

$S(2,3,9) = AG(2,3)$; $b = 12$. It is also the unital in $PG(2,4)$; unique. In each of the following matrices, the 12 triples formed by the rows, columns and six generalized diagonals form an $S(2,3,9)$. The 7 matrices together yield 7 disjoint designs, partitioning the set of all $\binom{9}{3}$ triples [109].

124	128	125	129	123	126	127
378	943	983	743	469	357	346.
956	765	476	586	785	489	598

There are exactly two inequivalent partitions [9].

$S(3,4,10)$; $b = 30$. Cyclic; unique. Exactly five disjoint designs exist, in a unique way [115]. For example, we may take the five designs defined by the columns of the following array. The six hexadecimal vectors in a column together with their images under the permutation $(1,2,3,4,5)(6,7,8,9,10)$ form an $S(3,4,10)$; the five columns give five disjoint designs (cf. Section X).

27	2*B*	2*D*	2*E*	36
3*A*	3*C*	33	35	39
69	65	74	72	6*A*
*B*1	*A*6	*B*8	*A*3	*AC*
*E*4	*F*0	*E*2	*E*8	*E*1
170	162	168	161	164

$S(4,5,11)$; $b = 66$. Unique. Exactly two disjoint designs exist, in a unique way [115]. For example, the designs $\{7A0^{55}, 712^{11}\}$ and $\{782^{55}, 748^{11}\}$ defined by the group of order 55 generated by $(1,2,3,\cdots,11)$ and $(2,5,6,10,4)(3,9,11,8,7)$ (cf. Table XV).

$S(5,6,12)$; $b = 132$. Unique. Exactly two disjoint designs exist, in a unique way [115]. An $S(5,6,12)$ may be obtained from an $S(4,5,11)$ by appending 1 and adjoining the complements of all blocks. Two disjoint $S(5,6,12)$'s arise from the two previous disjoint $S(5,6,11)$'s. Alternatively, two disjoint designs arise from the supports of the codewords of weight 6 in the two extended ternary quadratic residue codes of length 12 whose zeros are the residues and nonresidues modulo 11 respectively [2], [3].

$S(2,3,13)$; $b = 26$. There are exactly two inequivalent designs ([77], p. 237), one of which is cyclic (Table XI). Denniston [54] (see also [116]) partitioned the set of all $\binom{13}{3}$ triples into 11 disjoint designs.

$S(2,4,13) = PG(2,3)$; $b = 13$. Cyclic; unique. Chouinard [36] partitioned the set of all $\binom{13}{4}$ 4-sets into 55 disjoint designs.

$S(3,4,14)$; $b = 91$. There are exactly four inequivalent designs, one of which is given in Table XV [137]. A set of four disjoint designs (not hitherto known to exist) is obtained by applying the permutations $(1,2,4,7,10,13,12,14,3,6,8,5,9,11)$, $(1,4,8,10,2,11,3,5,7,6,12,13,9)$ and $(1,6,9,14,8,11,12,10,5,4,3)(2,7,13)$ to the design in Table XV. See also [116].

$S(2,3,15)$; $b = 35$. There are exactly 80 inequivalent designs [38], [78], one of which is $PG(3,2)$. Denniston [54], [55] partitioned the set of all $\binom{15}{3}$ triples into 13 disjoint designs.

$S(2,4,16) = AG(2,4)$; $b = 20$. The weight 4 words in a translate of the Nordstrom–Robinson code (see Table X); unique. There exists an $S(3,4,16)$ that is tiled by 7 copies of $S(2,4,16)$.

$S(3,4,16)$; $b = 140$. For example, the planes in AG(4,2), the weight 4 words in the [16,11,4] Hamming code. There are at least 31301 inequivalent designs [126], [127]. At least 8 pairwise disjoint designs exist (see [112], [124], [125] and the Appendix).

$S(3,5,17)$; $b = 68$. Unique (see Theorem 13 and Table XV). At least two disjoint designs exist [136], [185].

$S(3,4,20)$; $b = 285$. From the partitioning construction (Section VI). There are at least 10^{17} inequivalent designs [126], and exactly 29 cyclic designs [142]. At least 15 disjoint designs exist [67].

$S(3,4,22)$; $b = 385$. There are exactly 21 inequivalent cyclic designs, and exactly 5 disjoint cyclic designs [59]. At least 11 disjoint designs exist [144], [145].

$S(5,6,24)$; $b = 7084$. See Table XV. At least three inequivalent designs exist [56], [74].

$S(5,8,24)$; $b = 759$. The weight 8 words in the [24,12,8] Golay code. Unique [182], [183], [132], [45]. At least 9 disjoint designs exist [114].

$S(2,5,25) = AG(2,5)$; $b = 30$. Unique.

$S(3,4,26)$; $b = 210$. See Theorem 5 or [81]. At least 13 disjoint designs exist [146].

$S(3,6,26)$; $b = 130$. Unique (see Theorem 13, Table XI, [33]).

$S(2,3,27)$; $b = 117$. Many examples, one of which is $AG(3,3)$.

$S(2,4,28)$; $b = 63$. See [52], [84], Theorem 13. At least 154 inequivalent designs exist [24], [26].

$S(3,4,28)$; $b = 819$ [81]. At least 18 disjoint designs exist [66].

$S(5,7,28)$; $b = 4680$. See [56] or Table XV.

$S(2,6,31) = PG(2,5)$; $b = 31$. Cyclic: $\{0,1,3,8,12,18\} \bmod 31$; unique.

The prototype of these bounds is the following.

Theorem 14 ([72]):

$$A(n,4,w) \geq \frac{1}{n}\binom{n}{w}. \tag{31}$$

This is established by defining a map f from vectors $a = (a_0, a_1, \cdots, a_{n-1}) \in \mathbb{F}_2^n$ to $\mathbb{Z}_n$ by

$$f(a) = \sum_{i=0}^{n-1} i a_i \pmod{n}. \tag{32}$$

Then it is easy to see that the sets $C_0^{(w)}, C_1^{(w)}, \cdots, C_{n-1}^{(w)}$, where

$$C_k^{(w)} = \{a \in \mathbb{F}_2^n : f(a) = k, \quad wt(a) = w\}, \tag{33}$$

form a partition of the set of all $\binom{n}{w}$ binary vectors of weight w into n disjoint codes each with Hamming distance 4. Since one of the $C_k^{(w)}$ must contain at least as many words as the average, (31) follows.

Furthermore (31) can be replaced by

$$A(n,4,w) \geq \max_{k=0,1,\cdots,n-1} |C_k^{(w)}|, \tag{34}$$

although in practice this gives little improvement on (31).

Kløve [110] generalized Theorem 1 by replacing $\mathbb{Z}_n$ with an arbitrary abelian group G of order n, defining f by

$$f(a) = \sum_{g \in G} a_g g. \tag{35}$$

He found an explicit formula for the best lower bound for $A(n,d,w)$ (for given n and w) obtained from (34) using the optimal choice of G. Again these results do not greatly differ from those obtained from (31). See also Delsarte and Piret [50].

For $d \geq 6$ there are several competing analogues of Theorem 14.

Theorem 15 ([72]): If $\delta \geq 3$ and n is a prime power then

$$A(n,2\delta,w) \geq \frac{1}{n^{\delta-1}}\binom{n}{w}. \qquad (36)$$

Proof: Let $n = q$ be a prime power, and write $\mathbb{F}_q = \{\alpha_0, \cdots, \alpha_{q-1}\}$. The proof replaces (32) by the map f from $\mathbb{F}_2^n$ to $\mathbb{F}_q^{\delta-1}$ given by

$$f(a) = (T_1(a), T_2(a), \cdots, T_{\delta-1}(a)), \qquad (37)$$

where

$$T_1(a) = \sum_{a_i = 1} \alpha_i,$$

$$T_2(a) = \sum_{\substack{i<j \\ a_i = a_j = 1}} \alpha_i \alpha_j,$$

$$T_3(a) = \sum_{\substack{i<j<k \\ a_i = a_j = a_k = 1}} \alpha_i \alpha_j \alpha_k,$$

$$\cdots.$$

For any $\boldsymbol{k} \in \mathbb{F}_q^{\delta-1}$ it can be shown that the code

$$C_k^{(w)} = \{a \in \mathbb{F}_2^n : f(a) = \boldsymbol{k}, \quad wt(a) = w\}$$

has minimal distance 2δ, and (36) follows.

A subset $S = \{s_1, \cdots, s_n\}$ of an abelian group G is called an S_t-set of size n if all the sums

$$s_{i_1} + s_{i_2} + \cdots + s_{i_t}$$

for $1 \leq i_1 < i_2 \cdots < i_t \leq n$ are distinct in G (cf. [72], [73], [75], [76]). S_t-sets are relevant here because of the following result.

Theorem 16 (Compare [72]): If there exists an $S_{\delta-1}$-set of size n in an abelian group of order m then

$$A(n,2\delta,w) \geq \frac{1}{m}\binom{n}{w}.$$

Proof: Replace (35) by $f(a) = \sum_{i=1}^{n} s_i a_i$.

In 1962 Bose and Chowla ([17], [76]; see also [72]) constructed an S_t-set of size $q+1$ in $\mathbb{Z}_m$ for $m = (q^{t+1} - 1)/(q-1)$, for any prime power q. Another construction is the following.

Theorem 17: For any prime power q there is an S_t-set of size q in $\mathbb{Z}_m$ for $m = q^t - 1$.

Proof: Let ξ be a primitive element of the field of order q^t, and let F be the subfield of order q. The set $\{s \in \mathbb{Z}_m | \xi^s - \xi \in F\}$ is the desired S_t-set.

Furthermore the columns of a parity-check matrix for an $[n, k, d = 2t+1]$ binary linear code, together with the zero vector, form an S_t-set of size $n+1$ in the abelian group $\mathbb{Z}_2^{n-k}$. (The corresponding constant weight code is the collection of words of weight w in some coset of the code extended with an (anti-) parity check bit.)

Let $v_{\mathbb{Z}}(n)$ denote the smallest m such that an S_2-set of size n exists in $\mathbb{Z}_m$, and more generally let $v(n)$ be the smallest m such that an S_2-set of size n exists in some abelian group of order m. In [73] it is shown that $v_{\mathbb{Z}}(n) \sim n^2$ as $n \to \infty$, but for $v(n)$ it is known only that

$$\binom{n}{2} \leq v(n) < n^2 + O(n^{36/23}).$$

Table V gives the best upper bounds presently known on $v(n)$ for $2 \leq n \leq 28$.

TABLE V
UPPER BOUNDS FOR S_2-SETS
m IS THE SMALLEST NUMBER KNOWN SUCH THAT $v(n) \leq m$

n	m	n	m	n	m
2	2^a	11	99^e	20	381^h
3	3^a	12	123^f	21	?
4	6^a	13	152^g	22	?
5	11^a	14	183^h	23	?
6	16^b	15	222^i	24	512^l
7	24^c	16	255^j	25	624^j
8	40^a	17	?	26	651^h
9	52^d	18	256^k	27	728^j
10	72^a	19	360^j	28	757^h

Key to Table V

Elements of the elementary abelian groups $\mathbb{Z}_2^k$ are written in decimal; for example $0101 \in \mathbb{Z}_2^4$ is written as 5.

a = Optimal S_2-set in $\mathbb{Z}_m$ (see Table IV in [73]).
b = In $\mathbb{Z}_2^4$ use **0** and columns of parity-check matrix for [5, 1, 5] code.
c = $\{(0,0), (1,0), (2,0), (4,0), (0,1), (7,1), (0,2)\}$ in $\mathbb{Z}_2^3 \times \mathbb{Z}_3$.
d = $\{(0,0), (0,1), (0,2), (1,0), (1,4), (1,8), (2,2), (2,5), (2,8)\}$ in $\mathbb{Z}_2^2 \times \mathbb{Z}_{13}$.
e = $\{0,1,2,4,7,15,26,45,54,66,83\}$ in $\mathbb{Z}_{99}$.
f = $\{0,1,2,4,7,14,23,31,48,59,74,92\}$ in $\mathbb{Z}_{123}$.
g = $\{0,1,2,4,7,12,20,35,63,77,106,115,132\}$ in $\mathbb{Z}_{152}$.
h = Perfect difference set [8].
i = $\{0,1,2,4,7,12,20,29,46,69,92,116,140,170,191\}$ in $\mathbb{Z}_{222}$.
j = Theorem 17.
k = In $\mathbb{Z}_2^8$ use **0** and columns of parity-check matrix for [17, 9, 5] cyclic code.
l = In $\mathbb{Z}_2^9$ use **0** and columns of parity-check matrix for Wagner's [23, 14, 5] code [179].

For example, when $n = 25$, Theorem 16 and Table V imply that

$$A(25,6,12) \geq \left\lceil \frac{1}{624}\binom{25}{12} \right\rceil = 8334\ddagger.$$

(‡ indicates a code which does not yield the best lower bound known for this value of $A(n,d,w)$, but is easy to construct.) The entries labeled "*gs*" in Table I are obtained in this way. The entries in the third column of Table V appear quite weak, and the construction of better S_2-sets would probably improve several entries of Table I-B.

We also mention an unpublished general lower bound of Zaptcioglu [184], although in the range of our tables it does not lead to any new bounds. Suppose C attains $A(n,d,w)=M$, where $d=2\delta$. A *d-neighbor* of $c\in C$ is any $v\in\mathbb{F}_2^n$ such that $wt(v)=w$ and $\text{dist}(v,c)<d$. Let $GN(n,d,w,i)$ denote the number of d-neighbors of the ith word of C that are also d-neighbors of the jth word of C for some $j<i$. Then Zaptcioglu begins with the *equality*

$$A(n,d,w)=\frac{\binom{n}{w}+\sum_{i=2}^{M}GN(n,d,w,i)}{\sum_{i=0}^{\delta-1}\binom{w}{i}\binom{n-w}{i}}. \tag{38}$$

This is proved by multiplying both sides by the denominator, and using a straightforward counting argument. Since $GN\geq 0$ we immediately obtain the "Gilbert bound" of [72]:

$$A(n,d,w)\geq G(n,d,w),$$

where

$$G(n,d,w)=\frac{\binom{n}{w}}{\sum_{i=0}^{\delta-1}\binom{w}{i}\binom{n-w}{i}}.$$

Zaptcioglu shows that (38) has the following stronger consequence.

Theorem 18 (Zaptcioglu [184]): For $d\geq 4$ and $w\geq 3$ we have

$$A(n,d,w)\geq G(n,d,w)+\frac{\sum_{i=0}^{\delta-1}\binom{w}{i}\binom{n-w-1}{i}}{\sum_{i=0}^{\delta-1}\binom{w}{i}\binom{n-w}{i}}\cdot[A(n-1,d,w)-G(n-1,d,w)].$$

VI. The Partitioning Construction

The partitioning construction, used by several authors ([13], [49], [68], [149], [150]), produces good lower bounds for codes with minimal distance 4. It is related to a generalization of Theorem 14 (see subsection 4).

A *partition* $\Pi(n,w)=(X_1,\cdots,X_m)$ is a collection of disjoint sets or *classes* $X_1,\cdots,X_m$, each of which is a code of length n, distance 4 and constant weight w, and whose union contains all $\binom{n}{w}$ vectors of weight w. The vector $\pi(n,w)=(|X_1|,\cdots,|X_m|)$ with integer components is the *index vector* of the partition $\Pi(n,w)$, and

$$\pi(n,w)\cdot\pi(n,w)=\sum_{i=1}^{m}|X_i|^2$$

is its *norm*. We always assume $|X_1|\geq\cdots\geq|X_m|$. When there are several different partitions available for a given n and w we often denote them by $\Pi_1(n,w)$, $\Pi_2(n,w),\cdots$, and their index vectors by $\pi_1(n,w)$, $\pi_2(n,w),\cdots$.

The *direct product* $\Pi(n_1,w_1)\times\Pi(n_2,w_2)$ of two partitions $(X_1,\cdots,X_{m_1})$, $(Y_1,\cdots,Y_{m_2})$ consists of the vectors

$$\{(u,v): u\in X_i,\quad v\in Y_i,\quad 1\leq i\leq m\},$$

where $m=\min\{m_1,m_2\}$. This set (which is only part of the final code) clearly has length n_1+n_2, distance 4, weight w_1+w_2, and contains

$$\pi(n_1,w_1)\cdot\pi(n_2,w_2)=\sum_{i=1}^{m}|X_i||Y_i| \tag{39}$$

words.

The construction: To obtain a code of length n, distance 4 and weight w by the partitioning construction we write $n=n_1+n_2$, choose $\epsilon=0$ or 1, and take the union of the direct products

$$\begin{aligned}&\Pi(n_1,\epsilon)\times\Pi(n_2,w-\epsilon),\\&\Pi(n_1,\epsilon+2)\times\Pi(n_2,w-\epsilon-2),\\&\Pi(n_1,\epsilon+4)\times\Pi(n_2,w-\epsilon-4),\\&\cdots.\end{aligned} \tag{40}$$

It is apparent that this code does have the required properties, and contains

$$\pi(n_1,\epsilon)\cdot\pi(n_2,w-\epsilon)+\pi(n_1,\epsilon+2)\cdot\pi(n_2,w-\epsilon-2)+\pi(n_1,\epsilon+4)\cdot\pi(n_2,w-\epsilon-4)+\cdots \tag{41}$$

words.

As an illustration we construct a code showing that $A(18,4,7)\geq 2041$. We take $n_1=8$, $n_2=10$, $\epsilon=1$ and form the union of the direct products

$$\begin{aligned}&\Pi(8,1)\times\Pi_2(10,6),\\&\Pi(8,3)\times\Pi_1(10,4),\\&\Pi(8,5)\times\Pi(10,2),\\&\Pi(8,7)\times\Pi(10,0),\end{aligned} \tag{42}$$

(see below and Table VI). The corresponding index vectors are

$$\pi(8,1)=(1,1,1,1,1,1,1,1),$$
$$\pi(10,6)=(30,30,30,30,26,25,22,15,2),$$

so the first direct product contains

$$1\cdot30+1\cdot30+1\cdot30+1\cdot30+1\cdot26+1\cdot25+1\cdot22+1\cdot15=208$$

words;

$$\pi(8,3)=(8,8,8,8,8,8,8),$$
$$\pi_1(10,4)=(30,30,30,30,30,22,22,12,2,2),$$

so the second direct product contains

$$8\cdot30+\cdots+8\cdot30+8\cdot22+8\cdot22=1552$$

words;

$$\pi(8,5)=(8,8,8,8,8,8,8),$$
$$\pi(10,2)=(5,5,5,5,5,5,5,5,5),$$

so the third direct product contains

$$8\cdot5+\cdots+8\cdot5=280$$

words; and

$$\pi(8,7) = (1,1,1,1,1,1,1,1),$$
$$\pi(10,0) = (1),$$

so the last direct product contains $1 \cdot 1 = 1$ word. The total number of codewords is

$$208 + 1552 + 280 + 1 = 2041.$$

Codes obtained from the partitioning construction are indicated by "$p0$",$\cdots$,"$p3$" in Table I-A. The values of n_1, n_2, ϵ are as follows:

for type $p0$, $\quad n_1 = \left[\frac{n}{2}\right], \quad n_2 = n - n_1, \quad \epsilon = 0,$

for type $p1$, $\quad n_1 = \left[\frac{n}{2}\right], \quad n_2 = n - n_1, \quad \epsilon = 1,$

for type $p2$, $\quad n_1 = \left[\frac{n}{2}\right] - 1, \quad n_2 = n - n_1, \quad \epsilon = 0,$

for type $p3$, $\quad n_1 = \left[\frac{n}{2}\right] - 1, \quad n_2 = n - n_1, \quad \epsilon = 1.$

The partitions needed are listed in Table VI. (We do not take the space to indicate the particular partitions used in each construction.) Partitioning also gives $A(16,4,5) \geq 305$‡.

We next discuss the choice of a good partition. When applying the partitioning construction in a situation where several different partitions are available, we see from (39), (41) that we should choose pairs $\Pi(n_1,w_1)$, $\Pi(n_2,w_2)$ so as to maximize the inner product $\pi(n_1,w_1) \cdot \pi(n_2,w_2)$. (For example we use $\Pi_2(10,6)$ rather than $\Pi_1(10,6)$ in the first line of (42), so as to maximize the inner product with $\pi(8,1) = (1,1,1,1,1,1,1,1)$.)

We say that one partition $\Pi(n_1,w_1)$ *dominates* another $\Pi'(n_1,w_1)$ if

$$\pi(n_1,w_1) \cdot \pi(n_2,w_2) \geq \pi'(n_1,w_1) \cdot \pi(n_2,w_2) \quad (43)$$

holds for all choices of n_2, w_2 and all possible index vectors $\pi(n_2,w_2)$. If a partition is dominated it need never be used in the construction.

There is a simple test for dominance.

Theorem 19: $\pi(n_1, w_1) = (a_1,\cdots,a_m)$ dominates $\pi'(n_1,w_1) = (b_1,\cdots,b_{m'})$ if and only if

$$\sum_{i=1}^{j} a_i \geq \sum_{i=1}^{j} b_i, \qquad \text{for all } j = 1,\cdots,\max\{m,m'\}.$$

Proof: The components of the index vector $\pi(n_2,w_2)$ in (43) are nonincreasing positive integers, and any such vector is a positive combination of vectors of the form $(1,1,\cdots,1,0,\cdots,0)$.

A partition $\Pi(n,w)$ is *optimal* if it dominates all other $\Pi'(n,w)$ with the same n and w, or just *maximal* if it is not itself dominated by any other $\Pi'(n,w)$.

In the remainder of this section we describe techniques for finding good partitions.

1) By taking complements the existence of a $\Pi(n,w)$ implies the existence of a $\Pi(n,n-w)$ with the same index vector.

2) For $w = 0$ and 1 there are trivial partitions with index vectors

$$\pi(n,0) = (1),$$
$$\pi(n,1) = (1,1,\cdots,(n \text{ times})),$$

and for $w = 2$ there are well-known partitions

$$\pi(n,2) = \left(\frac{n}{2},\frac{n}{2},\cdots,(n-1 \text{ times})\right),\ n \text{ even},$$

$$\pi(n,2) = \left(\frac{n-1}{2},\frac{n-1}{2},\cdots,(n \text{ times})\right),\ n \text{ odd}$$

see [136]. All these partitions are optimal.

3) If we have a lower bound $A(n,4,w) \geq M$ there is always the partition

$$\pi(n,w) = \left(M,1,1,1,\cdots,\left(\binom{n}{w} - M \text{ times}\right)\right).$$

(This is useful when the inner product with $\pi(n,0) = (1)$ is to be maximized.)

4) The results of [72]—see Theorem 14—show that a partition $\Pi(n,w)$ always exists with $m \leq n$ classes. In many cases—for example if n is prime—the index vector for this partition is

$$\pi(n,w) = \left(\frac{1}{n}\binom{n}{w},\frac{1}{n}\binom{n}{w},\cdots,(n \text{ times})\right).$$

5) A number of optimal partitions with $w = 3$ are available in the literature. It is known that, if $n \equiv 1$ or 3 (mod 6), $n \neq 7$, then the set of all $\binom{n}{3}$ triples can be partitioned into $n-2$ mutually disjoint Steiner triple systems—implying that there is an optimal partition

$$\pi(n,3) = \left(\frac{n(n-1)}{6}(n-2 \text{ times})\right).$$

This result is due to Lu [130], [131]. (The manuscript of [131] was incomplete at the time of the author's death, but the six unfinished values of n have since been dealt with by Teirlinck [174].) Earlier results on this problem were given by Cayley [32], Denniston [54], [55], [57], Kirkman [109], Phelps [143], Schreiber [158], Teirlinck [171], Wilson [181]. When (as in this case) the set of all $\binom{n}{w}$ vectors of weight w can be partitioned into disjoint designs all having the same parameters, the designs are said to form a "large set." Further results on partitions may be found in [4], [19], [60], [65]–[68], [85], [113], [119], [144], [146], [159]–[162], [172], [173].

6) Van Pul [149], [150] and Etzion–Van Pul [68] found the partitions $\Pi(6,3)$, $\Pi(7,3)$, $\Pi(8,4)$, $\Pi(10,3)$, $\Pi_1(10,4)$ mentioned in Table VI. (However only $\Pi_1(10,4)$ is given explicitly in [68].)

7) In situations not covered by the preceding comments we use the computer to find good partitions. Our methods are based on the following considerations. a) Finding a good partition is a graph coloring problem. For if we construct the graph whose vertices represent $\binom{n}{w}$ binary vectors of weight w, and join two vertices by an edge if

and only if the vertices are Hamming distance 2 apart, then a partition $\Pi(n,w)=(X_1,\cdots,X_m)$ describes a coloring of the vertices using m colors, the classes $X_1,\cdots,X_m$ being the color classes. b) A useful heuristic for finding a good partition is to maximize the norm of the index vector. This is only a heuristic, for we already saw in the previous example that there are situations where partitions with less than the maximal norm are preferable. However, a partition with the greatest possible norm is always maximal. A second heuristic is to minimize the number of color classes. c) Good methods of choosing the initial classes $X_1, X_2,\cdots$ of $\Pi(n,w)$ are to use a maximal independent set algorithm, to use as many disjoint Steiner systems as possible (see Tables IV), or more generally to look for as many disjoint (or almost disjoint) copies as possible of the largest known code of length n and weight w. (Some partition obtained by repeatedly removing maximal independent sets is maximal.) We then look for a coloring of the remaining vertices with the greatest norm.

David Johnson [98] has developed a simulated annealing program for graph coloring, which attempts to maximize the sum of the squares of the color class sizes. Some of the partitions given in Table VI below were found (in part) using this program. Others were found by various iterative procedures. Johnson's program uses Kempe-chain interchanges for graph coloring; a recent alternative suggestion by Berge [10] may lead to better colorings and hence better partitions.

8) The methods of 7) are only successful for n up to about 14. For $n \geq 12$ we also made use of Etzion and Van Pul's "Construction B" for combining partitions ([68]). This construction works as follows. Given partitions $\Pi(l,u)$ and $\Pi(m,v)$ (for all u, $0 \leq u \leq \min(l,w)$ and v, $0 \leq v \leq \min(m,w)$) we construct a partition $\Pi(n,w)$, where $n=l+m$, by repeatedly using the partitioning construction (cf. (40)). More precisely, start with two sufficiently long rows of empty buckets—the "odd" row and the "even" row. For each pair (u,v) with $u+v=w$ we consider the given partitions $\Pi(l,u)=(X_1,\cdots,X_{r_1})$ and $\Pi(m,v)=(Y_1,\cdots,Y_{r_2})$, and distribute the r_1r_2 direct products $X_i \times Y_j$ into the first $r=\max(r_1,r_2)$ buckets in the row with the same parity as u, where each bucket gets $\min(r_1,r_2)$ codes $X_i\times Y_j$, such that no bucket contains two codes $X_i\times Y_j$ and $X_{i'}\times Y_{j'}$ where $i=i'$ or $j=j'$. The result will be that the buckets form a partition $\Pi(n,w)$. (At least r buckets are required because of the conditions $i\neq i'$ and $j\neq j'$. A distribution into r buckets is possible because the complete bipartite graph K_{r_1,r_2} has an edge coloring with r colors.) For example, take $(n,w)=(12,4)$. Using partitions with index vectors $\pi(6,0)=\pi(6,6)=(1)$, $\pi(6,1)=\pi(6,5)=(1,1,1,1,1,1)$, $\pi(6,2)=\pi(6,4)=(3,3,3,3,3)$, $\pi(6,3)=(4,4,4,4,2,2)$ we fill five buckets in the even row with $1\cdot 3+5\cdot 3\cdot 3+3\cdot 1=51$ words each, and six buckets in the odd row with $\pi(6,1)\cdot\pi(6,3)+\pi(6,3)\cdot\pi(6,1)=40$ words each, to obtain a partition $\Pi(12,4)$ with index vector $(51,51,51,51,51,40,40,40,40,40,40)$.

Table VI gives the index vectors of the nontrivial partitions used in constructing the codes marked "$p0$," "$p1$," "$p2$," "$p3$" in Table I. In view of 1) we only give partitions $\Pi(n,w)$ with $w\leq n/2$. The partitions marked "A" are given explicitly in the Appendix. Partitions known to be optimal are marked with an asterisk. For some values of n and w several different partitions $\Pi_i(n,w)$ ($i=1,2,\cdots$) are given (i is given in column 3), no one of which dominates any of the others. In many cases we have found other partitions besides those in Table VI, which although not needed for the partitioning construction, nevertheless are not dominated by the partitions in the table. It would be nice to replace these with smaller sets of maximal partitions. For $\Pi(10,4)$ there are at least two maximal partitions, and, as will be shown below, certainly no optimal $\Pi(10,4)$ exists.

Key to Table VI

- * = Optimal.
- A = See Appendix.
- B = Construction B.
- GS = From Theorem 14.
- (1) = Shortened $\Pi(9,3)$
- (2) = Shortened $\Pi(10,4)$.
- (3) = Shortened $\Pi(13,3)$.
- (4) = Shortened $\Pi(15,3)=(35,35,\cdots,(13\text{ times}))$ of [55].

Optimal and maximal partitions: We first show that the partitions marked with an asterisk in Table VI are optimal. Obviously $\pi(n,w)$ is optimal if all parts (except perhaps the smallest) have size $A(n,4,w)$, which proves the optimality of $\pi(8,3)$, $\pi(9,3)$, etc.

$\pi(6,3)=(4,4,4,4,2,2)$ is optimal.

Proof: Any code attaining $A(6,4,3)=4$ is equivalent to

$$C_1=\begin{bmatrix}1&1&1&0&0&0\\1&0&0&1&1&0\\0&1&0&1&0&1\\0&0&1&0&1&1\end{bmatrix}.$$

We must show that $(4,4,4,4,4)$ and $(4,4,4,4,3,1)$ are impossible. Since C_1 has column sums 2, the last four vectors of the partition also have column sums 2, and are therefore equivalent either to C_1 or to

$$C_2=\begin{bmatrix}1&1&1&0&0&0\\1&1&0&1&0&0\\0&0&1&0&1&1\\0&0&0&1&1&1\end{bmatrix}.$$

The vectors of C_2 cannot be partitioned $3+1$, so the last four vectors of the partition are also equivalent to C_1, and the partition is $(4,4,4,4,4)$. Let the first class be C_1. Then the four vectors 100011, 100101, 110001, 101001 have mutual distances 2 and each lies in a different one of the other four classes. It is now easy to check by hand that these classes cannot be completed.

$\pi(7,3)=(7,7,6,6,5,4)$ is optimal.

Proof: We know from Table IV that there do not exist three classes of size 7. Also $(7,7,6,6,6,1,1,1)$ is

TABLE VI
NONTRIVIAL PARTITIONS USED TO CONSTRUCT CODES IN TABLE I

n	w	i	m	Norm	Source	Index Vector of $\Pi_i(n,w)$
6	3	1	6	72	A*	4,4,4,4,2,2
7	3	1	6	211	A*	7,7,6,6,5,4
8	3	1	7	448	(1)*	8,8,8,8,8,8,8
8	4	1	6	844	A*	14,14,12,12,10,8
9	3	1	7	1008	Table IV*	12,12,12,12,12,12,12
9	4	1	8	2066	A	18,18,18,18,16,15,15,8
9	4	2	10	2036	(2)	18,18,18,18,18,14,13,7,1,1
10	3	1	10	1530	A*	13,13,13,13,13,13,13,13,13,3
10	4	1	10	5620	[68]	30,30,30,30,30,22,22,12,2,2
10	4	2	9	5614	A	30,30,30,30,26,25,22,15,2
10	5	1	8	8044	A	36,36,34,34,29,29,27,27
11	3	1	10	2731	A	17,17,17,17,17,17,17,16,16,14
11	3	2	11	2713	A	17,17,17,17,17,17,17,17,16,12,1
11	3	3	11	2705	A	17,17,17,17,17,17,17,17,17,10,2
11	4	1	11	10724	A	35,35,35,34,33,33,33,32,31,25,4
11	4	2	11	10616	A	35,35,35,35,35,33,33,32,28,21,8
11	5	1	10	25066	A	66,66,60,60,54,45,44,40,26,1
11	5	2	10	25046	A	66,66,60,60,54,45,44,42,22,3
12	3	1	11	4400	(3)*	20,20,20,20,20,20,20,20,20,20,20
12	4	1	11	22903	A	51,51,51,51,51,46,45,44,44,34,27
12	4	2	12	22843	A	51,51,51,51,49,48,48,42,42,37,23,2
12	4	3	12	22815	A	51,51,51,51,49,48,48,42,42,40,15,7
12	4	4	12	22795	A	51,51,51,51,49,48,46,44,43,37,20,4
12	4	5	12	22755	A	51,51,51,51,49,48,48,45,39,36,22,4
12	4	6	12	22663	A	51,51,51,51,49,48,48,45,41,32,22,6
12	5	1	12	55860	A	80,80,80,80,72,70,69,67,67,62,48,17
12	5	2	13	55350	A	80,80,80,80,75,72,71,69,63,55,40,23,4
12	6	1	10	99952	A	132,132,120,120,110,94,90,76,36,14
12	6	2	10	99776	A	132,132,120,120,110,94,90,72,42,12
12	6	3	10	99072	A	132,132,120,110,110,97,91,75,47,10
13	3	1	11	7436	Table IV*	26,26,26,26,26,26,26,26,26,26,26
13	4	1	13	42165	A	65,65,65,65,62,61,60,57,57,53,52,45,8
13	4	2	13	42163	A	65,65,65,65,62,61,60,58,55,54,52,45,8
13	4	3	13	42147	A	65,65,65,65,62,60,60,58,57,54,49,47,8
13	4	4	13	42015	A	65,65,65,65,62,60,60,58,57,56,53,37,12
13	4	5	13	41975	A	65,65,65,65,62,62,60,59,56,55,49,40,12
13	4	6	13	41795	A	65,65,65,65,62,61,61,59,58,54,50,32,18
13	5	1	13	135679	A	123,123,121,115,110,109,109,102,99,92,84,72,28
13	5	2	13	135557	A	123,123,121,115,110,109,109,101,99,93,86,68,30
13	5	3	14	135437	A	123,122,121,114,110,109,109,102,97,91,85,77,26,1
13	5	4	13	134757	A	123,123,123,116,110,109,106,100,98,92,81,68,38
13	5	5	14	134753	A	123,123,123,116,110,109,107,104,97,89,83,62,40,1
13	6	1	14	239106	A	166,166,160,156,143,143,139,135,131,122,107,100,46,2
13	6	2	14	239082	A	166,166,160,156,144,142,138,137,131,120,106,102,46,2
13	6	3	13	238832	A	166,166,160,156,143,142,138,136,130,120,111,97,51
13	6	4	14	238698	A	166,166,160,156,145,142,139,136,131,118,113,91,50,3
13	6	5	14	238384	A	166,166,160,156,145,142,139,136,131,119,112,88,52,4
13	6	6	13	238116	A	166,166,160,156,145,144,137,132,127,118,111,98,56
13	6	7	14	237556	A	166,166,160,156,145,142,140,136,131,118,106,86,59,5
14	3	1	13	10192	(4)*	28,28,28,28,28,28,28,28,28,28,28,28,28
14	4	1	13	79393	A	91,91,91,91,81,79,78,77,74,73,71,62,42
14	4	2	14	79357	A	91,91,91,91,80,79,78,78,75,74,71,60,41,1
14	4	3	14	79339	A	91,91,91,91,81,79,79,77,76,71,67,67,38,2
14	4	4	13	79269	A	91,91,91,91,82,79,78,77,75,72,67,62,45
14	5	1	15	291280	A	169,169,165,156,156,152,149,144,143,137,134,121,118,80,9
14	5	2	15	290646	A	169,169,165,156,155,153,151,147,143,137,134,120,112,76,15
14	5	3	16	290288	A	169,169,165,156,155,153,151,147,142,137,133,124,109,75,16,1
14	5	4	15	289872	A	169,169,163,156,155,152,149,148,142,139,132,131,102,76,19
14	6	1	14	680081	B	253,252,243,243,243,243,212,212,212,212,212,212,212,42
14	7	1	14	913176	B	282,282,280,280,271,271,271,271,271,271,271,271,70,70
14	7	2	15	887552	B	292,292,280,280,272,272,242,242,242,242,242,242,242,48,2

impossible, for it would shorten to either $\pi(6,3) = (4,4,4,4,4)$ or $(4,4,4,4,3,1)$. (Any class of size 6 must shorten in three ways to a class of size 4 and in four ways to a class of size 3.) Therefore (7,7,6,6,5,4) is optimal. This also implies the optimality of

$$\pi(8,4) = (14,14,12,12,10,8).$$

Second, we point out that there is no optimal partition $\pi(10,4)$. For $\pi_1(10,4) = (30,30,30,30,30,22,22,12,2,2)$ is maximal, as we now show, while $\pi_2(10,4) = (30,30,30,30,26,25,22,15,2)$ has fewer classes. Since $A(10,4,4) = 30$, by Theorem 7, no class has size greater than 30. From [115]—see Table IV—the maximal number of classes of size 30 is five, this can occur in an essentially unique way. We use the particular set of five given in Table IV. When these are removed the remaining $\binom{10}{4} - 5 \times 30 = 60$ vectors consist of

(01111)	(00000)	(5)
(00000)	(01111)	(5)
(00011)×(00011)		(25)
(00101)×(00101)		(25).

Each of the first five vectors (and each of the second five) must be a different color. Among the final 50 vectors there cannot be a color class of size 21, because if so then that class would contain at least three words of form $*****(00011)$ or three of form $*****(00101)$. Thus $\pi_1(10,4)$ is maximal.

Concluding remarks:

1) If $n = 2t$, $w = t$, t odd, we may take $\epsilon = 0$ and obtain

$$A(2t,4,t) \geq \sum_{w=0}^{\frac{t-1}{2}} \text{norm}\,\Pi(t,w). \tag{44}$$

Similarly if t is even we get

$$A(2t,4,t) \geq \max\left\{ \sum_{w=0,2,\cdots,t} \text{norm}\,\Pi(t,w), \sum_{w=1,3,\cdots,t-1} \text{norm}\,\Pi(t,w)\right\}. \tag{45}$$

2) Using their Construction B, Etzion and Van Pul [68] show that if n is of the form 2^k $(k \geq 2)$ or $3 \cdot 2^k$ $(k \geq 2)$ and w is even then Theorem 14 can be replaced by

$$A(n,4,w) \geq \frac{1}{n-1}\binom{n}{w}. \tag{46}$$

From the partitions in Table VI (especially $\Pi_2(10,4)$) this now also holds for $n = 5 \cdot 2^k$ $(k \geq 2)$.

3) Romanov's construction [155] showing that $A(16,3) \geq 2720$ (see Table II) also uses partitioning. We write the codewords in the form (a,b), where length $(a) = 9$, length $(b) = 7$. On the left side a has weight 0, 3, 6, or 9, and we make use of Kirkman's partition

$$\pi(9,3) = (12,12,12,12,12,12,12)$$

of the set of triples on 9 points into 7 disjoint copies $\mathcal{G}_1, \cdots, \mathcal{G}_7$ of $S(2,3,9)$ (see Tables IV, VI). On the right we partition the set of all 128 7-bit words into eight disjoint translates $\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_7$ of the [7, 4, 3] Hamming code. Romanov's code then consists of the vectors $(0, \mathcal{H}_0), \mathcal{G}_i \times \mathcal{H}_i$ $(1 \leq i \leq 7)$, and their complements.

VII. Lower Bounds Obtained by Modifying Codes with a Larger Minimal Distance

The following inequalities, due to Zinoviev [187], Van Pul [149], [150], and Honkala *et al.* [94], resemble those of Section V in requiring very little computation. They produce good lower bounds for codes with $d = 6$. We follow the treatment of Honkala *et al.* [94].

Theorem 20 ([187], [149], [150], [94]):

a) For $0 \leq g < \min\{w, \delta\}$ and $0 \leq k < n$ we have

$$A(n-k, 2\delta - 2g, w-g) \geq \frac{1}{\binom{n}{k}} A(n, 2\delta, w) \sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i}.$$

b) For $0 \leq g \leq w$, $0 \leq k < n$ and $k - g < \delta$, we have

$$A(n-k, 2\delta - 2k + 2g, w-g) \geq \frac{1}{\binom{n}{k}} A(n, 2\delta, w) \sum_{i=g}^{k} \binom{w}{i}\binom{n-w}{k-i}.$$

Proof: Suppose $\mathcal{C}$ attains the bound $A(n,2\delta,w)$. For any k-subset S of the coordinates let c_S denote the projection of $c \in \mathcal{C}$ into S, and let $c_{\bar{S}}$ denote the projection onto the other coordinates. A new code $\mathcal{C}_S$ with length $n' = n - k$, $d' = 2\delta - 2g$ and $w' = w - g$ is obtained by taking all words $c_{\bar{S}}$ for which $c \in \mathcal{C}$ and $wt(c_S) = i$ for some $0 \leq i \leq g$, and complementing any $g - i$ 1's. A counting argument shows that

$$\sum_{|S|=k} |\mathcal{C}_S| = A(n,2\delta,w) \sum_{i=0}^{g} \binom{w}{i}\binom{n-w}{k-i},$$

and a) follows. To prove b) we take all words $c_{\bar{S}}$ for which $c \in \mathcal{C}$ and $wt(c_S) = i$ for some $g \leq i \leq k$, and complement any $i - g$ 0's.

The lower bounds on $A(17,6,7)$ and $A(18,6,7)$ in Table I-B are obtained from Theorem 20a by taking $g = 1$ and $\mathcal{C}$ to be the Steiner system $S(5,8,24)$. The lower bounds on $A(15,6,7)$, $A(16,6,7)$ follow similarly using a particular choice of S.

Theorem 21 (Honkala et al. [94]):

$$A(n-2, d-2, w-1) \geq A(n,d,w).$$

Proof: We modify the codewords c for a code $\mathcal{C}$ attaining $A(n,d,w)$ as follows. If c ends with 00, complement the final 1, while if c ends with 11, complement the final 0. Now omit the last two coordinates of all words.

The lower bounds $A(22,6,7) \geq 759$, $A(22,6,11) \geq 2576$‡ follow by taking $\mathcal{C}$ to be $S(5,8,24)$ or the code attaining $A(24,8,12) = 2576$.

H. Hämäläinen [80] used a modification of this argument to show $A(21,6,6) \geq 269$. Start with the $S(5,8,24)$

TABLE VII
SUM-CONSTRAINED LEXICODES

Bound	s
$A(12,4,6)=132$	21
$A(24,8,18)\geq 78$	175
$A(25,8,18)\geq 254$	175
$A(26,8,18)\geq 760$	175
$A(26,8,19)\geq 256$‡	188
$A(19,10,11)=12$	84
$A(23,10,17)=8$	181
$A(24,10,18)=9$	203
$A(28,10,14)\geq 415$‡	130

TABLE VIII
LEXICODES WITH A SEED

Bound	Seed
$A(18,6,4)=22$	1422, 24410
$A(26,6,10)\geq 8189$‡	$A(25,6,9)\geq 4100$‡ from Table XV
$A(26,6,17)\geq 5407$‡	$A(25,6,16)\geq 4100$‡ from Table XV
$A(27,6,9)\geq 7198$‡	$A(25,6,9)\geq 4100$‡ from Table XV
$A(27,6,10)\geq 11656$‡	$A(25,6,10)\geq 5700$‡ from Table XV
$A(28,6,9)\geq 9577$‡	$A(25,6,9)\geq 4100$‡ from Table XV
$A(26,8,13)\geq 3004$‡	The lexicode $A(24,8,12)=2576$
$A(27,8,13)\geq 3601$‡	The lexicode $A(24,8,12)=2576$
$A(21,10,7)=13$	310CC, 54524
$A(23,12,10)=16$	58AA1C, 60E4A6

lexicode (see the following section), and take the $759-330=429$ words that do not end with 00. If such a word ends with 11 complement the final 0. By omitting the last two coordinates we obtain 429 words of length 22, distance 6 and weight 7. Using this set as the seed for a lexicographic code (see the following section) we get another code $\mathscr{C}$ showing $A(22,6,7)\geq 759$, in which (labeling the coordinates 1 to 22 from right to left) 269 words have a 1 in coordinate 2. Hence $A(21,6,6)\geq 269$.

VIII. LEXICOGRAPHIC CODES

Lexicographic codes are studied in detail in [44], and we refer to that paper for the general theory. Here we just consider constant weight lexicodes, which give easily computed lower bounds on $A(n,d,w)$ that are often reasonably good and in some cases give the best bounds known.

The *constant weight lexicographic code* (or *lexicode*, for short) with length n, Hamming distance d and weight w is obtained by starting with the empty code, considering all binary vectors of the given length and weight in lexicographic order (beginning with $00\cdots011\cdots1$), and adding them to the code if they have the desired Hamming distance from it.

This is a "no-input" construction. The most remarkable example of a lexicode is the Steiner system $S(5,8,24)$ (see [44], [45]); other examples are indicated by "x" in Table I.

Several variations are possible. The vectors of complementary weight $n-w$ may be used instead, or the vectors may be considered in Gray code order, or both. For example $A(25,12,9)=25$ and $A(27,12,18)=39$ also arise as Gray lexicodes.

Another modification, a *sum-constrained lexicode*, only considers binary vectors $(a_0,a_1,\cdots,a_{n-1})$ that satisfy the constraint

$$\sum_{i=0}^{n-1} ia_i \geq s, \qquad (47)$$

where s is specified in advance. For example the choice $s=21$ produces the Steiner system $S(5,6,12)$ ([44], [45]). Other examples are given in Table VII. Although a considerable amount of computing is needed to discover the best value of s, once found this gives a succinct definition of the code.

A more powerful modification is to start with an initial set of vectors (the "seed") instead of the empty set. Some codes (labeled "xy" in Table I) found this way are best described in the condensed notation introduced in Section XII, and are listed in Table XVI. Others (indicated by "xh" in Table I) are given in Table VIII. In the latter table, if the seed has shorter word length than the final code, we pad the seed by adding prefixes $0\cdots01\cdots1$ of the appropriate weight on the left.

IX. CONSTANT WEIGHT CODES FROM TRANSLATES OF LINEAR CODES

A number of good constant weight codes may be obtained from translates of linear codes (and from translates of the Nordstrom–Robinson code, which behaves in many ways like a linear code). If $\mathscr{C}$ is an $[n,k,d]$ binary linear code, let $B^{(w)}(u)$ be the set of vectors of weight w in the translate $u+\mathscr{C}$, $u\in\mathbb{F}_2^n$. Then

$$A(n,d,w)\geq \max_{u\in\mathbb{F}_2^n}|B^{(w)}(u)|. \qquad (48)$$

Furthermore (cf. [28]) the code

$$(B^{(w-1)}(u),1)\cup(B^{(w)}(u),0)$$

shows that

$$A(n+1,d,w)\geq \max_{u\in\mathbb{F}_2^n}\{|B^{(w-1)}(u)|+|B^{(w)}(u)|\}. \qquad (49)$$

A not very systematic search through known linear codes has yielded the following examples. The 14 vectors

$$(10000000000000)(11101011100000) \qquad (50)$$

span Karlin's [28, 10, 8] self-dual code ([132], p. 509, column 2, first code). We apply (49) to the [27, 10, 7] code $\mathscr{C}$ obtained by deleting the last coordinate. With $u=0$ we obtain $A(28,8,13)\geq 4668$‡, $A(28,8,14)\geq 5280$, with $u=1F$ we obtain $A(28,8,10)\geq 1652$‡, with $u=A$ we obtain $A(28,8,11)\geq 2666$‡, and with $u=15$ we obtain $A(28,8,12)\geq 3780$‡.

The generator matrix

011	011	011	011	011	001111111111
011	101	101	101	101	110011111111
101	011	110	101	110	111100111111
101	110	011	110	101	111111001111
110	110	101	011	110	111111110011
110	101	110	110	011	111111111100
110	110	110	101	101	010101010101

(51)

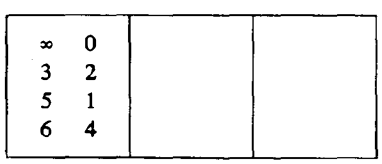
TABLE IX
WEIGHT DISTRIBUTIONS OF TRANSLATES OF [23, 12, 7] GOLAY CODE

#\i	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	0	0	253	506	0	0	1288
23	0	1	0	0	0	0	77	176	176	330	616	672
253	0	0	1	0	0	21	56	112	240	400	546	672
1771	0	0	0	1	5	16	48	120	240	400	560	658

defines a [27, 7, 12] code (a less symmetrical code with these parameters is given in [87]). Again we apply (49) to the code $\mathscr{C}$ formed by omitting the last coordinate. With $u=0$ we obtain $A(27,12,12)\geq 82$ and with $u=92120$ we get $A(27,12,13)\geq 81$.

Table IX gives the weight distributions of the cosets of the [23, 12, 7] perfect Golay code ([132], Chap. 2). The first column gives the number of cosets with the given weight distribution. (In both Tables IX and X the weight distributions are symmetric about $n/2$.) Using (49) we obtain $A(24,8,9)\geq 640$, as well as the other entries labeled "$t4$" in Table I.

The Nordstrom–Robinson code: The nonlinear Nordstrom–Robinson code of length 16, distance 6 and 256 codewords [132], [5], [107] produces a number of good constant weight codes, as was first observed by Semakov and Zinoviev [159]. We work inside the [24, 12, 8] extended Golay code $\mathscr{G}$ and represent codewords of $\mathscr{G}$ by 4×6 arrays called MOG's (or miracle octad generators). These have been described in several references (see [40], [42], [43], [46]–[48] and especially [45], pp. 303–304) and we do not repeat the definition here. We label the first 8 coordinates as follows (cf. [45], p. 316):

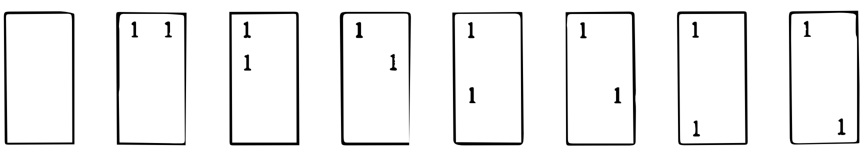


By deleting these 8 coordinates from the codewords of $\mathscr{G}$ we obtain (two copies of) the [16, 11, 4] Hamming code $\mathscr{H}$, while the codewords of $\mathscr{G}$ that vanish on these 8 coordinates yield the [16, 5, 8] first-order Reed–Muller code $\mathscr{R}$. We order the coordinates by reading down the columns, from left to right. When the Golay code defined by the MOG coordinates is read in this way it coincides with the lexicographic version of this code ([44], [45], p. 327).

Let $\mathscr{R}_i$ $(0\leq i\leq 6)$ denote the words of $\mathscr{G}$ that have 1's in coordinates ∞ and i, and 0's elsewhere in the first 8 coordinates, with the first 8 coordinates deleted. Each $\mathscr{R}_i$ is a translate of $\mathscr{R}$ containing 16 words of weight 6 and 16 of weight 10, and

$$\mathscr{N}=\mathscr{R}\cup\mathscr{R}_0\cup\mathscr{R}_1\cup\cdots\cup\mathscr{R}_6$$

is the Nordstrom–Robinson code. Thus $\mathscr{N}$ consists of the words of $\mathscr{G}$ that begin with one of



with these first 8 coordinates deleted.

Let $\alpha_\bullet$ denote any of the four octads (weight 8 words in $\mathscr{G}$) that have 1's at ∞, 3, 5, 6 and 0's at $0,1,2,4$. Similarly α_i $(0\leq i\leq 6)$ is any octad that meets the first 8 coordinates just in ∞ and $3+i, 5+i, 6+i \pmod 7$. The seven translates $\alpha_i+\mathscr{N}$ (with the first 8 coordinates deleted) together with $\mathscr{N}$ itself form a partition of the Hamming code $\mathscr{H}$. We remark that all seven translates $\alpha_i+\mathscr{N}$ are equivalent, all pairs of such translates are equivalent, and there are two inequivalent ways to choose three translates.

TABLE X
WEIGHT DISTRIBUTIONS OF TRANSLATES OF NORDSTROM–ROBINSON CODE THAT PARTITION THE SPACE

#\i	0	1	2	3	4	5	6	7	8
1	1	0	0	0	0	0	112	0	30
16	0	1	0	0	0	42	0	85	0
120	0	0	1	0	14	0	63	0	100
112	0	0	0	5	0	33	0	90	0
7	0	0	0	0	20	0	48	0	120

The Nordstrom–Robinson code also has the property that certain of its translates partition the whole space of vectors of length 16. The weight distributions of these translates are given in Table X; the last row describes the translates $\alpha_i+\mathscr{N}$. From Table X and (48) we obtain $A(16,6,6)\geq 112$, $A(16,6,8)\geq 120$.

The decomposition of $\mathscr{H}$ into 8 translates of $\mathscr{N}$ shows in particular that the 448 weight-6 words in $\mathscr{H}$ can be partitioned as

$$448=112+7\times 48,$$

where each part has minimal distance 6 (see the $i=6$ column of Table X). There is however a better partition of these 448 words. Let $\mathscr{S}_1$ denote the words of $\mathscr{G}$ that have exactly two 1's in the first 8 coordinates, in coordinates $(\infty,0)$, $(\infty,1)$, $(\infty,2),\cdots$, or $(\infty,6)$, with the first 8 coordinates deleted. Similarly $\mathscr{S}_2$ is obtained from the words of $\mathscr{G}$ that have 1's in coordinates $(0,1),(0,2),\cdots,(0,6)$; $\mathscr{S}_3$ from $(1,2),(1,3),\cdots,(1,6)$; $\mathscr{S}_4$ from $(2,3),(2,4),(2,5),(2,6)$; $\mathscr{S}_5$ from $(3,4),(3,5),(3,6)$; and $\mathscr{S}_6$ from $(4,5),(4,6),(5,6)$. Each $\mathscr{S}_i$ is a union of translates of $\mathscr{R}$. Let $\mathscr{S}_i^{(6)}$ denote the weight 6 words in $\mathscr{S}_i$. Then $\mathscr{S}_1^{(6)},\cdots,\mathscr{S}_6^{(6)}$ contain 112, 96, 80, 64, 48, 48 words, respectively. Since any two pairs defining an $\mathscr{S}_i$ have a point in common, and $\mathscr{G}$ has minimal distance 8, it follows that each $\mathscr{S}_i^{(6)}$ is a constant weight code of minimal distance 6. All the $\mathscr{S}_i^{(6)}$ are contained in $\mathscr{H}$, and so they partition the weight 6 words of $\mathscr{H}$ as

$$448 = 112 + 96 + 80 + 64 + 48 + 48.$$

There is a similar result for weight 5 words. Let $\mathscr{H}'$ denote the Hamming code translated by $00 \cdots 01$. Then $\mathscr{H}'$ is partitioned into 8 translates of $\mathscr{N}$ (one from the second row of Table X, seven from the penultimate row), which partitions the weight 5 words in $\mathscr{H}'$ as

$$273 = 42 + 7 \times 33$$

(see the $i = 5$ column of Table X). An alternative partition

$$273 = 42 + 36 + 4 \times 33 + 3 \times 21$$

where each part has minimal distance 6 may be obtained as follows. We denote the nine parts by $\mathscr{D}_1^{(5)}, \cdots, \mathscr{D}_9^{(5)}$, where $\mathscr{D}_i^{(5)}$ consists of the weight 5 words in the translate of Θ_i by $00 \cdots 01$ with the first 8 coordinates deleted, and $\Theta_1 = \mathscr{S}_1$, $\Theta_2 = \mathscr{S}_2$. For $j = 0,1,2,3$, Θ_{j+3} consists of the words of $\mathscr{G}$ having either two or four 1's in the first eight coordinates, such that these 1's are a subset or superset of $\{1,3,4\} + j \pmod 7$. Θ_7 consists of the words of $\mathscr{G}$ with two or four 1's in the first eight coordinates, such that these 1's are either the set $\{2,3\}$ or are a superset of $\{2,3,\infty\}$. Θ_8 and Θ_9 are defined in the same way as Θ_7, replacing 2,3 by 4,6 and 1,5 respectively.

Adding tails to translates: The remaining codes in this section are found by adding tails to translates of the Nordstrom–Robinson and Golay codes. We denote by B_t^w the set of vectors of weight w in a translate of either of these codes by a vector of weight t. Thus $|B_t^w|$ is given by the entry in Table IX or X in the column headed w and in the row in which the first nonzero entry occurs in column t.

The following codes in Table I are obtained from the Nordstrom–Robinson code $\mathscr{N}$.

$A(20,6,6) \geq 232 = 112 + 6 \times 20$ from $B_0^6 0000$, $(B_4^4)^i\{1100\}$, $0 \leq i \leq 5$, where $(B_4^4)^i$ $(0 \leq i \leq 5)$ represents the vectors of weight 4 in six translates of the type $\alpha_i + \mathscr{N}$ described in the last row of Table X, and $\{1100\}$ denotes all six binary 4-tuples of weight 2.

$A(20,6,7) \geq 462 = 30 + (112 + 96 + 80 + 64) + 4 \times 20$ from $\mathscr{R}^{(7)'}0000, \mathscr{S}_i^{(6)}(1000)$ $(1 \leq i \leq 4)$ and $(\mathscr{B}_4^4)^{(j)}(0111)$ $(0 \leq j \leq 3)$, where $\mathscr{R}^{(7)'}$ is obtained by complementing the final 1 in each of the 30 weight-8 words in $\mathscr{R}$, and $\mathscr{S}_i^{(6)}$ is defined above.

$A(20,6,8) \geq 588 = 120 + (112 + 96 + 80 + 64 + 48 + 48) + 20$ from $(B_4^8)^0 0000, \mathscr{S}_j^{(6)}\{1100\}, (B_4^4)^0 1111$, $1 \leq j \leq 6$.

$A(20,6,9) \geq 832 = 4 \times 120 + (112 + 96 + 80 + 64)$ from $(B_4^8)^i(1000), \mathscr{S}_j^{(6)}(0111)$, $0 \leq i \leq 3$, $1 \leq j \leq 4$.

$A(20,6,10) \geq 944 = 112 + 6 \times 120 + 112$ from $B_0^{10}0000, (B_4^8)^i\{1100\}, B_0^6 1111$, $0 \leq i \leq 5$.

$A(21,6,10) \geq 1382\ddagger = 112 + 7 \times 120 + 30 + (112 + 96 + 80 + 64 + 48)$ from $B_0^{10}00000, (B_4^8)^i\{11000\}$, $0 \leq i \leq 6$, $B_0^8 00011, \mathscr{S}_j^6(01111)$, $1 \leq j \leq 5$.

$A(22,6,8) \geq 1116\ddagger = 6 \times 90 + 42 + 42 + 7 \times 33 + 7 \times 33 + 6 \times 5$ from $(B_3^7)^i(100000)$, $B_1^5 111000$, $B_1^5 000111$, $(B_3^5)^i t_j, (B_3^5)^i \bar{t}_j, (B_3^3)^i(011111)$, $0 \leq i \leq 5$, $0 \leq j \leq 6$, where $\{t_0, \cdots, t_6\} = \{110001, 110010, 110100, 101001, 101010, 101100, 011001\}$, and $(B_3^7)^i$ $(0 \leq i \leq 5)$ represents the vectors of weight 7 in six cosets of the type described in the penultimate row of Table X, the cosets being chosen to have Hamming distance 4 apart. This code contains exactly 21 holes, all of which may be adjoined, yielding $A(22,6,8) \geq 1137\ddagger$. Further optimization by the methods described in Section XII gives $A(22,6,8) \geq 1139$ (see Table XVI).

$A(22,6,9) \geq 1736\ddagger = 6 \times 120 + 112 + 112 + 7 \times 48 + 7 \times 48 + 6 \times 20$ from $(B_4^8)^i(100000)$, $B_0^6 111000$, $B_0^6 000111$, $(B_4^6)^i t_j, (B_4^6)^i \bar{t}_j, (B_4^4)^i(011111)$, $0 \leq i \leq 5$, $0 \leq j \leq 6$. This can be improved to $A(22,6,9) \geq 1768\ddagger$ by first adding 28 holes in lexicographic order, then replacing the eight words FE3, 1DDA, 2A815A, 26419A, 30C523, 3300DA, 3304A3, 3C01A3 by the twelve words 208F23, 210CE3, 2403E3, 24199A, 26605A, 28155A, 2AA09A, 300CDA, 30E063, 33211A, 332223, 3C2823. By shortening this code we obtain $A(21,6,9) \geq 1092\ddagger$.

$A(22,6,10) \geq 2180\ddagger = 6 \times 90 + 85 + 85 + 7 \times 90 + 7 \times 90 + (42 + 36 + 4 \times 33)$ from $(B_3^9)^i(100000)$, $B_1^7 111000$, $B_1^7 000111, (B_3^7)^i t_j, (B_3^7)^i \bar{t}_j, \mathscr{D}_k^{(5)}(111110)$, $0 \leq i \leq 5$, $0 \leq j \leq 6$, $1 \leq k \leq 6$, where $\mathscr{D}_k^{(5)}$ is defined above.

$A(22,6,11) \geq 2636 = 2 \times (448 + 30 + 7 \times 120)$ from $\mathscr{S}_i^{(6)}(011111)$ $(1 \leq i \leq 6)$, $B_0^8 111000, (B_4^8)^j t_j$ $(0 \leq j \leq 7)$, and their complements.

$A(23,6,5) \geq 147 = 7 \times 20 + 7$ from $(B_4^4)^i(1000000)$, $000000000(1000000)(1110100)$, $0 \leq i \leq 6$.

The following codes are similarly obtained from the [23,12,7] Golay code:

$$A(26,8,11) \geq 1858\ddagger \quad \left(B_3^8 111, B_3^9 110, B_3^{10}100, B_3^{11}000\right),$$

$$A(27,8,11) \geq 2047\ddagger \quad \left(B_0^7 1111, B_0^8 1110, B_0^{11}0000\right),$$

$$A(27,8,12) \geq 3082\ddagger \quad \left(B_0^8 1111, B_0^{11}1000, B_0^{12}0000\right).$$

The final set of codes in this section come from the [24,12,8] Golay code $\mathscr{G}$. Now B_t^w denotes the vectors of weight w in a translate of $\mathscr{G}$ by a vector of weight t (see [46], [132], p. 69).

$A(25,8,9) \geq 829$ is obtained from the vectors $(B_0^8)'1$, $(B_0^{12})'0$, where $(B_0^8)'$ consists of the $759 - 210 = 549$ words of weight 8 in $\mathscr{G}$ not ending 000, and $(B_0^{12})'$ consists of the 280 words of weight 12 ending 111 with these three 1's complemented (Kaikkonen [105]).

$A(25,8,10) \geq 1232\ddagger = 960 + 272$ is obtained from a translate of $\mathscr{G}$ containing 360 words of weight 8 (denoted by B_4^8) and 960 words of weight 10 (denoted by B_4^{10}). We first take the 960 words $B_4^{10}0$. Any vector $u1$, where u is obtained by complementing any 0 in a vector of B_4^8, is at distance 8 from the initial 960 words; there are $360 \times 16 = 5760$ such vectors, and we must find a subset of them at Hamming distance 8 apart. We could take the 240 out of the 360 that have a 0 in a particular coordinate and complement that coordinate, obtaining $A(25,8,10) \geq$

$960+240=1200\ddagger$. However we can do better. Consider the subset of the 360 words with at most one 1 in a particular set of three coordinates, and in these coordinates replace 000 by 100, 100 by 110, 010 by 011 and 001 by 101. Consider first a random set of 3 out of the 24 coordinates. The probability that a vector containing 8 1's and 16 0's has at most a single 1 in three coordinates is

$$\frac{\binom{16}{3}+\binom{8}{1}\binom{16}{2}}{\binom{24}{3}}=0.7510\cdots,$$

so at least $360\times 0.7510\cdots = 270.36\cdots$ (hence 271) words can be added in this way. A particular choice of three coordinates gives 272. Thus $A(25,8,10)\geq 960+272=1232\ddagger$. By computer search it was found that 288 words can be added, yielding $A(25,8,10)\geq 960+288=1248$.

$A(25,8,11)\geq 1662=1218+444$ is similarly obtained from a translate of $\mathscr{G}$ containing 640 words (B_3^9) of weight 9 and 1218 words (B_3^{11}) of weight 11. We first take the 1218 words $B_3^{11}0$, and look for a subset of the $640\times 15=9600$ vectors $u1$ that can be adjoined, where u is obtained by complementing any 0 in a vector of B_3^9. If we take the 400 out of the 640 with a 0 in a particular coordinate we get $A(25,8,11)\geq 1218+400=1618\ddagger$. Again we can do better by using a subset of the 640 that have at most a single 1 in a particular set of three coordinates. By averaging we find that at least $640\times 0.6917\cdots = 442.68\cdots$ (hence 443) words can be added in this way. A particular choice of three coordinates gives 444, so $A(25,8,11)\geq 1218+444=1662$.

$A(26,8,10)\geq 1519=759+760$ is found by starting with the 759 words $B_0^8 11$, and looking for a subset of the vectors $u00$ to adjoin, where u is obtained by complementing any two 1's in a word of B_0^{12}. We take the $120+4\times 160=760$ words of B_0^{12} that have at most a single 0 in a set of four coordinates (see Fig. 2.15 of [132]), and replace 1111 by 0011, 0111 by 0001, 1011 by 1000, 1101 by 0100 and 1110 by 0010.

$A(26,8,12)\geq 3026\ddagger=2576+450$ is obtained in a similar way from $B_0^{12}00$ and $B_0^8 11$. There are $130+4\times 80=450$ words in B_0^8 with at most a single 1 in a set of four coordinates (see Fig. 2.14 of [132]), and we now apply the complementary transformation to the previous one (replacing 0000 by 1100, etc.). This construction can be improved as follows. We use the lexicographic version of $\mathscr{G}$, so that the octad $11\cdots 100\cdots 0\in\mathscr{G}$. There are 256 octads in $\mathscr{G}$ of the form $\{1000\}\{1000\}y$ (with each 1 in any of four positions). For each of these we form the vector $(1100)(1100)y11$, and adjoin these vectors to $B_0^{12}00$. This extends by "minimal degree lexicography" (see Section XII) to give $A(26,8,12)\geq 3070$.

$A(26,8,13)\geq 3328=2576+752$ is found by starting with the 2576 words $B_0^{12}01$. There are 35420 vectors of weight 13 at distance 8 from this set; they have the form $u10$, where u is the union of three words of weight 8 in $\mathscr{G}$ all at mutual distance 8 (see [45], Fig. 10.1, [46]). By computer it was found that 752 of these vectors can be adjoined to the 2576.

$A(27,8,12)\geq 3146\ddagger=2576+210+3\times 120$ is obtained from $B_0^{12}000$ and $B_0^8 111$, modifying the vectors in B_0^8 that have at most a single 1 in a set of three coordinates.

X. Codes from Permutation Groups

The codes in this section are unions of orbits under a nontrivial permutation group. Let G be a permutation group permuting the symbols $\{1,\cdots,n\}$. The *orbit* of a vector $x=(x_1,\cdots,x_n)$ under G is the set of all vectors $x^g=(x_{g(1)},\cdots,x_{g(n)})$, $g\in G$.

We first discuss groups generated by a single permutation π.

If π is a cycle of length n (equal to the length of the code), the code is a *cyclic code*, indicated by "c" in Table I. Orbit representatives are listed in Table XI.

If π is a cycle of length $n-1$, the code is an *extended cyclic* code, or "cyclic with a fixed point," indicated by "ec" in Table I. Orbit representatives are listed in Table XII.

If the permutation consists of a number of cycles of equal length the code is *quasi-cyclic* (see Table XIII). If there are i cycles of length n/i the code is indicated by "qi" in Table I.

The remaining codes defined by a single permutation ("polycyclic" codes) are listed in Table XIV, and indicated by "pc" in Table I.

The final table in this section (Table XV) lists codes that are defined by a group G (of order g) having more than one generator. These *group codes* are indicated by "g" in Table I.

The first column in Table XV gives the parameters of the code and [in brackets] the abstract type of G. The notation $q:r$ indicates that G is isomorphic to a group of permutations of $\mathbb{F}_q$ of the form $x\to ax+b$, where a belongs to the multiplicative subgroup of $\mathbb{F}_q^*$ of order r, and $b\in\mathbb{F}_q$. (The colon indicates a semidirect product as in [41].) $Q(16)$ is a generalized quaternion group ([97], p. 91).

The final column of Table XV gives orbit representatives for the code, written in hexadecimal and right justified, with the orbit size as superscript. For example the first orbit representative for $A(12,4,6)=132$ is BE^{55}, indicating that the vector

11	10	9	8	7	6	5	4	3	2	1
0	0	0	1	0	1	1	1	1	1	0

defines an orbit of size 55. The coordinates are numbered from right to left.

Of course the full automorphism group of a code constructed in this section may be much larger than the group we use to construct it. For example, the first code in Table XV has as automorphism group the Mathieu group M_{12} of order 95040.

TABLE XI
CYCLIC CODES

Bound	Other Representatives (in Hexadecimal)
$A(22,9) \geq 68$	0, C984F, 11BDB5, 284347, 3FFFFF (Ref. [105])
$A(25,10) \geq 151$	0, 33947, BC5D3, 1492D5, 23EEBF, 2D3ED3, 358D99
$A(8,4,3) = 8$	B
$A(13,4,3) = 26$	13, 85
$A(18,4,4) = 198$	17, 63, D1, 129, 303, 419, 445, 885, A09, 1089, 1421
$A(19,4,3) = 57$	43, 89, 405
$A(25,4,3) = 100$	D, 841, 2201, 8101
$A(26,4,3) = 104$	D, 441, 4201, 8101
$A(11,6,5) = 11$	97
$A(12,6,5) = 12$	97
$A(13,6,4) = 13$	B1
$A(13,6,6) = 26$	1AB, 279
$A(14,6,4) = 14$	53
$A(14,6,6) = 42$	BB, 4C7, 52D
$A(15,6,4) = 15$	8B
$A(19,6,5) \geq 76$	A7, 1503, 420B, 8449 (Ref. [108])
$A(20,6,5) \geq 84$	3043, 11111, 14025, 20017, 40883 (Ref. [108])
$A(21,6,5) \geq 105$	343, 1017, 21049, 28083, 40423 (Ref. [108])
$A(26,6,4) = 52$	20B, 10811
$A(27,6,4) = 54$	883, 4025
$A(15,8,7) = 15$	537
$A(16,8,7) = 16$	112F
$A(17,8,8) = 34$	B9D, 2DA3
$A(21,8,5) = 21$	985
$A(23,8,5) = 23$	410B
$A(24,8,5) = 24$	20B1
$A(26,8,6) = 130$	68B, 20139, 49015, 81843, 110A11
$A(19,10,9) = 19$	5793
$A(20,10,9) \geq 20$	1129F
$A(21,10,8) \geq 21$	1112F
$A(22,10,11) \geq 46$	12E6F, 3ED19, 155555
$A(24,10,7) \geq 24$	12E11
$A(24,10,9) \geq 56$	13A35, 84537, B0B0B
$A(24,10,10) \geq 72$	5348F, 85DC9, 88CB7
$A(24,10,12) \geq 96$	4BE2F, 519F7, 1A4EE5, 1DAC99
$A(25,10,10) \geq 100$	7B621, 8591F, 9AA4D, 151867
$A(25,10,11) \geq 125$	1D4B7, 42F37, B63A5, D954D, 1CF223
$A(26,10,10) \geq 130$	7B20B, 8165F, 1C7131, 235499, 654A49
$A(27,10,9) \geq 111$	19535, 85A2D, 1518E1, 923245, 1249249
$A(27,10,12) \geq 252$‡	5AE4F, 1322BF, 2A0EEB, 3A55C5, 43F195, 4668DB, 5AB943, 62DE51, 9493CD, 9C4E27
$A(23,12,11) = 23$	299AF
$A(24,12,10) = 24$	DE245
$A(24,12,11) \geq 24$	A65F1
$A(26,12,9) \geq 26$	289CB
$A(27,12,11) \geq 54$	2CC789, 42FA23
$A(28,12,12) \geq 84$	11D5E3, 532679, A17A1B
$A(28,14,12) \geq 28$	8C97C5
$A(28,14,13) \geq 28$	A2993F

TABLE XII
EXTENDED CYCLIC (OR "CYCLIC WITH A FIXED POINT") CODES

Bound	Orbit Representatives
$A(17,4,4) \geq 156$	(0000000000010111)0, (0000000001100011)0,
	(0000000110000011)0, (0000000000011001)1,
	(0000001000100101)0, (0000000101000101)0,
	(0000010000101001)0, (0000000010100001)1,
	(0000100100100001)0, (0000100100001001)0,
	(0001000100010001)0
$A(17,8,7) = 24$	(1110000100010110)0, (1011000010110000)1
$A(22,10,10) \geq 42$	(100100100011110101000)1, (101000111001101110000)0
$A(25,10,7) \geq 28$	(000000000100100010100111)0, (000100010001000100010001)1
$A(25,10,8) \geq 48$	(100010001010100000110100)0, (000000000110010010100011)1
$A(25,10,9) \geq 72$	(000000101000111100101001)0, (000001000110010000110111)0,
	(000000001101001100100010 1)1
$A(25,10,12) \geq 130$	(000001010101101111100011)0, (000001001110011101011011)0,
	(000010000111110110001011)1, (000100101100111010101001)1,
	(000001111010100010001111)1, (001001110010011100100111)0,
	(010101010101010101010101)0
$A(27,10,11) \geq 208$‡	(00001000110101010100100101)1, (00000011110011000010011011)0,
	(00010011001011010100010101)0, (00000100111001000111000011)1,
	(00000100001101010010011 0111)0, (00000010000111010011100111)0,
	(00000010001000001011011111)1, (00000001001000111110101011)0
$A(28,10,10) \geq 192$‡	(000000100101000101110010001)1, (000000001010100111010000110 1)0
	(000000101100101001001010001 1)0, (000000000100001111010111000 1)0,
	(000000010111000010001110001)1, (000000000111001001001100110 1)0,
	(000001101011000100001010101)0, (001001001001001001001001001)1
$A(28,10,11) \geq 270$‡	(000010001101000011100010111)0, (000000000011011010001110011)1,
	(000001000010001010100111101 1)0, (000100101010010100100010101)1,
	(000010010001100010111011001)0, (000001101000110010011010010 1)0,
	(000001010100001000011010111 1)1, (000010001110010000111110001)0,
	(000000110000110111001001001)1, (000000100101000010011101111)0
$A(25,12,11) \geq 36$	(001101000000110010101111)0, (000100101110000100101110)1
$A(27,12,10) \geq 39$	(00100000101110010000010111)0, (00000000010100010011100101 1)1

TABLE XIII
QUASI-CYCLIC CODES

Bound	Orbit Representatives
$A(16,8,6) = 16$	(00110101)(00011000), (00011000)(01010011)
$A(18,8,6) = 21$	(110100)(100000)(110000), (000010)(110100)(100001),
	(000011)(100001)(010100),(010101)(010101)(000000),
	(000000)(000000)(111111)
$A(18,10,8) = 9$	(000100111)(000101101)
$A(20,10,7) = 10$	(0000101011)(0000010011)
$A(26,10,6) = 13$	(0000000011001)(0000010100001)
$A(27,10,6) = 14$	(010)(000)(000)(001)(001)(010)(101)(000)(000),
	(010)(000)(000)(100)(000)(100)(000)(101)(010),
	(000)(010)(100)(000)(000)(001)(001)(000)(011),
	(000)(001)(100)(000)(110)(010)(000)(001)(000),
	(111)(111)(000)(000)(000)(000)(000)(000)(000),
	(000)(000)(111)(111)(000)(000)(000)(000)(000) (Ref. [105])
$A(27,10,7) \geq 36$	(000010111)(000000100)(010000100),
	(000100011)(001010000)(000110000),
	(000000001)(001000001)(100001110),
	(000001001)(011100100)(000000001)
$A(28,10,7) \geq 37$	(0000011)(0000010)(1000010)(1000001),
	(0000011)(0100000)(0011000)(0100100),
	(0000000)(0000011)(0001001)(0001101),
	(0001001)(1001000)(0000001)(0011000),
	(0001001)(0100101)(0000110)(0000000),
	(1111111)(0000000)(0000000)(0000000),
	(0000000)(0000000)(1111111)(0000000)
$A(25,12,8) = 10$	(11000)(11000)(10100)(10100)(00000),(00100)(00001)(11000)(00100)(11010)
$A(26,12,11) \geq 39$	(0001001111011)(0001101010000), (0001110001100)(0011000101011),
	(0011011001000)(0000111100101)
$A(27,12,8) = 15$	(001)(000)(010)(000)(000)(001)(010)(101)(101),
	(100)(010)(010)(001)(000)(110)(110)(000)(000),
	(011)(000)(010)(001)(101)(000)(001)(000)(010),
	(001)(011)(100)(101)(000)(000)(000)(010)(100),
	(000)(011)(000)(000)(011)(001)(100)(100)(010) (Ref. [105])
$A(26,14,12) = 13$	(0001001001111)(0001001010111)
$A(27,14,12) \geq 19$	(000)(000)(111)(111)(111)(000)(000)(000)111,
	(110)(110)(100)(110)(100)(000)(100)(110)100,
	(011)(101)(100)(001)(110)(110)(000)(100)010,
	(011)(110)(011)(100)(100)(100)(011)(000)001,
	(100)(100)(000)(011)(100)(010)(011)(011)011,
	(010)(100)(001)(000)(011)(110)(100)(011)101,
	(010)(001)(101)(100)(000)(011)(011)(010)110
$A(28,14,11) = 21$	(1000000)(0011101)(0110001)(1101000), (1010001)(1000000)(1001110)(1101000),
	(0101110)(0100011)(1000000)(1101000)

A number of different computer programs were used to find the group-invariant codes described in this section. The following seems to be the most efficient method. Given a permutation group G, we first find its orbits on the 0-element subset of the coordinates (there is only one!). Given representatives for the i-element subsets, we extend these in all possible ways by a singleton and find among the vectors thus obtained the (lexicographically minimal) representatives for the orbits on $(i+1)$-subsets. At the same time this tells us how often each type of i-subset occurs in an $(i+1)$-subset of given type. This process is continued until representatives for the i-element subsets with $i \leq w$ have been found.

For $t = w - d/2 + 1$, we form a matrix B indexed by orbits of w-sets and t-sets, specifying how often a t-set is covered by the vectors in a given orbit of w-sets. Orbits of w-sets for which the corresponding row of B contains an entry greater than 1 can be discarded.

We now define a graph on the remaining w-set orbits, joining two of them when they do not cover the same t-set, i.e., when the corresponding rows are orthogonal. The largest codes invariant under G are obtained as the largest weighted cliques in this graph, where the weights are the orbit sizes. This method requires enough space to store the $T_w \times T_t$ (0,1)-matrix B, where T_i is the number of orbits on i-sets.

XI. MISCELLANEOUS CONSTRUCTIONS

In this section we give some isolated constructions that do not fit into any other category.

The group code $\mathscr{C}$ showing that $A(16,4,8) \geq 1164$‡ (see Table XV) leads to three other good codes.

$A(16,4,8) \geq 1170$. In $\mathscr{C}$, replace the ten words FF, AF5, 11EE, 24DB, 7D82, BE41, C03F, CF30, F30C, FF00 by the sixteen words 1EF, 2F7, 4DF, 8FD, 10FE, 20FB, 07F, 7F02, 80BF, BF01, DF10, EF20, F704, FB08, FD80, E40.

$A(14,4,6) \geq 276$‡. Shorten $\mathscr{C}$ by taking the 275 words with a 1 in the first (i.e., left-most) and tenth coordinates, and adjoin 58B.

$A(14,4,7) \geq 317$‡. Shorten $\mathscr{C}$ by taking the 314 words beginning 10, and adjoin BF, B4B, 3F80.

$A(22,6,4) = 37$. Take a Kirkman triple system of order 15 ([14], [30]), i.e., a Steiner system $S(2,3,15)$ in which the 35 blocks are partitioned into seven "parallel classes"

TABLE XIV
OTHER CODES GENERATED BY A SINGLE PERMUTATION, AND RELATED CODES

A(11,4,4) = 35	(001000)(011)(10), (110001)(100)(00), (101000)(100)(01), (000011)(100)(01), (100100)(011)(00), (011011)(000)(00), (100100)(000)(11), (101010)(000)(10)
A(17,8,6) = 17	(111010000000)(100)(10), (100100100100)(011)(00), (101010101010)(000)(00)
A(21,10,9) ≥ 27	(1001)(0011)(1010)(0100)00101, (0001)(0101)(0011)(0011)00011, (1010)(0010)(0011)(0011)01100, (0110)(0011)(0001)(0101)10001, (0011)(0110)(1000)(0101)01010, (0010)(0101)(0110)(1001)11000, (1111)(0000)(1111)(0000)00010, (00)(00)(00)(11)(00)(00)(11)(11)10110, (00)(11)(00)(00)(11)(11)(00)(00)10110 (Ref. [105])
A(22,10,7) = 16	(11100)(10010)(10000)(10000)00, (10000)(10000)(00110)(01001)10, (10000)(10000)(01001)(00110)01, (00000)(11111)(00000)(00000)11
A(22,10,9) ≥ 35	(0001011)(1001000)(1001010)1, (0001001)(0100001)(1111100)0, (0000011)(0110011)(0100011)0, (0001101)(0010111)(1000000)1, (0011101)(1100000)(0010011)0
A(23,10,8) ≥ 33	(11100100000)(00101000100)1, (10100001000)(10100100011)0, (11100010010)(00000011001)0 (Ref [105])
A(23,10,9) ≥ 45	(111101001000100010000)10, (111010001100001000010)01, (100100100100100100100)11 (Ref. [80])
A(23,10,10) ≥ 54	(0000001011)(0001010011)111, (0011011101)(0101100001)000, (0001101001)(0000111101)100, (0001001011)(1111000100)010, (0000111001)(0110010110)001, (0101010101)(1010101010)000, (0000000000)(1111111111)000, then adjoin 3DE003
A(23,10,11) ≥ 63	(000001010110011101111)00, (000011111010110001001)01, (000101001111100100011)10
A(24,10,8) ≥ 38	00000000(11111111)(00000000), 00000000(00000000)(11111111), 111(1000)0(10001000)(10001000), 100(1000)0(00110100)(01000011), plus first 4 cyclic shifts right of each of 010(00110)(10001000)(01000011), 010(01001)(10001000)(00110100), 001(00110)(01000011)(10001000), 001(01001)(00110100)(10001000), 000(00110)(00110100)(00110100), 000(01001)(01000011)(01000011)
A(24,10,11) ≥ 90	(0010101100110100110101)(00), (0000100110001010011101)(11), (0000011101011000001111)(10), (0000010110111101100001)(10), (0101010101010101010101)(00)
A(26,10,8) ≥ 54	(000011101001)(000100000001)10, (000000000011)(100100101011)00, (000010000101)(000100110100)01, (000001001101)(100001000010)01, (000101000101)(011000011000)00
A(26,10,9) ≥ 84	(000000001000001001111011)(01), (000000001000101110000111)(10), (000010010110001101000101)(00), (000010001101000010001101)(01)
A(26,10,11) ≥ 168	(000001101000010111011001)(10), (000100110100100110010111)(00), (000001100010101111100101)(00), (000001011000110000011111)(01), (000010010101110001110001)(01), (000000110010101001010011)(11), (000001010100001100111101)(10)
A(27,10,8) ≥ 66	(001001010111)(100000000001)(000), (000000001111)(000101000100)(010), (000000100101)(100001001010)(100), (000010001001)(100010000001)(110), (000000010001)(100011110100)(000), (000011000011)(001000001000)(011)
A(27,10,10) ≥ 159	(000000100001001111011101)(000), (000001010011010010100101)(100), (000010001010001000011111)(001), (000001010100100010010111)(001), (000001101000101100100011)(001), (000000000110101100011001)(110), (000000110011000000110011)(011), (001001001001001001001001)(110)
A(28,10,8) ≥ 78	(00000000001010001101100001)01, (00000000100001000100100111)10, (00000001001001000011010101)00
A(28,10,9) ≥ 132	(000000000100001100100111)(0101), (000010010100011001001001)(0100), (000001100010101010000101)(0001), (000000000010110000101011)(1100), (000000101111010000110001)(0000), (000001011001000001011001)(0010)
A(22,12,9) = 8	(11010000)(11010000)(1100)(10)
A(25,12,9) = 25	(000000101101)(001100)(1100)(10)0, (000011000011)(100100)(1010)(00)1, (001100110011)(010101)(0000)(00)0, (010101010101)(000000)(0000)(11)1, (000000000000)(111111)(0000)(11)1
A(25,12,10) ≥ 28	(000000101101)(100001)(0011)(01)1, (000111000111)(100100)(1010)(00)0, (001100110011)(101010)(0000)(00)1, (001001001001)(110110)(0000)(11)0, (010101010101)(000000)(0101)(11)0, (000000000000)(111111)(1111)(00)0
A(26,12,10) ≥ 30	(111100010010001010000110)00, every 4th shift of (011100000000010000111010)11
A(28,12,8) ≥ 19	1100000000(100)(100)(010)(001)(001)(010), 0011000000(100)(010)(100)(010)(001)(001), 0000110000(100)(001)(010)(100)(010)(001), 0000001100(100)(001)(001)(010)(100)(010), 0000000011(100)(010)(001)(001)(010)(100), 1010101010(111)(000)(000)(000)(000)(000), 1001010101(000)(111)(000)(000)(000)(000), 0110100101(000)(000)(111)(000)(000)(000), 0101011010(000)(000)(000)(111)(000)(000) (Ref [80])
A(28,12,11) ≥ 63	(001001001001001001001001)(111)0, (000010111001000010111001)(100)0, (000010100100111101010001)(100)0, (000000010001000110100111)(011)1
A(28,12,13) ≥ 96	(000101100101001110011001)(011)0, (000000110011010101101111)(000)1, (000011100011110100010101)(110)1, (000001011111100011101001)(001)0

each containing five disjoint blocks. Add one further point "at infinity" to each parallel class, yielding 35 words of length 22 and weight 4, and adjoin $A(7,6,4)=2$ words on the 7 extra points.

$A(22,6,5) \geq 132$. In the weight 5 words of the [11,6,5] ternary Golay code, replace 0 by 00, 1 by 01 and 2 by 10.

$A(28,8,5)=33$. Start with the affine plane $AG(2,5)$, containing 30 5-sets (the lines) on 25 points, and adjoin three points X_1, X_2, X_3. Choose three noncollinear points P_1, P_2, P_3, and in the line P_1P_2 replace P_1 by X_3, in the line P_2P_3 replace P_2 by X_3, in the line P_3P_1 replace P_3 by X_3, and adjoin the 5-set $P_1P_2P_3X_1X_2$. Repeat this with three further noncollinear points Q_1, Q_2, Q_3 (replacing Q_1, Q_2, Q_3 in the three lines by X_2 and adjoining $Q_1Q_2Q_3X_1X_3$), and again with three noncollinear points R_1, R_2, R_3, making sure that $\{P_1, P_2, P_3, Q_1, Q_2, Q_3\}$, $\{P_1, P_2, P_3, R_1, R_2, R_3\}$ and $\{Q_1, Q_2, Q_3, R_1, R_2, R_3\}$ are conics in the affine plane. The final code is shown in Fig. 1.

$A(20,10,8)=17$ is constructed in Fig. 2.

$A(28,10,6)=16$ follows by shortening the Steiner system $S(2,6,31)=PG(2,5)$.

Finally Kaikkonen [105] observed that if n is even and $d'=\min\{n,2d\}$ then

$$A(2n,d',n) \geq A(n,d) + A\left(n,d,\frac{n}{2}\right).$$

This is obtained by replacing 0 by 01 and 1 by 10 in the code attaining $A(n,d)$, and 0 by 00 and 1 by 11 in the code attaining $A(n,d,n/2)$. For example $A(28,12,14) \geq A(14,6)+A(14,6,7)=106$. Many generalizations are possible, for example using ternary codes, but do not seem to lead to new records in the range of our tables.

XII. SEARCHING FOR CODES WITH A COMPUTER; CODES WITH NO KNOWN STRUCTURE

In preparing Table I we made use of several computer programs that searched for codes. Two kinds of programs were used, exhaustive search methods and heuristic (nonexhaustive) methods. In discussing running times, besides the usual variables n, d, w and M (the number of codewords), we use U to denote $\binom{n}{w}$, the size of the universe of possible codewords.

Exhaustive search methods: We explain our exhaustive search technique by describing the proof that $A(14,6,7) = 42$. A code with 42 words was constructed in Section III, so it suffices to show that no code exists with 43 words. In principle we must consider all possible subsets of 43 words from a universe of size $U = \binom{14}{7} = 3432$.

However, the size of this search space may be greatly reduced. First, from (5), any code attaining $A(14,6,7) = 43$ must contain a subcode C' with $n = 13$, $d = 6$, $w = 7$, $M = 22$, and C' must contain a subcode C'' with $n = 12$, $d = 6$, $w = 7$, $M = 11$. A previous exhaustive search has determined that there are precisely 95 inequivalent choices for C''. We may now restrict our search to codes that contain one of these codes as the first 11 words.

Next, we need not consider every subset of the weight-7 14-bit vectors as a possible code. Only sets of vectors with all pairwise distances ≥ 6 need be considered. All such sets may be generated by a standard backtracking algorithm, indeed in lexicographic order.

The search space may be further reduced by noticing that any M-word code C has $M!n!/|\mathrm{Aut}(C)|$ isomorphic versions (obtained by permuting the n coordinates and the M codewords). We wish our search to find exactly one (or at any rate very few) of the codes in each such equivalence class. The $M!$ factor is avoided by requiring that any code generated must be in lexicographic order—a condition readily incorporated into the backtracking algorithm. A large part of the $n!$ factor is automatically removed by the fact that the first 11 words form one of the subcodes C'' previously mentioned, and the first 22 words form a subcode C'.

More generally we may require all generated codes to be in "canonical form": namely lexicographically least under any permutation of coordinates and corresponding resorting of codewords, while preserving the property that the Johnson subcode $A(n-1,d,w)$ lies in the first $n-1$ coordinates and constitutes the first $\lceil (n-w)M/n \rceil$ words (and so on recursively for the subcodes of this code, ...).

Proving that a code is canonical is difficult, but some simple tests can readily show that a code is not canonical. If any departure from canonical form does occur during the backtracking, we may immediately prune that branch of the search.

The combination of all of these ideas made this initially intractable search problem solvable (in less than 18 minutes on an IBM 3090 model S computer, including the time to generate the 95 inequivalent codes C''), using a program written in a combination of Fortran and IBM-370 assembly language. The search tree contained about 90 million nodes, 309704 of them being codes C' with $n = 13$, $d = 6$, $w = 7$, $M = 22$ (we did not attempt to sort these into equivalence classes). Other exact values of $A(n,d,w)$ found in this way are given in Table III.

Heuristic search methods: For problems too large to be attacked by exhaustive search, we must be content with heuristic algorithms. Perhaps the most straightforward heuristic is simply to run the exhaustive search described above, or a variant of it, performing an incomplete search, and keep the best code found.

Often we are given a good partial code (obtained for example by shortening another code), and wish to complete it. Several heuristic methods are available. The lexicographically least code containing some given "seed" subcode is readily found in $O(UM)$ steps. A similar, but more powerful code-extension heuristic is "minimal-degree lexicography." Consider the H-vertex graph formed by the H holes in a partial code (two vertices being joined by an edge if the corresponding vectors differ in less than d places). Remove the lexicographically least vertex of minimal degree from this graph (as well as all its neighbors), and place it in the code; continue doing this (using the sequence of successively smaller graphs that arise) until no further augmentation is possible. This procedure takes $O(UM + H^2)$ time.

A simple probabilistic variant of these two methods uses "coin tossing." Namely, each time the extension heuristic could add a new vector to the current code, we toss a (possibly biased) coin, and add the vector only if the coin toss comes up "heads." Otherwise ("tails") we discard the vector and continue. Alternatively, deterministic skip-selection methods (such as a systematic backtrack search, or a greedy procedure) may be used.

Another kind of approximate optimization is based on "local search." We say that a code is "k-optimal" if its cardinality cannot be increased (while maintaining the minimal distance) by deleting k words and adding holes. Thus lexicodes (and other codes found by a greedy algorithm) are 0-optimal, while codes attaining $A(n,d,w)$ are k-optimal for all k. If k is small then an M-word code C may be tested for k-optimality (and if not k-optimal an improvement found) in $O(UM + \binom{M-1}{k-1}kT)$ time, where T is the size of the following bipartite graph.

This graph, which is constructed at the beginning of the search, has two sets of vertices, a blue vertex for every codeword of C and a red vertex for every vector not in C that "bites" (lies at distance less than d from) k or fewer words of C; an edge joins each such vector to the codewords that it "bites." Once this graph is constructed, no further hole-finding need be done.

We now consider all possible k-subsets S of C (the blue vertices), and see if their deletion will increase the size of C, using a trivial exhaustive search among the red vertices that are connected only to S.

The following "polishing" procedure was often able to improve even these k-optimal codes. We begin by constructing the bipartite graph described above, maintaining the red vertices (those not in the code) in a queue. If there is a red vertex h of degree 1 we exchange h with the codeword c to which it is connected, adding h to the code in place of c, which is placed at the end of the queue. When there are several choices for h we choose the one closest to the front of the queue, i.e., which has been out of the code longest. This procedure is then repeated. If after some fixed number (e.g., 1000) of itera-

TABLE XV
CODES DEFINED BY GROUPS WITH MORE THAN ONE GENERATOR, AND RELATED CODES

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(12,4,6) = 132$ [11:5]	55	(1,2,3,...,11), (2,5,6,10,4)(3,9,11,8,7)	$5F^{55}$, $83D^{55}$, ED^{11}, 897^{11}
$A(14,4,4) = 91$ [7:6]	42	(1,2,3,4,5,6,7)(8,9,10,11,12,13,14), (1,14)(2,12,5,13,3,10)(4,8,6,11,7,9)	186^{21}, 198^{21}, $1E0^{21}$, 17^{14}, $D1^{14}$
$A(14,4,5) \geq 169$ [13:12]	156	(1,2,3,...,13), (1,2,4,8,3,6,12,11,9,5,10,7)	$5D^{78}$, 2053^{52}, $6B^{39}$
$A(14,4,7) \geq 316$‡ [7:6]	42	(1,2,3,4,5,6,7)(8,9,10,11,12,13,14), (1,3,2,6,4,5)(8,10,9,13,11,12)	$19F^{42}$, $3B9^{42}$, $5D5^{42}$, 793^{42}, $7A5^{42}$, BAC^{42}, $FB0^{42}$, $5CB^{14}$, FE^{7}, $3F80^{1}$
$A(15,4,5) \geq 234$‡	42	same group as above then adjoin vectors 1F, 67, 79	$19A^{42}$, $1A6^{42}$, $418C^{42}$, $1D4^{21}$, 385^{21}, $7A0^{21}$, 4099^{21}, 4382^{21},
$A(15,4,6) \geq 382$‡	42	same group as above then adjoin 3F, FA0, 1790, 1B88, 1D84, 1E82, 1F40, 2F01, 6780, 7980, 7E00	$18F^{42}$, 399^{42}, $3AA^{42}$, $5C6^{42}$, 783^{42}, 4394^{42}, 4585^{42}, $40BC^{21}$, 4193^{21}, $43C8^{21}$, $5B4^{14}$,
$A(16,4,6) \geq 592$‡ [2^4:5]	80	(1,15,7,5,12)(2,9,13,14,8)(3,6,10,11,4), (1,16)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15)	BB^{80}, $1F1^{80}$, $2BC^{80}$, 297^{80}, $30F^{80}$, $94E^{80}$, $6F^{40}$, 378^{40}, $2CE^{16}$, 365^{16}
$A(16,4,8) \geq 1164$‡ [$PSL(2,7)$]	168	(1,2,3,4,5,6,7)(9,10,11,12,13,14,15), (2,3,5)(4,7,6)(10,11,13)(12,15,14), (1,8)(2,7)(3,4)(5,6)(9,16)(10,15)(11,12)(13,14)	$75D^{168}$, $76B^{168}$, $F78^{168}$, $1F23^{168}$, $1F2C^{168}$, $173A^{84}$, 737^{56}, $1F15^{56}$, $F0F^{42}$, $3FC^{28}$, $3FC0^{28}$, $17E8^{14}$, $1DE2^{14}$, $FF00^{1}$, FF^{1}
$A(18,4,5) \geq 516$‡ [3^2:8]	72	(1,5,9)(2,3,8)(4,7,6)(10,14,18)(11,12,17)(13,16,15), (1,2,3,4,5,6,7,8)(10,11,12,13,14,15,16,17) then adjoin 643, C86, 1A0C, 3418, 6814, C068, 14281, 1A090, 22111, 24422, 28844, 31088	$21B^{72}$, $64C^{72}$, $6A4^{72}$, $E14^{72}$, $E81^{72}$, $F02^{72}$, $1E40^{72}$,
$A(20,4,6) \geq 2280$ [$PSL(2,19)$]	3420	(1,2,3,...,19), (2,5,17,8,10,18,12,7,6)(3,9,14,15,19,16,4,13,11), (1,20)(2,19)(3,10)(4,7)(5,15)(6,16)(8,9)(11,18)(12,13)(14,17)	$5F^{1710}$, $F3^{570}$
$A(20,4,10) \geq 13452$	3420	same group as above	BDF^{3420}, FBD^{3420}, DFB^{1710}, $EF7^{1710}$, $12FF^{1710}$, $157F^{1140}$, $F6F^{342}$

tions the code has not been improved, the vector at the front of the queue is added to the code and its neighbors are removed and placed at the end of the queue (temporarily decreasing the size of the code).

In the range of our tables we are able to achieve k-optimality for values of k ranging 2–5. A less conservative attack would allow k-alterations for much larger values of k, considering only a small fraction of possible k-sets, without trying to achieve k-optimality. One such heuristic code-improver is the following.

1) Perform a permutation on the coordinates of the code.
2) Perturb every codeword by "pushing" it until it is lexicographically as small as possible.
3) Sort the (permuted and pushed) codewords into lexicographic order.
4) Remove the (lexicographically) last k words from the code, and attempt to replace them by more than k words, using some exhaustive or heuristic search method.
5) Go back to Step 1) (and repeat as many times as desired).

In this procedure one can use very large values of k, e.g., 20% of the codewords. This is one of a class of possible

TABLE XV (*Continued*)

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(24,4,6) = 7084$ [$PSL(2,23)$]	6072	(1,2,3,...,23), (2,3,5,9,17,10,19,14,4,7,13)(6,11,21,18,12,23,22,20,16,8,15), (1,24)(2,23)(3,12)(4,16)(5,18)(6,10)(7,20)(8,14). .(9,21)(11,17)(13,22)(15,19)	$6F^{3036}$, $1CD^{3036}$, 197^{1012}
$A(24,4,8) \geq 34914$	6072	same group as above	$5BD^{6072}$, $67D^{3036}$, $72F^{3036}$, $9DB^{3036}$, ADD^{3036}, $B67^{3036}$, $EE5^{3036}$, $F39^{3036}$, $FA3^{3036}$, $12BD^{3036}$, $149F^{759}$, $351B^{759}$
$A(28,4,6) \geq 15288$ [$PSL(2,13)$]	1092	(1,2,5,3,10,6,12,4,9,11,8,7,13). .(15,16,19,17,24,20,26,18,23,25,22,21,27), (1,3,5,7,9,11)(2,4,6,8,10,12)(15,17,19,21,23,25). .(16,18,20,22,24,26), (1,7)(2,6)(3,5)(8,12)(9,11)(13,14)(15,21)(16,20). .(17,19)(22,26)(23,25)(27,28)	$C01C^{1092}$, $C09A^{1092}$, $C0A6^{1092}$, $1C0B0^{1092}$, $2C112^{1092}$, $2C700^{1092}$, $3C041^{1092}$, $3C208^{1092}$, 4076^{546}, $412E^{546}$, $1C128^{546}$, $1C888^{546}$, $2C034^{546}$, $2C260^{546}$, $BC100^{546}$, $DC040^{546}$, $C033^{273}$, $C303^{273}$, $C330^{273}$, $AC00A^{273}$, $AC021^{273}$, $AD080^{273}$, $1C1C0^{182}$, $2E420^{182}$, DB^{91}, $6C000^{91}$
$A(17,6,5) = 68$ [$PSL(2,16)$]	4080	(1,6,13,5,4,2,15,10,14,12,3,9,7,11,8), (1,16)(2,3)(4,5)(6,7)(8,9)(10,11)(12,13)(14,15), (2,3)(4,9)(5,7)(6,8)(10,14)(11,13)(12,15)(16,17)	CE^{68}
$A(17,6,8) \geq 184$ [$Q(16)$]	16	(1,2,...,8)(9,10,...,16), (1,9,5,13)(2,16,6,12)(3,15,7,11)(4,14,8,10)	$195B^{16}$, $1C37^{16}$, $2D0F^{16}$, 5267^{16}, 6857^{16}, $8A4F^{16}$, $C41F^{16}$, 10337^{16}, $1492B^{16}$, 18547^{16}, $1A02F^{16}$, 6633^{4}, FF^{2}, 5555^{2}
$A(18,6,8) \geq 248$‡ [$2^4.2$]	32	(1,2)(3,4)(5,6)...(15,16), (1,3)(2,4)(5,7)(6,8)(9,11)(10,12)(13,15)(14,16), (1,5)(2,6)(3,7)(4,8)(9,13)(10,14)(11,15)(12,16), (1,9)(2,10)(3,11)(4,12)(5,13)(6,14)(7,15)(8,16), (3,4)(7,8)(9,13)(10,14)(11,16)(12,15)(17,18)	11593^{32}, $11A6C^{32}$, 31472^{32}, $147D^{16}$, $172B^{16}$, $17D4^{16}$, $1D47^{16}$, $1E1E^{16}$, $555A^{16}$, $3111E^{16}$, 31247^{16}, $11EE^{8}$, $12B7^{8}$, $333C^{8}$
$A(18,6,9) \geq 304$	384	(2,3)(6,7)(9,13)(10,15)(11,14)(12,16), (2,11,12)(3,16,14)(4,6,7)(5,13,9)(8,10,15), (2,5,3,8)(6,7)(9,16,15,11)(10,12,13,14)(17,18), (1,2)(3,4)...(17,18)	$103F3^{96}$, $11B1B^{96}$, $10F69^{48}$, $1FCA^{32}$, $307AC^{32}$
$A(19,6,9) \geq 504$ [$2^4.S_3$]	96	the generators for $A(18,6,8) \geq 248$ together with (2,3,4)(6,7,8)(10,11,12)(14,15,16)(17,18,19)	$3162E^{96}$, $103CF^{48}$, $11B27^{48}$, $11E4B^{48}$, 13369^{48}, 13535^{48}, $135CA^{48}$, 70356^{48}, $10F55^{24}$, $111BB^{24}$, $114D7^{24}$

optimizers that work on the principle of "cut a hole in the code, then refill it." The additional step of "pushing" codewords in order to artificially "expand" the hole can further increase the power of the heuristic.

The codes labeled "y," "ya," or "yd" in Table I were obtained by one or more of the previous methods. Those with fewer than 1500 words are listed in full in Table XVI. Most of these codes are at least 2-optimal. They are described in compressed notation, obtained as follows. The words are first sorted into lexicographic order, yielding a sequence of words $c_1, c_2, \cdots, c_M$ (say). The compressed notation for this code is $(\alpha_1, \alpha_2, \cdots, \alpha_M)$. To find α_i $(1 \leq i \leq M)$, let $u_1, u_2, \cdots$ be the list of all vectors, arranged in lexicographic order and following c_{i-1} in the

TABLE XV (*Continued*)

Bound [Group]	g	Generating Permutations	Orbit Representatives
$A(22,6,6) \geq 319$ [11:5]	55	(1,2,3,...,11)(12,13,14,...,22), (2,5,6,10,4)(3,9,11,8,7)(13,16,17,21,15)(14,20,22,19,18)	1925^{55}, $1B50^{55}$, 5868^{55}, 7881^{55}, $15D00^{55}$, $A3A^{11}$, $DC4^{11}$, $4B820^{11}$, 74804^{11}
$A(23,6,8) \geq 1265$‡ [23:22]	506	(1,2,3,...,23), (2,3,5,9,17,10,19,14,4,7,13)(12,23,22,20,16,8,15,6,11,21,18)	$A6F^{253}$, $2FC1^{253}$, $4B99^{253}$, 61E9[illegible], $B4A3^{253}$
$A(25,6,9) \geq 4100$‡ [$5^2 : 2A_4$]	600	(1,2,...,5)(6,...,10)(11,...,15)(16,...,20)(21,...,25), (2,6,25)(3,11,19)(4,16,13)(5,21,7)(8,10,20)(9,15,14)(12,24,22)(17,18,23), (2,11,20)(3,21,9)(4,6,23)(5,16,12)(7,8,18)(10,13,15)(14,25,24)(17,22,19)	$8CAF^{600}$, $A47B^{600}$, $EBC2^{600}$, $2CF6^{300}$, 9637^{200}, $9A9B^{200}$, $9D4B^{200}$, $C53D^{200}$, $C99E^{200}$, $CE4D^{200}$, $D4C7^{200}$, $D86E^{200}$, $1A8B3^{200}$, $2B869^{200}$
$A(25,6,10) \geq 5700$‡	600	same group as above	$A73E^{600}$, $B375^{600}$, $D2E7^{600}$, $9E2F^{600}$, $D733^{600}$, $19B96^{600}$, $D7F^{600}$, $11153D^{600}$, $1D5A6^{300}$, $1EAB2^{200}$, $11A176^{200}$, $2B969^{200}$
$A(25,6,11) \geq 7200$‡	600	same group as above	$B1EF^{600}$, $BE7A^{600}$, $CF1F^{600}$, $19D97^{600}$, $1A5BE^{600}$, $1C6AF^{600}$, $1D772^{600}$, $10B45F^{600}$, $119F2C^{600}$, $12274F^{600}$, $12CCB9^{600}$, $1086FB^{300}$, 113979^{300}
$A(28,6,7) = 4680$ [$PSL(2,27)$]	9828	(1,14,27)(2,10,4)(3,22,13)(5,19,8)(6,18,21)(7,12,11). .(9,16,26)(15,17,23)(20,24,25), (1,3,5,...,25)(2,4,6,...,26), (1,14)(2,13)(3,12)(4,11)(5,10)(6,9)(7,8)(15,26)(16,25). .(17,24)(18,23)(19,22)(20,21)(27,28)	$122F^{3276}$, $4F9^{1404}$
$A(28,12,10) \geq 48$ [3^2]	9	(1,2,3)(4,5,6)...(16,17,18), (4,5,6)(10,11,12)(13,15,14)(16,18,17)(19,21,20)(22,24,23)	$90C264C^{9}$, $911291A^{9}$, $9128A25^{9}$, 9249143^{9}, $E001C0F^{1}$, $1F8000F^{1}$, $007007F^{1}$, $000E38F^{1}$, $FC00388^{1}$, $038FC08^{1}$, $E07E004^{1}$, $0001FF4^{1}$, $03F0382^{1}$, $1C0E072^{1}$, $E380071^{1}$, $1C71C01^{1}$

lexicographic order, that have distance $\geq d$ from the subcode $\{c_1,\cdots,c_{i-1}\}$. If $c_i = u_r$ we set $\alpha_i = r-1$. Informally, given $c_1,\cdots,c_{i-1}$, we must skip α_i lexicographic words to get c_i. Koschnick [111], who independently discovered this compressed notation, refers to $(\alpha_1,\cdots,\alpha_M)$ as a *skip-vector* for the code.

For example the skip-vector for a lexicographic code itself is simply $0,0,\cdots,0$ (M times), which we abbreviate to 0^M.

The algorithm for decompressing this notation is equally simple. To recover a code $\{c_1,\cdots,c_M\}$, showing that $A(n,d,w) \geq M$, first form the sequence $\alpha_1,\cdots,\alpha_M$ by expanding each symbol a^k to $a,a,\cdots,a$ (k times). Then c_i $(1 \leq i \leq M)$ is the (α_i+1)st vector c in lexicographic order starting at c_{i-1} such that $wt(c) = w$ and the distance from c to $\{c_1,\cdots,c_{i-1}\}$ is at least d.

In a sense the codes in Table XVI are our failures. At least one of the authors (NJAS) believes that every value of $A(n,d,w)$ in the range of our tables should be attained by a code with some mathematical structure. Experience has shown that sooner or later most random codes in this range are superseded. We hope this will happen to the codes in Table XVI.

Errata in Earlier Works

In [13], Table II-D, $A(16,10,7) = A(16,10,9) = 4$ (not 3). In Table III-A, $T(1,2,7,16,10) = 8$ (not ≤ 6), $T(1,3,7,16,10) = 12$ (not ≤ 9), $T(1,4,7,16,10) = 16$ (not ≤ 12), $T(1,5,7,16,10) = 16$ (not ≤ 15). In Table III-D, $T(2,4,7,16,10) \geq 19$ (not ≤ 18). We do not at present know how these errors affect the upper bounds in [13] (nor papers such as [180] that make use of these bounds). Until further checks are made, all the upper bounds in [13] for codes with $d = 10$ obtained by linear programming should be regarded with suspicion. We have also been

01111	00000	00000	00000	00000	001
00000	11111	00000	00000	00000	000
00000	00000	11011	00000	00000	100
00000	00000	00000	11111	00000	000
00000	00000	00000	00000	10111	010
10000	00000	10000	10000	10000	001
01000	01000	01000	01000	01000	000
00100	00100	00100	00100	00100	000
00010	00010	00000	00010	00010	100
00001	00001	00001	00001	00001	000
10000	01000	00100	00010	00001	000
01000	00100	00010	00001	10000	000
00100	00010	00001	10000	01000	000
00010	00001	10000	01000	00100	000
00001	10000	01000	00100	00010	000
10000	00100	00001	01000	00010	000
01000	00010	10000	00100	00001	000
00100	00001	01000	00010	10000	000
00010	10000	00100	00001	01000	000
00001	01000	00010	10000	00100	000
10000	00010	01000	00001	00100	000
01000	00001	00100	10000	00010	000
00100	10000	00010	01000	00001	000
00010	01000	00001	00100	10000	000
00001	00000	10000	00010	01000	010
10000	00001	00010	00100	01000	000
00000	10000	00001	00010	00100	001
00100	01000	10000	00001	00010	000
00010	00100	01000	10000	00000	010
00001	00000	00100	01000	10000	100
11000	10000	00000	00000	00000	110
00000	00100	00000	00000	01001	101
00000	00010	00110	00000	00000	011

Fig. 1. $A(28,8,5)=33$.

unable to recover the "miscellaneous construction" of [13] that produced $A(18,6,6)\geq 144$, and in the present paper have replaced it by $A(18,6,6)\geq 132$. In [13], p. 89, col. 2, the words "equivalent to determining $D(t,k,v)$, where" are illegible in some copies. In (5), line 5, change 197 to 297.

In [44, (5)], the range is $a'<a, b'<b$. On page 348 the last line of Table XIII should read $A(24,10,12)\geq 80$.

In [45], p. 141, caption to Fig. 5.1, the length of code is 11 (not 10).

In [50], p. 12, line 21, $|\Gamma_{12}|=112952$ (not $|\Gamma_{10}|$).

In [72], p. 40, $T(1,6,6,15,10)=8$ (not 7). On p. 40, column 1, line 3, change 554 to 553. In Table IV, $A(16,10,7)=A(16,10,9)=4$ (not 3).

In [147], $A_9^2=56$ (not 99).

In [150], Table 2, $A(19,4,5)\geq 644$ (not 664).

Acknowledgment

We thank several correspondents who kindly sent us codes they had discovered. Since some codes were found independently by several people, and others have been transformed or "beautified" (and so do not appear in the final table in their original form) it is appropriate to record these constructions here. To keep this list to manageable size we mention only codes that are as good as the current record, were unpublished at the time of writing, and were found since the publication of [13].

1000	1000	1100	0101	1001
1000	0100	0011	1010	1001
1000	0010	0011	0101	0110
1000	0001	1100	1010	0110
0100	0110	1000	1100	0101
0100	0110	0100	0011	1010
0100	1001	0010	0011	0101
0100	1001	0001	1100	1010
0010	1010	0110	1000	1100
0010	0101	0110	0100	0011
0010	1010	1001	0010	0011
0010	0101	1001	0001	1100
0001	0011	1010	0110	1000
0001	1100	0101	0110	0100
0001	1100	1010	1001	0010
0001	0011	0101	1001	0001
1111	1111	0000	0000	0000

Fig. 2. $A(20,10,8)=17$.

T. Etzion and C. L. M. van Pul [68] showed $A(17,4,6)\geq 854$,

H. Hämäläinen [80] showed $A(18,6,9)\geq 304$, $A(19,6,9)\geq 504$, $A(23,10,9)\geq 45$, $A(25,10,8)\geq 48$, $A(26,14,11)=10$, $A(27,14,11)=13$, $A(28,12,8)\geq 19$, $A(28,12,10)\geq 48$.

I. Honkala [91] showed $A(22,12,9)=8$, $A(25,12,8)=10$. I. Honkala [93] showed $A(26,14,9)=6$, $A(26,14,10)=8$, $A(28,14,9)=7$.

M. K. Kaikkonen [105] showed $A(22,9)\geq 68$, $A(25,8,9)\geq 829$, $A(21,10,9)\geq 27$, $A(23,10,8)\geq 33$, $A(24,10,9)\geq 56$, $A(25,10,6)\geq 10$, $A(25,10,8)\geq 48$, $A(26,10,6)\geq 13$, $A(27,10,6)\geq 14$, $A(25,12,8)\geq 10$, $A(28,12,14)\geq 106$, $A(26,14,10)=8$, $A(26,14,11)=10$, $A(27,14,11)=13$, $A(28,14,9)=7$, $A(28,14,11)\geq 21$,

C. L. M. van Pul [149] showed $A(17,4,5)\geq 424$, $A(17,4,6)\geq 854$, $A(19,6,7)\geq 338$, $A(19,6,8)\geq 408$, $A(21,6,7)\geq 570$, $A(17,8,8)=34$, $A(18,8,6)=21$, $A(22,10,7)\geq 16$, $A(22,10,8)\geq 24$, C. L. M. van Pul [150] showed $A(17,6,7)\geq 166$, $A(18,6,7)\geq 243$, and

S. Rankinen [151] showed $A(20,6,8)\geq 588$.

After this paper was submitted we received a preprint by K.-U. Koschnick [111], which independently showed $A(12,4,5)\geq 80$, $A(18,8,6)=21$, $A(22,10,7)\geq 16$, and improved one of our entries by showing $A(23,10,7)\geq 20$. A code with the latter parameters and equivalent to Koschnick's is given in Table XVI.

Other codes were found by Chen, Jin, and Fan [34], Cheswick [35], Chung and Kumar [37], Darwish and Bose [49], Dueck and Scheuer [63], Lin [123], Zaptcioglu [184] and Zinoviev and Litsyn [188]. We thank all of these correspondents.

David Johnson kindly allowed us to use his simulated annealing graph coloring programs [98]. We also thank John Conway for many helpful discussions, Iiro Honkala, Heikki Hämäläinen and Markku Kaikkonen for informing us of a large number of codes that they had found, and Tuvi Etzion and Kevin Phelps for several helpful comments. Tuvi Etzion also helped us find some of the partitions in Table VI (cf. [65]). Aaron Grosky and Ralph Knag provided valuable assistance in running our programs at Bell Labs.

The automorphism groups of certain codes in this paper were computed using B. D. McKay's graph-automor-

TABLE XVI
CODES WITH NO KNOWN STRUCTURE, DESCRIBED BY SKIP-VECTORS

$A(12,4,5) \geq 80$

$0^3,1^2,3,1,0,3,2,0,2^3,4,0,2,0^2,1,0^9,1,0^6,1,0,2,0,1,0^{10},1,0^5,1^2,0^2,1,0,1,0^{16}$

$A(13,4,5) \geq 123$

$0^2,2,11,0,1^2,4,2,0,1,0,6,1,4,3,0^2,4,0,1,0^2,2,4,0,4,0,1,0^7,1,0^7,2,0^2,2,3,0,1,0^3,4,0,2,0^5,1,0^6,1,0^5,1^2,$
$0^9,1,0^3,1,0^6,1,0^{25}$

$A(13,4,6) \geq 166$

$0^3,1^2,3^2,1,0^2,1,0,1,2,1,2,1,0,1,0^4,2,3,0^3,1,3,0^2,1,0^2,3,1,0^4,1,0^5,1^2,0,1^2,0,2,0,1,0^{12},1,0^{12},1,0^{13},1,0,$
$1,0^2,3,1,0^2,1,0^3,1,0^{16},1,0^8,1,0^2,1,3,2,0^9,1,0^{16}$

$A(14,4,6) \geq 278$

$0^6,1,3,0,5,0^3,1,2,1^2,0^6,2,3,2,0,1,0,3,0^5,1,0^{12},4,0^4,1,0^{10},2,0^7,4,1,0^{20},1^2,2,0^3,1^2,0,1^2,0^{13},2,0,1^2,0^3,$
$2,0^{10},1,0^6,1,0^{23},1,0^5,1,0^6,1,0^3,2,0^2,2,0,1,0^7,1,0^2,2,0^7,1,0,1,0^7,1,0^{20},1,0^8,1,0^2,1,0,3,0,1,0^{22}$

$A(14,4,7) \geq 325$

$0^5,6,7,0,1,0^2,4,7,2,3,0,4,2,1,0^5,3,0^2,1,0^2,2,0,3,1,0,1^2,2,3,0^4,1^2,0^6,1,0^2,1,0^4,1^2,0,1,0^2,1,0^8,1^2,5,0,$
$4,0,1,4,3,0,1,0,1,0,2,0^3,4,0^2,1,0,2,1,0,1,0^{20},1^2,0^5,1,0^{31},3,0,6,1,0^5,1,0^5,1,0^5,2,0^3,1^2,0^4,1,0^{11},1^2,0^4,$
$1,0^{40},1,0^3,1^2,0^{69}$

$A(15,4,10) \geq 237$ (has a shortening to **$A(14,4,10) = 91$**)

$0^7,1,0^3,3,2^3,0,1,0^2,1^2,0,1,0^3,2,0^4,5,4,0,2,0^4,1,0^6,1,2^2,1,0,2,0^5,1,0^{34},3^2,1,0^2,1^2,2,0,1^2,0,3,2,0,5,$
$1^2,0^2,1,0^4,1,0^4,2,0^5,1,0,3,0,2^2,0^4,1,0^7,1^4,2,0,1,0^3,1^2,0^2,1,0,2,0^2,2,0,1,0^6,1,0^5,2,0,1,0^2,1,0,1,0^2,$
$1^2,0^5,1,0^4,4,1,0^7,1,0^{25}$

$A(15,4,6) \geq 389$

$0^2,11,3,4^2,9,5,1,0,5,0,2,1^2,2^2,0,1,4,0,3,2,0^4,2,3,1,4,0,1,0,2,0,2,11,0,3,5,2,0^3,2,1^2,3,0^3,2,1,0^2,1,0^3,$
$1,0,1,2,0^2,2,1,0^3,1,4,0,4,1^2,5,2,0^3,9,0,2,0,1,0^2,6,0,3,0^6,4,0,1,0^3,1^2,2,0,1,0^3,1,0^5,1^2,0^5,1,0,1,0,1,$
$0^3,4,5,1,4,3,0^4,2,0^4,2,0^2,1,0,1^2,0,1,3,0^2,1,0,1,0^2,1^3,0^3,1,4,0,1,0^2,1^2,0^{11},1,0^3,3,0,1^2,0,1,0,1,0^{16},1,$
$0^{16},2,0,1,2,0^2,1,0,2,1,0^2,1^2,2,1,0^2,1,0^4,1,0,1,0^5,1,0,1^2,0,1,0^5,7,0^2,1,0,1,0^{12},1,0^{24},1,0^{17},1,0^4,1,0^{14},$
$1,0^{30}$

$A(16,4,11) \geq 312$

$0^{105},2,3,4,1,0,1,4,3,2,0,4,1,0,3,0^4,1,0^3,2^2,0,5,1,3,0,1,0,3,0^2,1,0^3,2,1,4,0^9,1,0^{15},1,0^{10},1,3,0,2^2,0,1,$
$0,1^2,0^2,1,0,5,0,1^2,0^2,1,0^3,1^2,0,1,0^2,1,0^8,1,0,1,0^2,1,2,1,0,3,0^2,2,0^5,1,2,0^5,1,0^2,1,0^{10},1^2,3,0^3,1^2,$
$0^{15},1,0^{15},1,0^5,1,0^6$

$A(16,4,6) \geq 615$

$0^3,2^2,4,9,2,14,0,2,0,2,0,5,2,7,1^2,0,1,5,0,2,1,4,1,4^2,2^2,0^4,1,5,1^2,0^4,2,0,2,0,5,0,1,2,0,2,5,3,1^2,0^4,1,$
$0,1,0^3,1^5,3,4,0,1,0,6,3,0^3,4,2,1,0^5,1,0^2,3,1^2,0^2,1^3,0,2,0^6,1,0,1^3,0^4,1,0,1^2,0^3,1^2,0^6,1,0,4,0^2,1,0,2,$
$1,0^2,3,0,5^2,1,7,0^3,1,0^3,1,0^2,2,0,1,0^2,1,0^3,1,0,1,2^2,0,2,1,0,2,0^6,1,0^4,1,0,1,0^7,1,2,0^{20},1,0^4,3,0^5,1,3,$
$0,1^2,0,5,0,1^2,0^2,1,0,4,2,0^2,1^2,0^3,4,0^2,1^2,0^7,1,0^4,1,0^2,1^2,0,1,0^7,2,4,1,0,1^2,0,1,0,2,0^6,1,0,2,0^6,1,$
$0^{17},1,0^3,1,0^{19},1,0^{18},1,0^4,1^2,0^3,1,0^6,1,0^{20},1,0^7,1,0^2,1,0^4,1,0,1,0^6,1,0^6,2,0,1,0^{11},1,0^2,1,0^5,1,0^3,1,0^4,$
$1,0^7,1,0^{28},1^2,0^5,1,0^6,1,0^{10},1,0^{47}$

$A(18,4,13) \geq 518$

$0^7,10,18,7,3,6,4,1,4,2^2,3,4,0,2,3,4,1,2,1^2,0,2,1,0^2,1,0^2,4,15,4,2^2,0,2,4^2,3,0^2,3,2,1,3,6,1^2,2,1,4,1,$
$0^2,1^2,21,0,5,3,2,1,2,3^3,2,0^6,4,3,0^3,6,0^2,2,1^2,3^2,0,1,3,0^5,1,5,1,3,0,1,0^3,2,0^4,1^3,0^5,1,0^2,1,2,0^2,1,0^3,$
$1,0^{10},32,9,0,2,5,2,0,7,1,0,4,1^3,0^2,3,0^3,5,0^4,1^2,6,3,0,1,0,1,2^2,1^2,3,1,3,0,2,1^3,4,0,1^2,0^2,1^3,3,1,0,4,1,$
$0,2,0^4,1,0,3,7,3,0^2,3,0,1,0,2,0^3,4,1,0^2,1^2,0^2,3,5,0^5,1,0^4,1,0^8,3,6,2,4,0,1,4,1,2,0^2,2,0,1^3,0,1,0^2,1,0,$
$4,0^4,1,2,0^7,4,0^5,3^3,0,2,0^6,1,0^2,1,0^8,1^2,0^7,4,0^4,2,1,0^5,2,0^2,3,0,1^2,0,4,0^2,2,1^2,0^5,1,2,0,4,0,3,0^5,2,0,$
$3,1^2,2,0^3,3,0^7,1^2,0^{11},1,6,0,4,1^2,0^2,1,0^2,2,0,1,0,1,0,1,0^5,5,0^{20},1,2,0^{16},1,0^5,1,0^6,1,0^2,1,0^6,1,0^{10},1,2,$
0^{26}

$A(20,4,5) \geq 874$

$0^2,13,5,8^2,5,18,21,1,3,0,1,5,0,15,2,7,0,8,0,2,1,2,1,5,2,11,15,3,5,0,9,1,7,3,6,7,4,2,0^2,2,1,2,1,9,5,3,$
$6,0,3,1,3,0,11,1,4^2,3,0,1,0,1,4,1^2,0,1^2,0^4,2,1,6,0,1,15,2,1,3,0,3,1,0^2,7,0,1,3,1^2,3,1,8,2,0,2,0,2^2,4,$
$3,1^3,0^7,1,0^2,2,0,9,2,4,1,0,3^2,2,0^2,4,0^2,1,3,1,0^2,1,2,0,1,0,4,0^3,2^3,3^2,0^2,2,0^2,2,0,1,6,1,0,1,0^2,1,0^2,2,$
$0^3,1,0,3,2^2,1,0^2,2^2,0^2,4,1,0^3,2,0,4,0,4,0,2,0^2,1,0,1,0^4,4,2,0^3,1,0,1,0^5,2,0^3,2,0^{13},2,0,1,0^5,1,0^5,1,0^4,$
$3,0,3,0,2,0^4,2,3,1,0,1,2,0,3,0^3,1,0,1,0^3,1,0,1^2,0^2,1,0^6,1,2,0,1,0^2,1,0^{14},2,0^3,1,0^4,1,0^4,1,0^7,1,0^2,3,0,$
$1,0^{20},6,2,0^2,2,3,0,1,2,0^3,4,1,0,2,1,0^2,3,0^2,1,2^2,0^{12},1,0,1,0^8,1,0^5,1,0^4,1,0,1,0^2,2,0^7,1,0^9,1^2,0^8,1,$

TABLE XVI (*Continued*)

0^{17},2,0^3,1,0^{27},2,1,0,4,0^3,3,0^3,2,1,0^5,1,0,1,0,1,0^2,1,0^7,1,0^9,1^2,0,1,0^{11},1,0,2,0^{19},1,0,1,0^3,1,0^{26},1,
0^{34},1,0^{29},2,0^3,2,1,0^{10},1,0^7,1,2,0^3,1,0^3,1^2,0^{10},1,0^{17},1,0^{21},1,0^5,1,0^3,1^2,0^5,1,0^8,1,0^{16},1,0^{21},1,0^{39}

A(18,6,8) ≥ 260

0^2,10^2,13,8,9,30,8,1^2,0,9,10,9,6,0,2^2,5,4,2,0^4,1,0,1,0,2,1,3,2,0^{26},2,3,1,0,1,0^3,1,0^{51},31,16,7^2,17,24,
4,1,0,4,1,10,9,2,7,0^2,2,0,21,0,7,2,7,3^2,0,3,2,3^2,1,32,7,14,16,0,10,2,1,8,4^2,10,1^3,4,3,4,12,1,5,2,10,
0,3^2,1^2,0^2,7,2,0,1,0^4,1,0^9,1,0^{59}

A(21,6,8) ≥ 774

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,
16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^5,4,0,1,0^{24},86,7,54,12,1,
0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},11,79,0,8,
23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,3,0^4,3,0^8,1,0^9,5,0,2,0^5,2,1^2,0^{29},6,2,16,3,1,10,6,1,5,3,0,4,0,
4,1,0,3,1,0^2,2,1,0,1,0^3,1,0^2,3,0^6,3,0^4,1,0,13,5,0,1,0,9,1,0,5,3,0^4,3^2,1,4,0^2,1,0^7,7,0^{21},2,0^{15},1,0^2,1,
0^{21},3,1,2,0,3,1,2,6,0^3,7,2,9,3,1,3,1,3,6,0,15,4,0,1,0,7,1,4,1^2,10,1,10,12,13,9,0,1,0,2,0,2,0^3,17,0^2,6,
0^2,5,0,2,0^5,1,0^{17},1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1^2,0^5,1^2,0^4,1,0^{24},1,0^{25},10,7,20,3,11,2,1,5,0^4,1^2,
0^3,2,1^2,0,1,0^8,14,15,0,1,8,1,0^2,1,2,0^4,2,0^6,1,0^2,1,0^3,1,0,4,0^2,1,0^9,1,0^{15},1^3,2,0^3,2,0^{25},2,0,1,0^{61}

A(21,6,9) ≥ 1184

11591,0,66,15,8,78,9,8,19,0^2,3,16,10,2,11,0,5,0,3,2,0^7,17,5,1^2,2,0,1,5,0,2,1,0^{21},4,1,0^2,1^3,0^9,1,0^{43},
46,1,49,1,29,8,28,2,0^3,1,6,0,1,6,0^6,1,0^4,1,4,0^2,1,0^4,1,0^{27},1,0^{56},1,4,0,1,2,0^2,1,0,2,0^2,1,0,3,14,0,11,
4,18,0,2^2,1,0^{111},16,1^2,0^3,6,0,1,2,0,3,0,1,0^2,5,6,12,4,6,10,2,0^2,8,2,0,1,4,3,0^2,1,0,4,0,2,0,1^2,0,19,0^5,
1,2,0^5,1,0^{106},61,0,7,0,1^2,0,2^2,0^4,1^2,2,0^3,3,1,0,1,0^2,1,0^2,1^2,0,1,0,1,0,1,0^4,1,2,0^7,1,0^3,2^2,0^3,2^2,1,
33,3,0,2,5,0^2,2,1^2,0^2,1,0^8,1,0^6,1,0^4,1,0,1,0,1,0^5,1^2,0^4,2,0^4,1^2,0,2,1,0^6,2,0^2,1,0^{12},1,0^3,2,0,1,0^{15},1,
0^{31},1,0^{28},1^2,0,1,0^{119},1,0^2,1,0^2,2,0^7,1,0^{12},1,0,6,3,2^3,7,1,2,0,5,1,0^2,3,2,6,0,2,0,1,0^{10},1,0^3,1,6,1,0^8,1,
0^7,1,0^4,3,0^2,1,0,1,0^5,1,0,1,0,2,6,3,0,1^2,0,1,2,0^5,2,4,1,0^7,2,0^6,1,0^{33},1,0^5,1,0,1,0^{11},1,0^{13},2,0,3,0,1,
0^3,1,0^{37},1,0^{50}

A(21,6,10) ≥ 1454

0^6,6,7,2,0,4,19,10,0,10,5,28,8,5,3,59,3,0,1,6^2,0,3,1,8,0,4,2,3,0^2,1,0^4,2,0,2^2,0,74,0^2,24,3,0,3,2,7,3,
5,0^2,6,1,0^5,9,0,1,0,1,0,13,5,4,7,12,2,3,4,6,0^{11},1,0^7,4,2,1^2,0^4,47,2,4^2,0^2,2,0^5,1,0^2,1,0,1,0^2,5,0,2,0,
1,0,14,4,8,7,0,3,0^4,3,1,0^6,1,0^9,1,2,0^5,5,3,1,2,0,1^3,0^{13},1,0^5,2,0^7,4,1,0^{17},1,0^3,1,0^{12},50,1,2,1,0,1,0^7,
1,0^7,3,0,2,0,2,1,6,0^3,3,0^9,2,0^4,1,0^5,1,2,0^2,1,0^8,1,0^3,1^2,0^{20},1,0^2,1,0,1,2,0^{35},1^2,0^{17},1,0^{18},3,0^{10},1,0^5,
1,0^{15},1,0^{60},10,35,2,5^2,0,8,4,0,2,4,2,0,1,4,18,3,9,2,5,1,0^3,1,3,2,0,2,1,0,10,13,1,4,0^2,2,4,0,1,0,1,0,1,
6,0,2,1,3^2,1,3,0,1,0,2,0^2,4,0^7,10,6,0,3^2,5,1,0^2,2,1,0^2,1^2,2,0^2,4,0^4,2,3,2,0^4,2,1,0^3,1,0^3,1,0,1,0^6,1,0,
2,1,0^{18},1^2,5,4,7,0,1,0^2,3,0,1,0^3,1^2,0,1,0,1,0,3,1,0,1,0,1,0^2,1,0^{11},2,0^3,1,3,0^5,1,0,1,0^2,1,0^6,1,0^5,1,
0^{15},1,0^{10},1,0^4,2,0^{10},1,0,1,0^2,1,0^8,8,2,0^2,4,0^2,1^2,3,0^6,2,0,1,0,3,1^2,3,0,4,5,0,1^2,0^2,1,0^{10},2,0,1^2,2,0^2,
3,1,0,1^2,2,0^{13},1^2,0^2,1,2,0^7,1,0,1,0^5,1^2,0,1,0,1,2,1,0,2^2,0^{15},1,0^2,1,2,0^{45},1^2,0^{19},1,3,0,1,0,2^2,0^{33},1,
0^{55},2,1,0^{14},1,0^{49},4,0,1,0,1,0^{24},1,0^{29},1^2,0^{39},1,0^{42},1,0^{53},1,0^{137}

A(22,6,8) ≥ 1139

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,
16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^5,4,0,1,0^5,1,0^{17},86,7,54,
12,1,0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},99,0,
8,23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,3,0^4,3,0^8,1,0^9,5,0,2,0^5,2,1^2,0^{29},39,5,3,18,8,1,6,5,1,4,0,4,
1,0,4,1,2,0,2,1,0,1,0^3,1,0^2,5,0^6,3,0,1,0^2,1,0,13,5,0,1,0,9,1,0,5,3,0^4,3^2,1,4,0^2,1,0^7,7,0^{21},2,0^{15},1,0^2,
1,0^{20},57,0,11,1,3,1,0,1,0,1,3,0,1,0^{11},2,0^8,19,2,5,0^3,1,0^6,1,0^9,1,0^{10},1,2,1,0,1,0^2,1,0^5,1,0^3,1,2,0^2,1,
0^6,1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1^2,0,1,0^3,1^2,0^4,1,0^{24},2,0^{25},49,2,0^2,1,0^{20},1,0^6,28,0^2,3,0^4,1,0^3,1^2,
0,1,0^4,2,0,1,0^9,3,0^2,2,1,0^{13},1,0^{12},2,3,0^{27},1,0^3,9,1^2,7,3,0,1,4,0^4,1,0^2,1,0^3,3,0,3,1^3,0,1,0^3,1,0^2,1,0^5,
1,0^5,1,0^2,1,0^2,1,0^{57},7,21,5^2,3,2,0,4,24,31,17,7,5,15,1,0,3,1,0^2,1,0^{28},23,0^2,2,3,1^2,0^3,1^2,0^1,1,0^{11},1,
0^2,1,0^2,1,0^{31},2,0^{26},3,0^4,4,0^{98},1,0^{55}

A(23,6,8) ≥ 1436

139,694,443,214,933,307,286,778,117,622,712,220,438,242,328,471,22,4,55,10,0,62,23,2,1,2,10,0,
16,12,1,4,12,0,2^2,6,3,0^3,22,10,7,2,4,2,1,7,0,2,0,1,0^3,1,0^8,17,0,1^2,11,0^2,1,0^5,4,0,1,0^5,1,0^{17},86,7,54,
12,1,0,1,30,0,19,2,0,2,22,0,1,10,8,7,5^2,2,3,0,2,1,0,1,9,5,0^2,2,0^2,1,0^{14},4,2,0^2,1,0^2,1,0^{16},1,0^{15},99,0,
8,23,1,15,3,5,2,4^2,0^2,5,1,0^2,2,0^2,2,1,0^4,8,0^3,5,2,0^2,3,0^4,1,0^9,5,0,3,0^5,2,1^2,0^4,1,0,1^2,0^{12},1,0^6,65,5,
3,19,13,6^2,1,6,0,6,1,5,6,3,0,7,1^2,3,0^3,2,1,10,0^2,1^2,0^2,5,0,1,0^2,1,14,5,0,1,0,9,1,0,5,3,0^4,4,3,1,4,0^2,
1,0^7,8,0^{21},3,0^{12},1,0^2,1,0^2,1,0^8,1,0^{10},15,3,43,20,11^2,5,3,7,3,0,4,0^3,4,1,2,1^3,0^2,1,0^4,10,18,5,6,2,0^2,
2,0^2,1^6,0^5,1,0,1,0^7,1,2,1,0,1,0^3,1,0,1,0,1,0^2,1,0,2,0^2,1,0^7,1,3,5,1,0,4,2^2,0,1,0^9,1,0^2,1,0^2,1,3,0,1^2,

TABLE XVI (*Continued*)

$0^2,1^2,0^4,1,0^{24},2,0^{25},52,2,1,0,1,0^2,2,0,1,0^{12},3,1,0^2,1,35,0^2,3,0^4,1,3,0,2,1,2,0,1,0,1,2,1,0^5,1^2,0^2,3,2,$
$0,3,1,2^2,0^4,1,0^2,1^2,0^{12},1,0,1,0,1,3,0^2,1,0,1,2,1,0,1,0^6,2,0^4,6,11,4,0^2,6,0^3,1,0^2,1,0^2,3,0^2,2,0^4,1,0,1,$
$0^3,1,3,0,3,0,3,1^3,0,1,0^3,1,0^2,1,0^5,1,0^5,2,0^2,1,0^2,1,0^5,1,0^{16},2,0,1,0^2,2,0^{15},1,0^9,2,4,2,11,0^2,3,2,17,$
$1^2,6,0^2,7,1,0^3,14,4,2,6,7,2,7,20,0,5,1,0,1,11,2,5,0,1^2,0^3,1,0^{12},1,0^4,31,0^3,1^2,2,0^3,1,0^2,2,0^2,1,0^3,3^2,$
$0^{15},2,0^4,1,0^2,1,0^2,1,0^3,2,0^2,1,0^2,1,0,2,0^2,2,1,2,0^2,2,1,0,2^2,0,6,0^4,1,0,1,0^6,1,0,1,0^4,2,1,0^5,1,0^2,1,$
$0^{12},1,0^8,1,0^7,1,0^{12},2,1,0^{22},1,0,1^2,0^{20},1,0^2,1,0,1,2,1,5,1,0^2,7,2,0,1,0^2,1,2,0^2,1^2,0,3,0^4,1,0,1,0^2,2,$
$0^5,1^2,0^2,2^2,1,0,1,0,1,0^3,1,0^2,2,0^2,1,0^4,1,0^3,1,2,1,2,3,0,2,4,3,2,4,0,2,0,3,2,1^3,0,1,3,1,0^2,1,0,2,0^3,1,$
$0,2,0,1,2,0,3,0^3,1,0,4,0^2,1,0^7,1^2,2,1^4,0^7,1,3,0^4,3,1,0^2,1,0^2,1,2,0^3,1,3,2,0,4,0,1,0^2,1^2,2,0^5,3,0^3,1,0,$
$1,0,1,0^3,1,0^4,1^2,0^4,1,0^2,7,2,0^3,1^4,0^5,1,0^2,1,0^{10},1,3,2,0^{10},2,0^{21},2,0^4,1,0^3,5,0,1,0,2,0^4,1^2,0^4,1^2,0^2,$
$1^2,2,0^{21},1,0^{14},1,0^5,1,0^{32}$

A(28,6,5) ≥ 272

$0^2,148,162,22,123,80,17,71,291,299,220,5,37,183,19,28,123,9,90,55,9,260,121,70,8,101,1,11,46,48,$
$75,2,4,31,21,6,24,2,14,0^2,48,7,58,4,1,41,35,1,17,2,12,0,9,19,7,1,2,9,66,35,3^2,5,9,0^2,7,2,8,3,2,15,6,$
$0,3,1,15,3,13,0,3,1,0,4,0,9,2,0,1,0^3,15,16,3,11,0,15,7,0,18,0,7,2^2,4,1,0,1,0^3,1,0^3,11^2,5,2,4,11,2^2,3,$
$1,0,3,5,2,0,2,0^2,1,0^3,1,0,1,0^6,7,0,5,0,5,3,10,12,4,1,0,3,0,4,0,2,0,1^2,0,1,0^4,1,0^2,1^2,0^4,1,4,0,1^3,7,2,$
$0,3,1,0^5,1,0^2,1,0^4,1,0^2,1,0^6,1,0^4,3,0^3,1,2,1,0^2,1,0,2,1,0^{12},3,0^{24}$

A(25,8,10) ≥ 1248

$196702,124,0^2,22,0,5,3,0,8,0^5,6,0^2,3,0^5,18,7,0^2,2,0,1,0^2,2,0^5,2,0^8,6,0^{23},12,0^{87},850,0^{23},10,0^{23},4,$
$0^{23},8,0^{87},478,0^{23},4,0^{23},2,0^{23},4,0^{87},444,0^{15},8,0^{15},4,0^{47},192,0^{159},240,0^{79},120,0^{159},1138,0^{15},162,0^{135},$
$17,15,7^2,0,2,3,0^9,12,0^{39},16,6,0,14,0^3,1,0^7,14,0^{24},23,1,4,5,0^{12},12,0^{23}$

A(26,8,19) ≥ 257

$30027,1557,7514,6608,3164,1865,3262,695,400,12,243,460,241,614,94,0,391,195,111,212,252,1065,$
$316,234,17,32,140,1833,132,8,0,483,34,5,0,3,9,4,0^2,10,0^2,13,188,0,7,8,21,1,0,3,0,1,2^2,0^3,1,4,0,1,4,$
$1,2,1,0,1,0,1,0^3,1,0,2,0^5,1,0,11,0,1^2,0^5,1,0^{10},1,0,1,0^5,3,0^{17},1,0^{18},1,0^{23},1,0^{47},1,0^3,1,0^{10},1,0,2,0^8,$
$1^2,0^8$

A(26,8,9) ≥ 883

$0^4,1,9,2^2,0,6,2,1,7,0,113,65,138,32,409,5,0,53,0,5,11,0,3,9,51,43,0,1,33,0,25,4,0^2,39,0,4,6,0^9,5,52,$
$30,0^2,5,0^2,1,0^6,2,4,2,4,0^2,6,0^{15},30,3,0^5,1,0^3,2,0^7,1^2,0^7,1,0^{13},1,0^{18},21,14,0^3,1,0,1,0^3,1,0,1,0^{17},1,$
$0^7,1,0^8,1,0^3,1,0^{10},1,0^{24},1,0^5,1,0^5,1,0^4,2,0^6,1,0^6,1,0^8,1,0^8,1,0,2,0^{16},1,0^7,1^2,0^2,1,0^5,1,0^{12},1,0^{24},1,$
$0^5,1,0^{16},1^2,4,0^2,1,0^2,1,0^3,1,0^3,1,0,2,0^9,1,0,1,0^{32},1,0^5,1,0^4,1,0^6,1,0^7,1,0^{11},1,0^3,1^2,0^{11},2,0^5,1,0^{20},1,$
$0^{17},1,0^{11},1^2,0^{12},1,0^2,1,0^{10},1,0^{15},5,10,0^2,1^2,0^2,4,2,1,0,1,5,0^2,2,1,0,1,2,4,1,0,2,1^2,3,1,4,0,2,0^2,1,5,$
$1^3,0,1^2,0,3,1,0^3,1,2,0^2,2,0^2,1,2^2,3,0,1,0,1,0,1,0,3,1,0,2^2,0,2,0^3,1,3,4,1,0^8,2,0,1,0,1,0,1,0^2,4,3,0,$
$1^3,0^2,1,0,1,2^2,1^4,0,1^2,0^3,1^2,6,1,0^2,1,0^4,1,0^3,2,1,0^2,1,0,1^2,0,3,0,2,1,0,1,3,2,0,1,3,0,1^2,0,2^2,0^3,3,$
$0^2,3,0,2,0,3,0^3,3,0^3,2,0^6,3,0^2,1,3,0^4,3^2,0^4,1,0^6,1,0,1,8,1,6,0,6,0,1,0^2,1,0,1,0^{33},1^2,0^{22}$

A(27,8,7) ≥ 278

$2273,164,5,224,161,154,40,2,44,135,0^2,45,4,9,15,2^2,8,4,7,12,100,13,0^2,8,0^2,2,4,3,2,0,4,0,9,0^3,2,0^3,$
$1,3,2,0^8,9,22,0,4,0^4,1,0,1,0,1^2,0,1,0^7,1,0^3,1,0^3,12,0^{10},1,0^{11},1,0,2,0^8,1^2,0^4,1^2,0^5,11,5,0,1^3,0^3,1^4,0,$
$1,0^3,1,0^4,1,0^4,1,0,1,0^3,1^2,0^2,2,1,0^3,1,0^3,1,0^2,1,0^2,1,0,15,48,13,4,6,8,12,17,26,1,53,43,16,8,14,13,$
$0,4,0,6,7^2,4,5,0,1,0,15,3,8,25,6^3,0^2,12,8,1,3,0^5,2,1,0^6,4,0^3,4,0^3,1,0,2,0^7,1,0^{18}$

A(27,8,19) ≥ 766

$1,0,1,0^7,1^2,0^{59},1,0^{12},1,0^{681}$

A(27,8,9) ≥ 970

$0^4,1,9,2^2,0,6,2,1,7,0,3,0,1,9,1^2,0,3,1,0^2,2,0,49,17,84,12,44,64,3,50,163,1,0,43,0,12,8,0,5,0^3,7,2,0,$
$159,6,10,21,0,23,1,7,0^3,1,0^6,30,0,3,0^3,4,0^{10},1,8,5,21,30,0^2,1,2,0^4,3,0^4,1,0^3,1,0^{11},1,0^7,1,0^{15},2,16,1,$
$22,0^2,1,0^3,1,0,1,0^7,1,4,0,1,0^{10},5,0^{23},1,0,1,0^{10},1,0^6,1,0^{13},11,1^2,4,0^{10},1,0,1,2,0^2,1,0^{10},2,0^{21},1^2,0^{14},$
$1,0^3,1,0^2,1,2,1,0^8,1,0^3,1,0^8,2,0^5,1,0^5,1,0^{13},1,0^{10},1,0^2,6,0^{10},1,0^8,1,0,1,0^6,1,0^{20},1,0^6,1,0^9,1,0^7,1,$
$0^{16},1^2,0^7,1,3,0^5,1,0^6,1,0^4,1,0^2,1^2,0^5,1,0^2,1^2,0^9,1,0^4,2,0^4,1,0,1,0^8,1,0^4,1,0^4,1,0^2,1,0^6,1,0^8,2,0^{12},5,$
$0^2,1^4,0,1,0,2,0^3,1,0^2,3,0,1,0,1,5,0^3,1,0^4,1^3,2,0,2,0,3,15,0,1^6,0^4,2,1,2,0,1,0^2,4,0^6,2^2,3,1,0^2,3,0,1^3,$
$5,0^3,2,0^2,1,0,1,0,3,2,0^5,3,0^2,2^2,0,1,0,2,1,2,4,0,1,3,0^3,3,1^2,0,1,0^2,1,3,0,2,1,2,1^4,2,3,2,0^2,1,0^2,1,0^2,$
$3,1^2,0^3,3,9,0^2,7,1,0^3,1,0,13,8,15,3,0,1^2,11,0,15,3,4,3,2,8,4,8,3,0,1,0^2,1,6,0^2,1,0,1,3,4^2,2,1^2,4,0^2,$
$3,2,5,0,4,2^3,1^2,0^4,2,0^3,3,0^4,2^2,0,1,0,2,0^2,3,0,1,6,0,2^2,0,1^2,0^6,1,0^8,2,0^5,1,5,1,7,4,1,7,2,0,4,2,0^3,20,$
$17,0,1,3,2,1,0,4,3,0^5,4,0^3,13,0,8,2,1,0^2,12,0^2,3,1^3,2,0^2,2^2,0^5,1,0^5,1,0,1,0,1,0^6,9,1,3,0,1,2,1,5,0^5,$
$3,0^4,1,0,1,0^4,1,0,1,0^5,1,0^3,1,0,1,0^3,1,0^{13}$

A(28,8,21) ≥ 296

TABLE XVI (*continued*)

$2616,169,1019,39,293^2,484,166,831,33,369,134,377,17,34,69,1,215,201,406,7,220,1,91,52,39,5,193,$
$219,0,89,2,61,1,40,26,13,5,3,10,50,7,60,4,38,57,1,0,2,1,7,20,1,6,0,11,5,6^2,3,21,6,22,0,16,6,5,10,3,$
$16,10,2^2,6,1,2,3,1,4,1,50,434,82,4,6,79,17,20,38,29,166,1,7,3,8,6,7,40,24,12,9,27,25,5,12,6,24,6,$
$11,2,11,0,4,1,4,5,51,0,2^2,11,0,1,2,0,2^2,1,7,13,0,6,1^3,0,2^2,4,14,18,85,28,1,35,31,14,64,0,6,7,1,0,3,1,$
$6,19,0,2,3,11,1,8,0,4,17,1,10,11,7,0,1,2,8,1,13,8,12,2,0,3,0^3,1,2,0^2,1,9,2,0,7,1^2,2,14,0,2^2,4,8,0^2,2,$
$0,1,3,4,1,0^3,1,3,0^4,2,0^2,1,0,8,6,0,2,0^2,2,0,4,0,2,1,0,2,0^2,2,0,2,1,0^5,3,1,0,1,0^5,1^2,0^9,1,0^8,1,0^{14},1,0^2$

A(28,8,20) ≥ 833

$0^5,5,21,15,2,23,30,8,2,19,8,2,8^2,2^2,8,1,2,51,6,0,4,17,7,2,9,24,7,9,2^2,7,1,0,20,0,14,13,48,12,2^2,0,$
$14,4,2,0^2,17^2,6,12,24,10,0^2,1,9,1,5,33,22,9,17,0,2^2,3,2,1,2,1,3,0,1,0,1,2,23,10,2,4,1^2,7,1,21,2,4,5,$
$1^2,0,2,0,59,2,9,12,3,1,4,2,6,12,9,0,5,2,1,7,0,16,0,25,10,0,5,3,24,0,3,11,1,2,12,0,3,19,3,2^3,7,0,1,22,$
$0,1,0^2,2^2,1,0^2,20,0,2,0^2,2^2,8,0,1,0,1,11,3,0,3^2,8,0,1^2,0^4,1,3,2,0^2,1,0^4,1^2,19,2^2,0^3,1,0,4,0,2,0^4,3,$
$0^2,2,0,4,1,4,1^2,0^3,3,0^3,4,0,1,0^2,1,0^2,9,0^2,1,0,11,3,1^2,2,1,0,1,2,1^2,0^4,1,0,1^3,7,6,0,1,0^2,2,1,0^2,1,0^2,$
$2^2,0^8,6,0,1,2,1,14,2,3,0,1^2,2,1^2,2,7,12,1,8,7,1,0,1,0^2,8,0,2,1,0^2,5,2,12,0,8,0,14,11,3,0,2,3,0,1^2,0,3,$
$0,4,2^2,0,5,0^2,2,9,0,3,0,3,0,1^2,2,0,2,0^2,2,0^2,2,6,0^2,5,10,1,3,0,5,1,0^2,9,0,1,2,0,1,0,4,0,1^2,0,1,2,0,2,$
$7,0^2,3,1,4,0^3,2,11,0^2,1,0^4,8,0^2,12,1,0,10,2,0,1,0^2,1,0^4,4,0^4,3,0,1,0^6,4,2,1,0,1,0^{11},1,0^8,1,0,1,0,3,$
$0^3,6,0,2,0^3,2,7,0,18,0^2,5,18,2^3,0,1,0,3,1,2,3,0,2,1,2,1,4,3,2,1,0^2,1^2,0,12,7,1^3,2,1,2,3^2,1,0^7,2,0,4,$
$0^2,2,0,1,0,1,2,0,2,1,0^2,5,0^3,8,0^2,4,1^3,0^3,3,0^3,2,0^2,6,0,3,1,2,0^7,6,0^3,3,0^5,1,0^2,2,1,0^4,1,0^4,2,0,1^2,$
$0^2,2,0,5,4,0,4,1,0^7,1^3,0,1,0^3,1,0^3,3,2,3^2,1^2,0^2,1,0^3,6,0,4,2,0,1,0,1,0^7,1,0,1,0^3,2,0,4,1^2,0^2,1,0^2,1,$
$0^2,1^2,0^5,2,0,3,0^3,1,4,0^2,1,0^2,2,0^7,1,4,1,0^7,1,0^3,2,0,1^3,0^{22},1,0,2^3,0^3,1,0,1,0^5,1,0^8,1,0^{10},1,0,1^2,0^2,$
$2,0^2,4,0^{13},1,0^2,1,0^{10},1,0^3,2^2,0^{18}$

A(28,8,19) ≥ 1107

$0^5,28,0^3,33,12,13,0,9,5,0^4,13,11,17,9,2,19,0^4,4,3,0,7,0^2,10,0^2,7,0^3,8,1,2,0,1,0,19,0^5,9,0,12,0^{12},5,$
$0,4,0^3,103,1,0,4,0^4,132,0,2,0^3,14,392,13,0^3,56,0^4,28,0,2,0,4,46,0^2,4,0,52,0^2,8,1,0^3,335,0^3,42,0,16,$
$11,0,22,0,6,0,2,0,14,0,1,0,291,0^3,69,0,21,0^2,5,3^2,0,9,0^2,5,0^2,104,586,14,12,8,0^2,3,4,0^6,1,0,17,0^2,$
$13,0^3,3,0^2,1,3,0,4,0^2,37,5,90,2,0^3,273,13,0,49,150,0^7,407,98,22,46,57,55,0^3,82,0^7,7,312,46,281,9,$
$47,36,0^7,173,73,459,0,2,0^7,2,0^9,214,43,45,133,14,65,30,15,50,67,64,56,24,6,1,253,0^3,3,0,3,0,237,$
$161,176,0,94,38,0^2,1,0^3,1,82,61,42,51,95,11,0^3,2,0^{11},28,82,41,120,0,27,21,25,0^2,32,0^2,18,0,1,0^5,$
$28,107,81,62,2,14,1,0^2,9,0^4,1,0^3,4,0,94,2,59,40,28,10,12,10,0,9,7,9,2,0^6,3,0^{12},2,0^2,1,21,47,15,9,26,$
$3^3,8,9,1,3,1^2,0^{16},1,0^5,2,1,29,9,23,9,18,20,1,0,13,1,0,11,6,1^3,25,0,15,91,1,0^2,30,1,0^2,2,3,0^3,6,16,9,$
$12,0,22,0^3,11,2,0,23,286,0^5,84,44,5,8,16,0^2,6,0^4,1,3,0^2,1,0,4,0,15,75,2,35,1,0^3,61,36,0^2,12,2,0^2,$
$15,76,4,12,6,9,3,0^3,10,2,1,8,47,15,25,0^3,9,0^2,2,0,1,0,5,1,0^5,31,71,82,26,19,5,0^3,4,0^4,56,4,70,0^2,10,$
$9,3,1,0^2,3,8,3,9,8,0,9,97,3,6,0,11,24,2,8,9,6,5,6^2,0^2,1,0,5,0^6,3,0^3,1,0^3,1,83,5,20,8^2,2,11,2,0,1,5,$
$0^8,1,0^2,2,1,0^2,6,0^2,1^2,0^9,2,1,6,40,3,0,4,0,1,0,5,15,0,4,2,12,2,3,308,7,2,1,23,10,3,124,1,12,35,0^2,$
$17,0,6,0,4,0^2,11,9,12,42,0,2,1^2,0^3,2,0,1,0^2,10,4,0,3,0,9,3,0,5,0,4,0^4,1,4,1,0,1^2,5,2^2,0,5,49,10,0^3,4,$
$1,0,2,0,3,15,7,2,0,8,2,0^5,2,0,10,5,0^3,4,7,9,8,1,2,0^3,12,5,9,0,65,12,7,19,0,3,0^2,6,0,5,0^7,4,1,2,0,1,5,$
$3,0^9,1,0^2,5,1,0,1,0^2,2,0^{12},1,0^3,8,39,1,0^5,4,0,11,1,7,1,6,2,0^6,1,3,0^4,3,9,3,0,3,2^2,0,3,0^8,3,1,0^{11},1,5,$
$1,0,3,1,2,1,0^{12},1,0^3,1,0,1,0^2,1,0^4,4,0^3,1,0^3,2,0,13,2,1,4,0,1,0^3,3,0^2,2,1,0^2,1,0,1,0^3,2^2,0,1,0^5,1,0^{13},$
$1,0^{13},1^2,0^2,1,0^4,1,0^{11},2,0^3,1,0^4,1,0^3,1,0^{33},1,0^{10},1,0^{25}$

A(23,10,7) ≥ 20

$31030,8098,707,3722,291,84,959,43,9,2,39,0^3,3,0^5$

A(26,10,12) ≥ 185

$0^4,9,804,1634,31,0,653,287,18,333,12,7,19,1108,2,108,41,8,10,177,2,0,9,11,13,3,1,6,23,45,6,16,1,$
$0^2,21,0,18,4,25,0,5,21,1,0^2,1,3,1,2,1^2,0^2,1^2,0,52,18,3,6,7,0,12,4^2,2,0,1^2,0,7,0^2,2,0,2,0^6,15,9,2,1^2,$
$3,1^2,6,12,2,0,10,0,10,58,1,2,9,1,9,6,7,0,7,0,1,5,2^2,0^9,4,3,0,28,9,10,9,24,7,0^2,8,4,3,0,6,5,0^2,1,0^7,$
$54,36,0,13,5,1,18,1^2,2,4,1,14,1,0^5,1,0^{13}$

A(26,10,13) ≥ 191

$0^2,56,420,4213,616,3042,732,98,10921,2166,577,1260,0,299,606,249,4566,1544,579,478,0,143,145,$
$150,224,361,38,431,267,241,0,34,2033,404,108,1062,0,1172,89,242,160,371,0,74,22^2,17,129,101,$
$105,31,14,64,14,31,27,12,0,67,3,33,0^2,11,3,94,119,63,15,1^2,6,40,75,69,30,0,45,31,7,93,71,24,41,14,$
$98,40,7,6,2,3,2,5,2,18,14,2,39,22,1,3,0,11,3,0,1,2,0,2,3,1,6,0,2,0^2,2,0^5,3,62,0,23,150,5,6,79,29,8,6,$
$72,8,13,2,12,62,29,42,0,4,0^6,4,0,2,0,5,1^2,0,9,7,1,2,0,2,4^2,1,0^2,2,7,0^2,1,0^{17}$

A(27,10,11) ≥ 213

$52263,46674,39014,12594,7191,49805,257,1176,8505,2624,2904,1024,411,5583,9562,3356,214,1045,$
$3611,2612,291,935,641,227,406,430,2926,3213,1039,2077,44,486,200,22,156,171,481,183,51,412,27,$

TABLE XVI (*continued*)

93,42,38,91,9,17,3,1410,278,244,96,100,17,33,132,24,10,6,31,106,11,7,4,15,74,11,55,6,3,22,4,8,0,1, 0,29,0,10,6,11,0,1^{3},0^{11},190,10,52,15,5,37,3,0,5,2,8,21,1,3,1,0^{2},15,0^{2},9,5,0^{4},2,0,5,0,1,0^{4},1,0^{36},47,3, 1^{3},2,5,1,0,1^{2},3,2,1,0,1,0^{2},2,0^{13},1,0^{5},1,0^{3},1^{2},0

A(27,10,12) ≥ 257

0^{4},1212,865,887,143,95,906,134,674,11,56,2,0,44,58,634,6,0,39,157,8,47,14,8^{2},23,9^{2},50,31,11,0,3^{2}, 5,17,0^{3},5,0,7,0,4,6,0,3,0,5,0^{6},1,0^{4},38,20,9,5,13,3,11,17,2,17,4,0,2,13,4,0^{2},3,0^{5},1^{2},0,2,11,2^{4},1,0^{3},2, 0^{2},23,42,20,16,10,0,6,3,7,0,2,0^{2},13,3,0,1,0,6,0^{6},1^{2},3,0^{3},36,35,4,0,5,0,3^{2},4,0^{4},7,0,1,0^{6},8,11,6,7,1,4, 3,1^{2},4,1,0,8,2^{2},0,1,0^{2},3,0,1,0^{6},23,8,2,1,4,2,3,0^{3},18,8,12,5,12,1,2,1,4,0,4,1^{2},0^{3},1^{2},0,1,2,0^{3},17,6,13, 2,26,4,18,3,6,2^{2},0^{4},1^{2},2,0,2,0^{5},2,0^{4},2,0^{9}

A(27,10,13) ≥ 283

0^{2},56,420,4213,616,3042,732,98,10921,2166,577,1260,0,299,606,249,4566,1544,579,478,0,143,145, 150,224,361,38,431,267,241,0,34,2033,404,108,1062,0,1172,89,242,160,371,0,74,22^{2},17,129,101, 105,31,14,64,14,31,27,12,0,67,3,33,0^{2},11,3,94,119,63,15,1^{2},6,40,75,69,30,0,45,31,7,93,71,24,41,14, 98,40,7,6,2,3,2,5,2,18,14,2,39,22,1,3,0,11,3,0,1,2,0,2,3,1,6,0,2,0^{2},2,0^{5},3,62,0,23,150,5,6,79,29,8,6, 72,8,13,2,12,62,29,42,0,4,0^{6},4,0,2,0,5,1^{2},0,9,7,1,2,0,2,4^{2},1,0^{2},2,7,0^{2},1,0^{17},43,14,13,20,7,0,1,9,14, 19,10,4,0,10,0,3,1,0^{5},1,16,4,3,0^{5},6,0,4,1,0^{4},83,23,1,76,8,0^{3},46,0,10,1,0^{3},21,2,7,0^{3},3,2,0^{5},2,0,4,1, 0^{2},2,0^{5},1,0^{12}

A(28,10,18) ≥ 195

0^{3},56,286,1296,3684,5186,1441,138,0,833,5,115,4737,1382,956,86,373,96,51,2263,1799,21,135,242, 103,342,104,49,48,113,75,17,8,2,1,667,1,269,55,58,19,35,365,86,2,92,15,37,4,21,0,11,0^{2},12,15,12, 0,10,16,1,3,16,0,31,8,2,1^{2},9,0,1,0,2,5,1,2,0^{2},10,0,4,0^{3},2,4,0^{3},6,0^{4},1,0,1,0^{5},1,0^{5},649,116,343,515, 44,27,21,20,33,258,58,39,14,49,30,3,7,6,21,31,49,29,30,2,9,1,8,0,22,23,1,6,13,7,93,5,55,10,6,1,0, 10,3,0,2,1,2,0,1,0,16,0^{14},1^{2},0,9,0^{2},1,0^{12}

A(28,10,11) ≥ 280

52263,46674,39014,12594,7191,49805,257,1176,8505,2624,2904,1024,411,5583,9562,3356,214,1045, 3611,2612,291,935,641,227,406,430,2926,3213,1039,2077,44,486,200,22,156,171,481,183,51,412,27, 93,42,38,91,9,17,3,1410,278,244,96,100,17,33,132,24,10,6,31,106,11,7,4,15,74,11,55,6,3,22,4,8,0,1, 0,29,0,10,6,11,0,1^{3},0^{11},190,10,52,15,5,37,3,0,5,2,8,21,1,3,1,0^{2},15,0^{2},9,5,0^{4},2,0,5,0,1,0^{4},1,0^{36},47,3, 1^{3},2,5,1,0,1^{2},3,2,1,0,1,0^{2},2,0^{13},1,0^{5},1,0^{3},1^{2},0,11,4,5,0,3,1,0,3,0,1^{2},0^{2},1,0^{2},1,0^{5},1,0,1,65,17,13,16, 6,4,2,0^{4},3,5,0^{3},1,2,4,12,2,1,2,1,2,0,2,0^{5},1^{2},0^{8}

A(28,10,12) ≥ 356

0^{4},1212,865,887,143,95,906,134,674,11,56,2,0,44,58,634,6,0,39,157,8,47,14,8^{2},23,9^{2},50,31,11,0,3^{2}, 5,17,0^{3},5,0,7,0,4,6,0,3,0,5,0^{6},1,0^{4},38,20,9,5,13,3,11,17,2,17,4,0,2,13,4,0^{2},3,0^{5},1^{2},0,2,11,2^{4},1,0^{3},2, 0^{2},23,42,20,16,10,0,6,3,7,0,2,0^{2},13,3,0,1,0,6,0^{6},1^{2},3,0^{3},36,35,4,0,5,0,3^{2},4,0^{4},7,0,1,0^{6},8,11,6,7,1,4, 3,1^{2},4,1,0,8,2^{2},0,1,0^{2},3,0,1,0^{6},23,8,2,1,4,2,3,0^{3},18,8,12,5,12,1,2,1,4,0,4,1^{2},0^{3},1^{2},0,1,2,0^{3},17,6,13, 2,26,4,18,3,6,2^{2},0^{4},1^{2},2,0,2,0^{5},2,0^{4},2,0^{9},6,1,0,10,7,4,3,1,2^{2},5,1,7,1,17,7,6,13,19,0,16,8,3^{2},0^{3},1,3, 1,0^{2},1^{2},3,1,0^{3},1,2,0,87,16,64,31,6,28,23,3,2,18,9,16,7,8,1,0^{5},1,0^{2},2,1,4,0,6,1,3,0^{2},2,1^{2},0^{2},3,0,1,0^{17}

A(28,10,13) ≥ 414

0^{2},56,420,4213,616,3042,732,98,10921,2166,577,1260,0,299,606,249,4566,1544,579,478,0,143,145, 150,224,361,38,431,267,241,0,34,2033,404,108,1062,0,1172,89,242,160,371,0,74,22^{2},17,129,101, 105,31,14,64,14,31,27,12,0,67,3,33,0^{2},11,3,94,119,63,15,1^{2},6,40,75,69,30,0,45,31,7,93,71,24,41,14, 98,40,7,6,2,3,2,5,2,18,14,2,39,22,1,3,0,11,3,0,1,2,0,2,3,1,6,0,2,0^{2},2,0^{5},3,62,0,23,150,5,6,79,29,8,6, 72,8,13,2,12,62,29,42,0,4,0^{6},4,0,2,0,5,1^{2},0,9,7,1,2,0,2,4^{2},1,0^{2},2,7,0^{2},1,0^{17},43,14,13,20,7,0,1,9,14, 19,10,4,0,10,0,3,1,0^{5},1,16,4,3,0^{5},6,0,4,1,0^{4},83,23,1,76,8,0^{3},46,0,10,1,0^{3},21,2,7,0^{3},3,2,0^{5},2,0,4,1, 0^{2},2,0^{5},1,0^{12},1,7,6,19,3,2,8,4,6,0,6,5,1,5,0,7,3,4,1,0,14,30,15,10,20,3,0,2,4^{2},7,0^{2},2,3,0,3,0^{2},1,4^{2},0, 2,1,0,5,0,1,0,1,0^{3},26,67,37,6,28,22,19,4,7,2,9,2,19,0,5,18,3,7,24,0^{2},8,1,0,1,0,1,0,9,0,2^{3},0^{2},6,1,3,0, 2,1,0^{2},1,0,2,1,0^{2},2,1,0^{3},1,2,0^{5},1,0,1,0^{13}

A(28,10,14) ≥ 435

0^{3},1150,96,5557,4719,2069,4941,1017,10572,1545,2879,3746,675,0,842,229,295,389,1949,163,2268, 2130,9212,0,47,1563,161,174,210,429,0,218,1161,3037,254,700,65,4877,681,5297,706,0,330,820,60, 34,411,436,1620,124,873,885,1639,503,513,0,408,643,160,1301,1860,396,5^{2},28,444,558,161,286, 344,544,305,183,0,65,2416,95,3964,1032,1720,38,105,851,58,1061,312,32,980,1003,187,198,0,60, 508,221,249,117,53,187,28,107,1,424,210,12,460,305,176,6,4,0,214,19,0,8^{2},84,0,100,165,42,26,32, 5,13,0,34,2,16,24,0,25,0,22,8,11,2,6,0^{3},2,4,1,0,130,398,240,208,257,40,52,232,106,6,2^{2},106,3,17, 523,4,215,115,17,20,6,142,3,1,12,93,1,0,8,4,71,43,15,2,24,5,13,16,26,0,3,1,0,4,2,13,1,0^{2},2,3,2,4,3, 7,8,2,1,0,2,0,1,3,0^{3},42,1,35,0,12,0,1^{2},2,0^{2},6,1^{2},2,0,2,1,2,1,0,1,0,10,0,1^{3},0,1^{3},0^{5},1,0,3,1,0^{3},1,0,1, 0^{22},69,270,157,82,24,3,97,54,9,184,7,32,28,21,47,48,5,6,52,21,7,13,2,50,8,3,54,0^{2},31,2,11,67,2,21, 42,3,2,11,1,9,1^{2},5,21,0,34,0,6,0^{2},2,0,2,1,0,11,6,2^{2},3,1,3^{2},0^{12},7,4,2,6,1^{3},3,0,1,3,4,8,0^{3},3,2,0^{2},2,7,2, 0, 1, 2, 0^{2}, 2, 1^{3}, 0, 2, 0^{2}, 1, 0^{5}, 2^{2}, 0, 1, 0^{7}, 1^{2}, 0, 1, 0^{19}

APPENDIX TABLE

$\pi(7,3) = (7,7,6,6,5,4)$, norm=211: 321654261315435223146215341462423 15

$\pi(8,4) = (14,14,12,12,10,8)$, norm=844
5214334215143251362435261246153612442163516421625342631523415124334125

$\pi_1(9,4) = (18,18,18,18,16,15,15,8)$, norm=2066
17623428635817463454125177316422563873426148517253184325231283746465712458534721
613357412618726363545241734561216732764574152 3

$\pi(10,3) = (13,13,13,13,13,13,13,13,13,3)$, norm=1530
5942793687314568856 96A81592413413272798134652672415367898672313124925 7A794658148
593213674552812973487 2A96981635445986137

$\pi_2(10,4) = (30,30,30,30,26,25,22,15,2)$, norm=5614
54712763112843623648312585431276459317626584345225436176184725341716326423357614
21487575321755814263362415413281825637142436751247573365365542824145671317242816
392463141821738532756473215624174183663425786 54321

$\pi(10,5) = (36,36,34,34,29,29,27,27)$, norm=8044
28657343817173624561257134426131247583724512686231547863743812415158264512867245
38645273718164382552658738264312368411347415623164551244873216281724536372485163
72416563782738537214614567332718412585622361476418737325413364815225814354218471
312648782563

$\pi_1(11,3) = (17,17,17,17,17,17,17,16,16,14)$, norm=2731
7A4532881759623119349123864659 64A5768914257463279AA1372816496A7593475825382A12A1
586423157964A27A8658571324358916A82413974536989A1581276279483786A34532714A691643
79A25

$\pi_2(11,3) = (17,17,17,17,17,17,17,17,16,12,1)$, norm=2713
5A1386975463487251793729 8A2483491561863279716289436A41579274151 63A7286485627353A
8219748246325B93A18569712A9254686A13954729156843A11A632A7549874398523847591685 7A
64312

$\pi_3(11,3) = (17,17,17,17,17,17,17,17,17,10,2)$, norm=2705
25816741797459633968436A34985212857193625 5A2328741891746369782851454A12217639478
9635817496A385612747A9163364A952925B78419583A742368312567 29A4A5618931645978224A8
1B573

$\pi_1(11,4) = (35,35,35,34,33,33,33,32,31,25,4)$, norm=10724
89546341268B21776242659 8A84675BA7131A59832B319A46B37A545933761472218A9A291324175
169568453A452893718693A42645742596832791A815365163718A4227A375534938152949371426
842798A619585612765 4A393645782A3A81266A41571386742293745172979583619251A84A58341
52396376154978648123692857513 68AA6742A478376291819325484926 13A686184795732A52374
917429138A

$\pi_2(11,4) = (35,35,35,35,35,33,33,32,28,21,8)$, norm=10616
A951842513754A3B3251A2874739146568289462617355936 72A9486145874B3921AB7517464329A
962389871528196338614A57973A8219576148A2B324A638B978431261775828A995642163758B43
1175243569A426133A8625428A3612575134675984269458717632856947 18AA3B62915437182913
9875442B15937827614 5825A6331947571282546364A2515863497261 5A44536345271729 8693A71
8486719A32

$\pi_1(11,5) = (66,66,60,60,54,45,44,40,26,1)$, norm=25066
93521728164179693243516743257448136274353299216532181547423817125643546365741675
25683212178596213428531647372912411395784842638345269411247539513641279275842586
14327215816738125637418268474354913212365681944526373712724418825616354873541462
32148535621771625945236881453412935878271537412329657214814563127359148724153326
48651385742923196815231822463941435764293146317587247635212164789931513289425643
67251524614A36514533842842712371653458722135796185813721286934

$\pi_2(11,5) = (66,66,60,60,54,45,44,42,22,3)$, norm=25046
63521827184179693243519843258446137264353287218532171548423718125743546365941695
25673212187576213426531648382712411385794742637345268411249539513641289285A42576
14328215617937125738417287494354613212365691744526383612824417725 61A354683541482
32147535621881725745239661453412635676281538412329758214614573128357146824153326
47651385742823196715231722463941435864293146317578248635212164978831513279425A43

APPENDIX TABLE (*Continued*)

69251524714638514533742642812381653457822135896175713821276934
$\pi_1(12,4)$ = (51,51,51,51,51,46,45,44,44,34,27), norm=22903
864134B319A5564215638972A89AA4241791542B782763566978421634839A51B2B856396B5978A2
678235123BB413872456596182A57683145471A91A24737694241A575231863B824152A89634B743
586A31513B97294A59762115247179A38AB94639512B432658B275754318A82479236A5A36941825
3618B497A81623415A377152948291434617B1465978B3A53B9826938532BA96645B182617926475
83A5782341532713269526B9A8781397845157481356278314B469AA1425B2673941A519A3843527
686329469732B5AB97238461467593129A895187342846517A6418B69572593142493673A2516872
5AB81417A4325B9
$\pi_2(12,4)$ = (51,51,51,51,49,48,48,42,42,37,23,2), norm=22843
A234B42B3B37A42659717886A9851B65A19185698567959B71A186B7392484321624375B898A1156
17A68596873427136452279436274313248A43255869A189156BA518A6187979562A43829534437A
2473A242B1336254815769736247432952431BA15861596782B6717199858A659865B1453A242B63
312743421A3A4722754395168A2643725A34431A219A87B718596412743632479432876891569A85
13A85167CB7168596B916599A781A58214353247743262A43527A3443162167A9541352427B33B21
495861A895B673342763147225843596BA197B18AC61A8597493524A432612437679158586719BB6
79852A8A1596423
$\pi_3(12,4)$ = (51,51,51,51,49,48,48,42,42,40,15,7), norm=22815
BC41343A1136B94758626897A8952C75A28295789576858962B2B7A63A4191342741365A989A2257
26A7B587963146237154468137461323419A13455978A2982578A529A7296868574A1394B531136A
4163A41492337451C25678637416134A541329A25872587694A7626298B59B75897582153A414C73
324613142A3A1644651385279A4713645A31132A428A96B629587C24613734168134967982578A95
2BA952763162795B7A827588A692A59421353416613474A13546A3113274276A8512354146833942
185972A985B761314673216445913587BA286C2BA972A9586383541A1347241367682595976283C7
68954A9A258714B
$\pi_4(12,4)$ = (51,51,51,51,49,48,46,44,43,37,20,4), norm=22795
31AC2821CB233186579498B6A9A54865A4747568B569757A849476AA372181324621395B977A4456
4B96A58679312843615227813628134321891325596A74A845687549764B9798562B1372A531139A
2193A212A43362518459679362181327521347A458645A698B768494B785A9657B65A4153A212763
342913124A3A1922851375468B2613925A311348248B79A8475B61429136321A813279678456B875
4AA754692394685768746577BAA4957241353219813262813529B31134624697A514352129833824
1A576497A596B831296341A225813586C74B9748A3649A5712835217132642138698457576849296
AB753C7A4596B18
$\pi_5(12,4)$ = (51,51,51,51,49,48,48,45,39,36,22,4), norm=22755
3C24B4C92327B43658717996ABA51965A181856BA567858971B186A72843932416432759A8891156
17A6B5A6872347126354479326473212439932455B6A81AB156B8519861A787956493284A523327B
43729434A1226453B157687264373248543218A159615967A4867171A895AB658A6591352A434862
214732341A29374475328516B94632745B233219419A87B7185A6C14732624379324876891569A85
12A851673B7169586A816588A791B58413252437732464932547923321641678A531254347A22941
395861A8A5B672234762137445A32596A81A781B9C61B9587392543832461432767915858671A3B6
7A85498A1596B42
$\pi_6(12,4)$ = (51,51,51,51,49,48,48,45,41,32,22,6), norm=22663
A24C34392B352946785159B6A9A71B67C181876AB765878951B186A52843932416432579B8891176
15A6C7A685234512637445932645321243993247796A819B176A8719861A585A764B328495233259
4352943491226473A175685264353248743218B17A617965AC865151B8A7AB678A67913729434862
214532341929354457328716A94632547923321941AB85C5187B62145326243593248568B176BA87
13A87165945169786A816788A591B78413272435532464932745B233216416589731274345A22941
397861A8A7B654234562135447A32796A81B581A9261B9785392743832461432565917878651ABB6
5A87398A17964C2
$\pi_1(12,5)$ = (80,80,80,80,72,70,69,67,67,62,48,17), norm=55860
965317A143671252437CB371B58274431A6998C15246383524931865165783A927244BA4953895B2
156A364A179A427312893B4953895B1238AB976A487123B362414C99765235247762915A8B63135A
84782B983A65BA185672B61354A897221974283A481A7B6A7216135C3761525B3694CA21659612C4
5723484AB815782614583A92B84B6143798528319A667512A4865394AC1561B89731673254A1774A
8253B9B4C1A527BC4156377A248724631985B98675CA1923A294B3415691565A7438B68372241642

APPENDIX TABLE (*Continued*)

A793164A527498312B972B1934A5A5863BA1362438294598A815B73547361242783A8311B9245839
26A718491457A8A261B5996C4217C69A253484BA267319A6581B72A6A3B9124851976813C2164B43
8772385693384B45621AB71965246518845B9A293764AB811398A624134852732761B16378413972
495A95443972683A254B7AC91CB82A63392541A36B657429756AA46B1852392CB345741362785A83
718B246916857389749346235921A421A357782A47619219AB393C586BA5842126B15749

$\pi_2(12,5) = (80,80,80,80,75,72,71,69,63,55,40,23,4)$, norm=55350

9A5416712486135324B9A46185B37224167BA691532A474532861A851756A46838322B92854985C3
1579472A1689236413984A2854AC57134B7A8679286134C4732129896753453266739158BA741459
62963A584CA5891A5763871452C8963317923A49291B687A6317145C4671535A47928A31758713B2
56342A28A6156B37125B4A93C82891246A5C3B418A77651372A9A4B2CC15A188B641764352816629
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3756152A1256898371A5887A2316C7853542A28537441B87CA166397A488132B51867614B31A2B24
A6634955A44B2B257319C8197532751BB25A8C3D46728CB114889732142B53A4367191746A214963
2B5885224A637949352769C819A83C74483521C47875623A6579627719834A39C42B621473695984
615A327B1795649B628427345831A2318456C93B2671A31A8C4A4B597895A23137C1562B

$\pi_1(12,6) = (132,132,120,120,110,94,90,76,36,14)$, norm=99952

31542766754122374813646275134216576238594A15381925782131482843564917156629872265
43178621454652 3A71238164931254452183673891854723612534871542157328374463281325 88
14417127856725324963481 9A71562351245617322735348162485618563431277124381438A2635
17327474532611525467213456361987916325723154 24A327125438981658264512465491532 69A
132581A784317342534161293654392236114772547556428316942725613387146421581427 9538
46123746272715345324151786786143352893124153351846226473427615516724374622648153
35142139825334168768715142354351727264732164835972418512464178331652724961382465
57452774116322934563921614352437134 87A185231A962351945642154628561898345217 23942
451327523 61978A163654312764525116235474723715362A834183421772134365816584261843 5
37223716542153265 17A918436942352765872171441885231823644738237512451784352163274
58198376381254452139461832 17A325645412687134562278926651719465348284131287529183
51A4958326756124315726463184732214576672 4513

$\pi_2(12,6) = (132,132,120,120,110,94,90,72,42,12)$, norm=99776

31542766754122374913646275134216576239594815381925782131482843564917156628872265
43178621454652 3A71238164931254452183673891854723612534971542157328374463281325 88
14417127956725324963481 9A71562351245617322735348162485618563431277124381438A2635
17327474532611525467213456361A879163257231542483271254389916582645124654915326 9A
132581A784317342534161293654392236114772547556428316942725613387146421581427 9538
46123746272715345324151786796143352883124153351846226473427615516724374622648153
35142138825334169768715142354351727264732164835972418512464178331652724961382465
57452774116322934563921614352437134 87A185231A962351945642154628561998345217 23842
451327523 61978A163654312764525116235474723715362A834183421772134365816584261843 5
37223716542153265 17A918436942352765972171441885231823644738237512451794352163274
58198376381254452139461832 17A325645412687134562278826651719465348284131287529183
5184959326756124315726463194732214576672 4513

$\pi_3(12,6) = (132,132,120,110,110,97,91,75,47,10)$, norm=99072

32541677654211364923747165234127567139596825382415681232481843574926257718861175
43268712454751 3A82138276932154451283763892854613721534962541256318364473182315 88
24426216957615314973482 9A62571352145726311635348271485728573432166214382439817 35
26316464531722515476123457372896927315613254 14A3162154389826581645214754925317 9A
231582A684326341534272193754391137224661596557418327941615723386287412582416 9538
47213647161625345318252687687243351843219253352847117463916725527614364711748253
3524123A81533427867862524135435261617463127483596142852147426833275161697238147 5
56451669227311934573912724351436239 86A285132A971352945741254718562898345126134 91
65231651372969827375432167451522713546461362537189342834126612343758275841728435
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58298367382154651239472831 26A315745421786234571168817752629475348184232186519283
5284959317657214325617473294631124567761 4523

APPENDIX TABLE (*Continued*)

$\pi_1(13,4)$ = (65,65,65,65,62,61,60,57,57,53,52,45,8), norm=42165
B326A54137863947341925B4383296BD3A16124572A89C9A7B6245C41952831C7674AB2C588C491B
349A721A75ABC85A67822164C9836A7461352B197859A34975CB95B11ACC29482A6C32A612495B18
775883B9A3636751B5248CB914632D68658B33471591CA27469A361584522B761ACB185A418C769A
4A852BC1975834B183262CB35961B729A9368453D932B4735642A132ABC86473796458157B34291C
BA574A12C816C249D83792B5A861CBC68A755A16759248453CA17B5B314A963128576B913B54C836
212439A249D87B23718476828453949AB36C169427A327C18578324BB261A64733B79816A2895441
A5796C9516AC23BC897B4116C6782C5523D25A69DA87514C79A3B451698281237C36B451A5346732
ADB194463C18B748562516A1237549346C7C836224C7BA182B597A34625349CB7359BAB7123676AC
1482A718C59C839214563B47B325A9476188124B3527639A9C8265716A9C1B584195A42CB37
$\pi_2(13,4)$ = (65,65,65,65,62,61,60,58,55,54,52,45,8), norm=42163
B24685312B962A3A23194573272496B827161435C487A87CB96435C31954C216B6B3A94759C7391C
23A68419858C7A586B944163CA82CB7361254B177C56823C758A95B11B884937466824A614385917
B957A286D2926A51B54378B913624A67C57B223B15A18C4B3989261573544A9D1CCD175631B9B6CA
3A854CB17A592361724848725C61C84AA82B7352692473B256349124A68B73829C635A15C923491C
B65739148A16843BAA299475CA61D8B76C755C16D5C4373528B1975A213B9A2147568BA127538B27
41432A943A9C6B42B19396C47352A389B26816734CA24C8175A924378461B639229A8716B4CA5331
B59AC8C51AB8426789CA3116768B475542B45869AC7B51398AB2A35178949142682D7351A5236824
ABC193362B1767385A4516B142C53A23C89B726443B6A71748596C23C45236BA8257A97C1426A68B
1394A91785C89274135B2D37B2458C3A617A1439254CA286C974658167A81B593185634B72D
$\pi_3(13,4)$ = (65,65,65,65,62,60,60,58,57,54,49,47,8), norm=42147
B326A54137963D47341825C43932867B3A16124572AC8C8A786245941852931C7A74B62C599C481B
348A721A75ABC95A67922164C8936A7461352B187958A34875CB85B11ACC28492A6C32A612485B19
775993B8A3636751B5249CB814632D69659B33471581CA27468A361594522B761ACB185A419C768A
4A952BC1875934B1932D2CB35861B728A83C94536832B4735642A132ABC96473786459157B34281C
BA574A12C916C248C93782B5A961DCC69A755A16758249453BA17B5B314A863129576B813B54C936
212438A248D97B23719476929453848AB36C168427A327C19579324BB261A64733B78916A2985441
A5786C8516AC23BC987B4116C6792C5523B25A68BA97514CA8A3B451689291237C36D451A5346732
ADB184463C19B749562516A1237548346C7C936224C7BA192B587A34625348CB73589AB7123676AC
1492A719C58C938214563B47D325A8476199124B3527638A8C9265716A8C1B594185A42CB37
$\pi_4(13,4)$ = (65,65,65,65,62,60,60,58,57,56,53,37,12), norm=42015
A236B54127A62847241835942923 86CC2916134573B98A8D7B6345C41A5392167674AD3B599B481A
24AC731C75C98A5B67C33164A8C26A7461253B187958B24875B985A11CBA3849376A23C613485A19
B75992B8C2626751B53498A814623D69659B224715C1BA374689261594533B761A8A195C41AD76AB
4B853BA1875924A1923738B25C61B738A8269452BB2394725643A123ACB96472786459157C24381B
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313428B348BC7A32719476B39452848CA26A168437C237A19579234BB361C64722978D16A3985441
A578CD85169A32B9C87A41169678395532A35C68AB97514B78D2B45168A39132BA28B451A5246723
BDB184462C19B749563516B1327548246C7B926334C7DA193B587A2463524CB87259B687132A76DC
1493A719C58A928314562B47B235C8476199134A2537628C8C9365716A8C1B594185A436B27
$\pi_5(13,4)$ = (65,65,65,65,62,62,60,59,56,55,49,40,12), norm=41975
B235861429C52C1A214C3671272395A827454316A387B87CAB53169148639245A591CD3765A71D49
219C834D86897A6859C33451AC82B97154263A477B65821A7689B6944B883C17355823C543186A47
976A9285D2C25964A631789A41B23C57567A221D46B48C3A19892A4691633AB548BA476514BCA5B9
18D63B94786C215472383A726B54C83B9B2A71625B2371C2651394237B597182C9516946AC213B49
A56A1C43874C831DAA2CB3769B5498975D766845A6831A1625A4C86C241BBD24376589A427618C27
34312B931BC95A329481C5737162A1BA925845713CB23584767C23178354951A22C9874593BA6114
A6CB58A64DB832978A9B1445758A376632A36C5AB97864198B92B1647893B432582A7164D621C823
B9B451152947571A683645C432A61A2198B8725331D5B7473B69CA21B36215CA8267B9794325A589
41C38947A6B8B27341652917B236B918547C431A263C82A5A9735684578A4B6914A6513B72C
$\pi_6(13,4)$ = (65,65,65,65,62,61,61,59,58,54,50,32,18), norm=41795
C24765312ABC293A23194583282497B6271714359468B689CB7435A31654821A97A3DB495ABA3C1D
237A641D65689B567AB441C39A627C9371254A18985A623AB568A5B11A6649384C7624C714365918
A958826BD2727B51A543868913724C78758B223A15D16D493B682715935449D716CB185731ACD78B
36A54CB1A95823D182464B925C71864B96278352AA2483725734B124B9687362BA735815D9234A1C

APPENDIX TABLE (*Continued*)

97593A146817643D892AB4A5C88196D97A955617C564383526C1B65D2137BC21485768B129536827
414328B43BDA97429183A7C49352B3A8B27617A349B24961C59824396471A73B228B6A17A4895331
A5BC96C5178642D968B8311797684A5542B45C78AAB6513968B2A3517684A1429627B351B5237624
ACB18337291A8936574517C1429538237696B27443C9AD1A465B9823A4523BAD625C97C9142B976C
1384C91865A682B413572C3BD245B83971A814392549726A9A8475617BA61C583165A347B29

$\pi_1(13,5) = (123,123,121,115,110,109,109,102,99,92,84,72,28)$, norm=135679

1C4269A6943D73559B1427352C6DA3BCA9172491189765318294673B8B6AD2B9A978371531473225
B647C28A16D4ACB2174987D1162549C8A65D79B43BC5A247C35185BAB376D3289B1A587C62431294
CA91889B63A52CB794215654CA33521A1768C172D6184C3A62D3152BA2867A43D98BC57B83356765
2C948A1294C3BAD7486213155BA4562821A792B39A776C89356642B81174C2B29837351C853A24B6
511237596132C69A8C4A3997B415BC8614235DB864981C535A762A1B7827A39C8416B47158849219
634C625B58AD3C612B41A39965A789B74265A9CB2B477216CC532194721DB6327CAB512C3143B815
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C77514353CB7463D1A26374186596A9748C192275314986A541BCD76931739B3B2465C14764A239B
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5A4899A25618283B9192285B5364379837C1A65361642278794C2A5B749BCA5484363A58147D8C2A
9CB1863D23197B23746998A5BC8137163246AB7125619AB3762B4C2B3598613582631A4589142813
45CD457A798C917A637258B1A9D4D642762A73C5C63214A8B58775132139A47219632A684C13CA65
894B85513A8496A267749C73B491912B52436CA693B4275B134A1D973865812C2B6337281B3754A7
41A9521297433ABD43152C8A25C5781694854B76932713728CA9158A3A4687B694D1254A36591896
5C473B2

$\pi_2(13,5) = (123,123,121,115,110,109,109,101,99,93,86,68,30)$, norm=135557

1C4269A6943D73559B1427352C6DA3BCA9172491189765318294673B8B6AD2B9A978371531473225
B647C28A16D4ACB2174987D1162549C8A65D79B43BC5A247B35185BAB376D3289D1A587C62431294
AA91889B63A52CB794215654CA33521A17689172C6184C3A62D3152BA2867A43D98BC57B83356765
2C948A1294C3BAC7486213155BA4562821A792B39A776C89356642B81174C2B29837351C853A24B6
511237596132C69A8D4A3997B415BC8614235DB864981C535A762A1B7827A39C8416B47158849219
634C625B58AD3C612B41AD9965A789B74265A9CB2B477216CC5321947213B6327DBB512C3143C815
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41A9521297433ABD4315268A25C5B81694854B76932713728CA9158A3A4687B694B1254A36591896
5C473D2

$\pi_3(13,5) = (123.122,121,114,110.109.109,102,97,91,85.77,26,1)$, norm=135437

1C427A97A43B63559C1426352E78939ABC1624A118A6753182A476388B79C2BA9C68361531463225
D746C28817A4ACB2164D86B1172549B8975C6B943AB5D246B351A5BDC36783287C195868724312A4
BA9188D7739539B6A4215754CB335219167CC1629718A43972831529A2876943BCBCA56B833576C5
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B3B472842AAC7C965127CA58843C4669815279633522B1CB7914A4218455BA1837A5A7C227C8731A
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C982C4A6392C5B64192861B5B4A271752843B347215368CA2B12985B465B32818174A36928457394
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5C48B9A25718283B91A228585374369A36D1C7537174226B6B4C29AC64B9AA5484373958146C8A29
BAC1873D23196B23647AB8D58BC1361732479B612871D9D367284C2B35587135827319458A142A13

APPENDIX TABLE (*Continued*)

45CB45696A8CC16A736258B19AB43792672A63C5C73214CAB5866513213D94621A73297845138C75
BA4A85513B8487B27664AD63D491912A52437BC793A4265A134B1A56387581292C73B6281C3654D6
41C9521296433AC94315278B25A5681794A54B67932613628B9C15893A4786B784D12949375A1B97
5A463C2

$\pi_4(13,5) = (123,123,123,116,110,109,106,100,98,92,81,68,38)$, norm=134757

B23968695A47B912864A516A4C937B4C1173823D315D16624C9A589281497A739574B313D1284D6B
8158A4D392745C67B38D28152C46285C9B7A52673B36149478161A48275B92A73C3B162419826654
3365A59224778C451B3A61A7984498A3A571B325659A83C419671A6258641B72A927358278179314
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A178B215621DCC3B62468B8776513A5292C48759A37942C8B14A9154C6229B283C8697546B1879DA
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67452C7A185423C7983C732B1A8C827254358B63A1C96D49466417ACB127535B7A3DB5146271288D
3926436C9C5AC51B618243BC8A73496B57391A754816A935A92673341AB5129AAD74D6819419726D
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1A35624

$\pi_5(13,5) = (123,123,123,116,110,109,107,104,97,89,83,62,40,1)$, norm=134753

C319686A5D478923864B526A49917C4B2271831A125926634A985893824B7A719574A121B2384C69
A258D4C1A3745A67C18B38253A46385A9B7C53671C1624A478262C4D375CA3C71B1926342BB36654
1165B59334778A452A1D62A7D844A8B1C572B13565A981C42867296358642D73BC37158378279124
A9B31764235A4723A8B21659834215996 3AD8CB53A7D412C785DAB612347C461A3B47D263874592A
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C286DBC3112775932A64843B6219DC54BA6BA9C35124981537A27B134653AB2C4D6897138D569AA8
2C15D34

$\pi_1(13,6) = (166,166,160,156,143,143,139,135,131,122,107,100,46,2)$, norm=239106

14B2365C352A813897972A1B6493B27856617298514BA3CA784A3164753821C4C3A267554624B619
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APPENDIX TABLE (*Continued*)

2B1C5437A67A6592C4B168524317A5381724CA984361D627B8ABD649148A7A425152DB62756A8B43
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21369576156794C359732C7641BC54214A9718524AB32623869142973AA762324B11D38B9B675384
3DC2135AB7152A884A5C3726C19B6819472A
$\pi_2(13,6)$ = (166,166,160,156,144,142,138,137,131,120,106,102,46,2), norm=239082
14C2356B362A813897972A1B5493C27865517298614CA3BA884A3154763821B4B3A257664524C519
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$\pi_3(13,6)$ = (166,166,160,156,143,142,138,136,130,120,111,97,51), norm=238832
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59D426C139B3A786A534B12568CD61395D46A37C84B51973A2446C1C7227A73A482B56C187B23941
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3DC2135AB7152A88495C3726C19B6819472A
$\pi_4(13,6)$ = (166,166,160,156,145,142,139,136,131,118,113,91,50,3), norm=238698

APPENDIX TABLE (Continued)

1472356B36AC71927AA82B1BC493A2896551827C614DA3BA89473154B63921B4B3D258664524751A
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317B2542A412B9A664523958D13B7827335917B4C28656179591496CB327A4381AB556A9D498C216
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$\pi_5(13,6) = (166,166,160,156,145,142,139,136,131,119,112,88,52,4)$, norm=238384

1472356C36AD71927AA82D1CB493B2896551827C614DA3BA89473154B63921B4B3C258664524751A
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$\pi_6(13,6) = (166,166,160,156,145,144,137,132,127,118,111,98,56)$, norm=238116

14D23599368CB182A747281BA4C3B27C659172AB614D93A87C4A3154BA3A21D9A34257664B28951D
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APPENDIX TABLE (*Continued*)

31BB25426412CD8664923A56813BCC2933B7168482B7561A96714B658329543719655CAA84CB8217
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$\pi_7(13,6) = (166,166,160,156,145,142,140,136,131,118,106,86,59,5)$, norm=237556

1482356D36AC813287472A1A94C3A27C6551728B614AC39D79483154763D2194A3B257B64529851C
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$\pi_1(14,4) = (91,91,91,91,81,79,78,77,74,73,71,62,42)$, norm=79393

61B8473919C812581BA53DA7141B2976148C74235A9D3A6586B472A2736BC19B25C94342A7C7349B
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54C89AA7962451D9428AB27365ADBA3861CC643D928747325DB16C5764D83241B918A5673B16491C
52C7844512AB582B3A78735CA194291543A6C1B847B3AC2B297863134591B5B3A17829B2AC148A96
2D3C963758424186B95A1D7A213D894B9655938C1235417D84AC2369CA6B7248568254C7B133CDB8
293A76B248A5C36D4592374653781B184531A5C42B791B6278AB1A716D6529D43859741242987634
971646189731248CAB29C95A381283CA56B958672124BC195A2318D63CAD64984B937A5B5637D421
7A6249C318195B761DA9498B62353C254A8152C7B63494A3152B7682C189547A1D46721AB63C7D32
954638AB1DC3B2D5A764A67483512CB543682A719

$\pi_2(14,4) = (91,91,91,91,80,79,78,78,75,74,71,60,41,1)$, norm=79357

61B8472A1AC813581B952D97141B3A76148C743259AD296586B473937268C1AB35CA424397D724AB
156C4329D763CD25B38C9826AAB12CD3B417285C849A61392A796D8CB496583D7BA5174342A6C362
95D83168317142C62945DB265AB154328AC911DB8749674A3C981246735A3247C6D2B47CA564AC32
895B763B945718CB81254561935A3B9898164B91321D74C1723DAB18D4372951563AC6B2981A2CB4

APPENDIX TABLE (*Continued*)

977C45B2385A7C34B61383257494622A3456C95BB6C9342A198B675C193A9712489CDE8217A9631A
B7B4125D567843614825A5242CB18387A1D7B3DC5641836792CB12B98AD7A57D1148362534A26C32
54C8A997A63451DA4389B372659DB92861DC642BA38747235CB16C5764D82341BA1895672B164A1C
53C78445139B583B2978725C91A43A154296C1B847B29D3B3A78621245A1B5B291783AB39C1489A6
3C2CA62758434186BA591D79312C8A4BA655A28D1325417B849C326AC96B734856835497D122CDB8
3A2976B34895C26D45A3274652781B18452195C43B7A1B63789B19716C653A54285A741343A87624
A7164618A721348C9B35CA59281382C9B6DA58673134BC1A593218D62C9D64A84BA2795B5627D431
79634AD2181A5B761DCA4A8B63252C35498153C7B624A492153B7683C18A54791D467319B62A7D23
A546289B1DC2B3D59764967482513CB542683971A
$\pi_3(14,4) = (91,91,91,91,81,79,79,77,76,71,67,67,38,2)$, norm=79389
71C9462818C913591BA52CDE141B3867149C64325E8B2A7597B463A36279C18B35C842435696248B
1D7C43258673DA28B39C8927A8B12CA3A416295DC45B713D286B7A9CB4A7593D6B8516434287C372
55A9317931D142672A456B2758B1C43298CA116A9D4D764538A9124763583246C7D2B46C85748D32
9A5B673BA45C196B91254571A3583DA9AC174BA1321B64C1623B8D19C4362A515738C7B2A9182DB4
AC6D458239586C34A713932564A4722834579B56B7DC342D1A9B765C1A38A61249AC8B9216DA7318
B6C41259576C4371492585242CB1939681A6B3CA57419376B2CB12BA68CD85691149372534827B32
549D8AA6873451CD439A836275A8BA2971DC7426D39646235CB17C5674D92341B819A5762B174D1C
53C69445138B59382A69625CA1843D1542A7C1B946B2C83B386972124581B5B2A16938B3AC149A87
3C2C872659434197B85A196A312CB84BC755829A1325416C84AC327BCA7B6349579354A6D1225ABD
382A67B349C5C27B4583264752691B194B21A5C43B681D7369AC1A6178753AD42958641343896724
861747198621349CDB38C85A291392CA57B859763134BC185A3219C72DAC74894C826A5B5726D431
6D7348C219185B671CA8489B73252B354C9153A6B72484A2153D6793C198546A1847631A872D6C23
854729AB1CD2B3D5A674A76492513CB542793A618
$\pi_4(14,4) = (91,91,91,91,82,79,78,77,75,72,67,62,45)$, norm=79269
638C49283B7C315B3A55279A343617CC348794125BA62865BCA49171927DB3CD15B742415B7924C7
3C764125AD618B2D8165AD29BCB32891A4372856D4A6731B2C9DA67864BA5D1869A53841429B7182
55C913771353426D284A69285D635412BA9C339B7A467D4815DA3246815D12496782947C9584AD12
6A5B7B1C64593A8A73254583B15A17D8D93B48631237A4C3621CB93B741A2C535918BD62C8382694
C79B45621A5CBD149531A12DB464822814677B5C98BA142835C8769C391A6A324DB6A9723DB65138
CC54327A97B541D34725A5242CA3717D9369618C5B43C19D82893296B67A75DC334A192514726C12
54896D6A8B14537C419D8172A5D8CA297369742B818B4B215A63A876B47C21438A3C956826354635
819B744731989D1A2CB79258B36418354265A37C4C829A1A179B623245B3C5B293DA1C81673478BD
192CA9275D41436A79583AB6132ACB467955C2D6312543B8748C12A6B96A81475D71548BA32257AA
152B97A14865D2C84C612D4A725D36384D2368C416B93B7189CD39A3CC6818B4297A54314167BA24
6839483BAC23149B751AAD5B2731C2875699568713146B385A123DC92A9C74664B9278575C296413
6C7148923D3957BA36B847CA612528154693519C6824A4B235178CD1B39D5496364CB1369A27CB21
A5452A6D3C82A1759864AC7482531B85427A19B36
$\pi_1(14,5) = (169,169,165,156,156,152,149,144,143,137,134,121,118,80,9)$, norm=291280
864C73E574DB519391AB23D415CE4B1A7B989A7CC34E2B68AD132D6CAC65219B8A358D1219448B35
ADE62AC62747B29568714453327DB9A3AD571BE2998B659126BAB828C695A4C276AC189978345B73
656A8A9E3727245AEE2F11E2B3D8B75451793841BD9AC7297736142B5CE6939352A14DCD1C95B463
D6EDC87A98121EBA7382528A64B72B616289EC5493DD8C5275D873DE9A6816C95425C16844A81372
AEB37CD41283ED241A53E9918ED5A6DB7538169AA4812BCBC1D78B7234853492DC386E9AD14157AD
371A3514126BA3B742664AB8B39C9C5E2D46516D7AC538B26B72C892143956DA19C7A5E62B37469A
3B17D481326874A51B5623492B7215C4A58D67588D996C3A94B7543727DB3634C16D9182ED172B59
ECB495D8C4DA5A71E7D2832C3481C3754825A3D6321B791829A461C1B75BC929311C4FD59B6734D4
62A87B6CB235418A54E187A63E38629596CA7384515B17281E6DD453B2239686A4B486913BC28E39
5A12B75C123594845AAB73CCA617235DBF69E1BD7EA672C4D636985F758CA12392A8D1475B3C9213
45684251D82E235A14A7C6E19D86C917826839ED756361D84B245D597B2C9A47638A614B67BAC227
43C3A48771C6531BE2B5E6D937281A4C19834D7A16C34E1D3B1697AB9CB659E442838C2A54F3DBC2
9B4185275DA16395DB286E194531D24B5A6372A5BCA4E86159D47C65A9BC3139C153C4A221B76B39
83251DAD47B87E6CB5587DB139271D56A62495C48C562A1422C587E69EA32E67D94A517368C394B7
1D324B951762B9AD318D7AAC9B166849758231A7398C4654C3E2E32A16D1295A4A38C867B865D2A6
B91D5752884913ABE74563FECD172DF8ABA1C37294864AE7D2C3392C55718ED2D567147CB2143EBA
314829627A3483A1B5218934C82D6615E4C82B758D48D7621536D762C58437B659E219A85394BD51

APPENDIX TABLE (*Continued*)

69B3DEC769E4346A5A57B3DC8171B29318B4DA3ADBC5E12E24C67615A3E8891435C646C72819D565
42A2ADE12CD743D54C91BDA378FE6C723792164BD18946A1B5A85273D5D434C9E6219BBE6371A68A
5719EB3A97C491C35EAB5F42D86243B7183492B65712AC78D1942673264957A1687123DD325BCB42
68E69187A598AE43C17B9865CDA9D4452A3CB52B1A4EC32C7136CE3A6172B43A8914175D269A2181
B2952BA73DC648B96354AA4792CBE9618A75735B846B283C9D435A62137284551476C312B8D9A5DE
A29B6471C3DE82563C914B967D14C258C1397CDABE7E43C98B746851235A12BBDE9AB235C61D3EB2
1895529A3E8D19562D4D43786BEDB4329657526B8A138C98C751421A463C792BEC1A736B854294D2
61367842C9A6D3CB8134C9A15B248658337D121E64D2A75B954CA78E3746912C91A236E81B9D8AC3
57

$\pi_2(14,5) = (169,169,165,156,155,153,151,147,143,137,134,120,112,76,15)$, norm=290646

964D72E574CB518281AB32C415DC4B1A7B898A76D24C3B69EE123A6DAE65318B9A259E1318449B25
ACE63AD63747B385697144522 37CBDA2AC571BE3889B658136BAB939D685A4D376AD1988C9245B72
656A9A8E2737345E8E2B11E3B2C9B75451782941BC8AD7387726143B5DE6828253A14CDC1D85B462
C68AD97A89131DBAE293539A64B73CB1639A7D5482EC8D5375C972FD8A6916D85425D16844C91273
AEB27DC41392ED341A528C819EC5A6CB7529168AA4913BEBC1879B7324952483CC29678EA14157AB
271A2514136BA2C7D3664AB9B24D8D5E3C46516C7AE529B36C73D983142856CA18D7A5E63B27468A
2C17D491236974D51B5632483B7315D4A59C67599E886D2A84B7542737CB2624D16C8193EC173B58
ECB485A9E4CA5A71D7E3923D2491D2754935A2C6231B781938A461E1B75BD838211C4FD58B6724C4
63A97B6EB325419A54E197A62D2963C586EA7294515B17391C68B452D3328696A4B496812BE39C28
5A13B75D132584945AAB72DDA617325DBC68E1BB7DA673D4C626895C759EA13283A9B1475B2C7312
45694351C938325D14A7D6C1CE96D817936928EC756261C94B345A587C3D84476298614B67BAC337
42D2A49771D6521B83B5E6DD2739184E1892CA7A16C24B1D2B1687AB8EB658E443929D3A54D2CBE3
8C4195375CA1628ACD396E184521C34B5A6273A5BDA4B9615E847C65B8BD2128D152D4A331B76B28
92351DAD47E97A6DB5597CA12B3718E6A63485D49F562E1433D597B68CA23E67C84A517269B284D7
1C234BD51763B8AC219E7AAD8E166948759321A7289C4654D2F3B23C16C1385A4A29D967B965D3C6
BC185763994812ACE74562FCBD173BD9AE51D27384964AE783C22F3D55719E83C567147DB3142ACA
214938637A2492C1B5319824F93D6615E4D93A759C49D7631526F763A59427B658C318A95284BC51
68B2ECD7A864246E5A57B2CC9171B38219B4EA2A8BD5C13E34D67615D2E9981425A646D72918C565
43A3ABE13BC742C54D819BA27DFE6E732783164BC1984691B5A95372C5C424C8D6318BB86271A69A
5718DB2AC7D481D25EAB5C43A96342B7192483B65713A579F1843672364E57A1697132EC235BDB43
69E68197A58DAC42B17B8965CEA8F4453A2DB53B1A4BF23D7126EC2A6173942CE814175C368A3191
B3853BA72CD649B86254AA4783CDC8619A75725D946B392D8C425A63127394551476D213E9C8A5DA
F38B6471D2CB93562F814E867A14D359C1287BD8A67B42B89A746951325913CBCE8AB325D61A2EB3
198553CA2E9D18563E4C42796B5CB4238657736C9A129C89A751431D462A783ECF1D726B9543C483
61267B438CD6A28B912498A15D34965D227B131C94C3A75B854BA79A2746813B81A326C91B8C9AD2
57

$\pi_3(14,5) = (169,169,165,156,155,153,151,147,142,137,133,124,109,75,16,1)$, norm=290288

964E73E574DB518381AB23D415CD4B1A7B898A76C34D2B69EE132A6CAC65218B9A359C1218449B35
ADE62AC62747B28569714453327DBCA3AD571BE2889B658126BAB929C685A4C276AC1988F9345B73
656A9A8E3727245A8E2B11D2B3C9B75451783941BD8AD7287736142B5CE6838352A14DCD1C85B463
E6EAC97A89121EBA6392529A64B72DB1629A7C5483ED8C5275D973EC8A6916C85425C16844D91372
AEB37CD41293ED241A538E819ED5A6DB7539168AA4912BDBC1879B7234953482DD3967CAE14157AE
371A3514126FA3B7C2664AB9834C8C5D2B46516D7AC539B26B72C982143856DA18C7A5E62B37468A
3D17C491326974C51B5623482E7215C4A59D67599B886C3A84D7543727DB3634C16D8192ED172B58
EDB485A9C4DA5A71C7D2932C3491C3754925A3E6321B781928A461E1B75BC828311D4DC58B6734C4
62A97B6CB235419A54E197A63E39628586CA7394515B17291D68B453C2238696A4B496813CB29D38
5A12B75C123584945AAB73CCA617235DBE68D1BB7EA672C4D636895D759CA12382A9B1475B3C7213
45694251D928235C14A7C6D1DE96C817926938DA756361D94B245A587B2C84476398614B67BAD227
43C3A49771C6531B82B5E6DC3729184E1893DA7A16D34E1D3B1687AB8CB658E442939C2A54E3DBF2
8C4195275DA1638ADB296C184531D24B5A6372A5BCA4B9615F847E65B8BC3138C153C4A221B76B38
93251DAC47E97A6CB5597DA13B271856A62485C49C562E1422C597B68DA32D67B84A517369C384E7
1D324BC51762B8DE319E7AAC8B166948759231A7389B4654C3B2F32E16D1285A4A39C967B965C2D6
BE185762994813ABD74563FEDC172DC9AB51C37284964AD782C33F2C55719E82E567147CB2143AEA
314928627C3493E1B5219834C92D6615E4C92A759D49D7621536F762C59437B658E218A95384BD51
68B3DEC7A86434 6D5A57B3DC9171B28319B4DA3A8BC5E12E24C67615C3E9981435C646C72918D565

APPENDIX TABLE (*Continued*)

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42A2ABE12CA743B54D81DBA379FC6C723782164BD19846A1B5A95273D5A434C8GD218BB86371A69A
5718CB3AD7C481C35EAE5F42F96243B7193482B65712AC79D1842673264D57A1697123CE325BCB42
69D68197A589AD43B17B8965EBA8D4452A3EC52B1A4BE32C7136CE3A6172B43A9814175D268A2191
C2852BA73DC649B86354AA4782DCD8619A75735B946B293C8D435A62137294551476C312E9D8A5FD
A28B6471D3EB92563E814C867A14C259D1387BC8A67B43B89A746951235912CEDF8AB235C61A3EB2
198552EA3C9E18562F4D43796B5DB4328657726D9A139D89C751421E463A782FEC1D736B9542D482
61367B428DE6A38B9134C8A15E24965C337D121C94D2A75B854DA79A3746812B81A236F91B8E9AC3
57
```

$\pi_4(14,5) = (169,169,163,156,155,152,149,148,142,139,132,131,102,76,19)$, norm=289872

```
B4763834A858B2AC52D415EC2DB86C294F956DB47DA9134A772E1EB69D5A12AB346A8B2127B6B743
679416531898F1797C32835D419C637A496C25B1EF568A721949C51386E57A91D8C326B5EBA7469D
3C4366B57F17139B8D1A2281C8AAC74362583B42AE9C751A57962E1DCD65A89D41423C552BAEC436
B8A439B457212976E391D14CC3BB146231587A7985C9DAE1379A53B4A58A2646871892A784962CB1
F5C6EA38219CBD1328A5945289D3CA5B6CE724368B52146792ADD3C1848AC3C1B4D86B7D42523645
C629752A216B4B4E51D37CEAC57BEAB31976425BD8A393C17391468127956FD82C6537BD1357AC65
732EB942419C6869239E1B4713B129BA3AC6857D8BF64CAE85D7A43C1546A9C8625472C1862C1DA5
385B734D9576A8E2D641E51784924E6A7A1D83A3C12D9B24198352426774A81852274CD8945763F3
E15B6B78313CD2D8EB62EC6B9E9351CA75844BD3B2862A19295D9A57A11786D3B4C6C8426D51C37D
5421493A213D8796E54DA63B492D1ACB3785326B956A71CE4388B3C9594A8216518A427D6C37A129
55A9B1625C1B16672DBC3982A7C7A42481DBCC3857B7324E5B1373598914AF578D89326E64BC411C
DA93EA8472C5832BE164E53BA51A2C682DF947D829937A2DB72F634D586457C651A9C81DB3EABD51
8432961CD7C26E4C831A6524579231B7E869C187B34D5B6259C37C64A7DB8236724A8E41125639AD
681C2B5B493EFC3BADA79B32E4182835641BC9A57843162911E4B36735A91AF85D7762C9848D56C4
2A91A3A628514D76D23849784D299C5EA531D2B634859378BAC14A152632159B47839A58BE6A71CC
87263B51E9B724B6E3857B9D432619EAD8726C9154B3AE6D71945C1BACD23B9135A8276541278947
C2561B31AE53C5829B12A4786F1E872A6D3C1745C8D69831247EA7514AEBDC93E5A1264D8D9B3432
67C5D8B83967C537EB5CE598E2C2A1462E3B9DA4959C821C173FEB2A546C392D4AB39A8E13294BCB
7151C8621F4EC75A3862BD7756F9AE719431286C62A4A8B23737A1C5D64D7B74A51298EC33627C8D
3926394574AB72CCEB9D834168A145672DAB5144C32145B87268196517B3A532786215B8B15C96A1
83D3924ECBAEC9D5827BE5399C7BED4616483E16243DFB13925A46BD527156C3492B2A781FD81252
B14A1BD9934A785AC83769C3518864A276C56A59CDAE13B844597E312A5163AC26875C21487B5689
31EBD3729AEC41CB43A276893C24B148324A8D873549AB65C9D37B6215C92166C7AD61E9A824BCC1
24B73169B5A326781AD7EBCD45A566717E9C91C46823A454B892312345B8961E382958C669B14D51
7283A4D17996DA7C32C473A2B51675CA4B3A21297C51C86497DB453843E5621382D1B7CA2E85D3A6
B9
```

phism program NAUTY ([134], [135]). We thank him for letting us use this program.

Appendix
Tables of Partitions

This Appendix gives a number of partitions of the set of all $\binom{n}{w}$ binary vectors of length n and weight w into disjoint codes with distance 4 (see Section VI and especially Table VI). To explain the notation,

$$\pi(6,3) = (4,4,4,4,2,2), \text{ norm} = 72\text{: } 23541364214136122543$$

indicates that there is a partition of the 20 6-bit, weight-3 binary words into 6 disjoint distance-4 codes, of respective sizes 4, 4, 4, 4, 2, and 2. The norm is $4^2+4^2+4^2+4^2+2^2+2^2=72$. The lexicographically first word of $\Pi(6,3)$ (namely 000111) is in code 2, the second word (namely 001011) is in code 3, the third is in code 5. $\cdots$ and the 20th word (namely 111000) is in code 3.

The digit-string giving the code-numbers (or "colors") is in a constant-width font with 80 characters per line, and the colors are represented by $1,2,3,\cdots,9,A,B,C,\cdots$.

References

[1] E. F. Assmus, Jr. and J. D. Key, "On an infinite class of Steiner systems with $t=3$ and $k=6$," *J. Combin. Theory*, vol. 42A, pp. 55–60, 1986.

[2] E. F. Assmus, Jr. and H. F. Mattson, Jr., "Disjoint Steiner systems associated with the Mathieu groups," *Bull. Amer. Math. Soc.*, vol. 72, pp. 843–845, 1966.

[3] ——, "New 5-designs," *J. Combin. Theory*, vol. 6, pp. 122–151, 1969.

[4] R. D. Baker, "Partitioning the planes of $AG_{2m}(2)$ into 2-designs," *Discrete Math.*, vol. 15, pp. 205–211, 1976.

[5] R. D. Baker, J. H. van Lint, and R. M. Wilson, "On the Preparata and Goethals codes," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 342–345, 1983.

[6] E. Balas, V. Chvátal, and J. Nešetřil, "On the maximum weighted clique problem," *Math. Oper. Res.*, vol. 12, pp. 522–535, 1987.

[7] E. Balas and C. S. Yu, "Finding a maximal clique in an arbitrary graph," *SIAM J. Comput.*, vol. 15, pp. 1054–1068, 1986.

[8] L. D. Baumert, "Cyclic difference sets," *Lecture Notes Math.*, vol. 182. New York: Springer-Verlag, 1971.

[9] S. Bays, "Une question de Cayley relative au problème des triades de Steiner," *Enseignement Math.*, vol. 19, pp. 57–67, 1917.

[10] C. Berge, "A new color change to improve the coloring of a graph," *Discrete Appl. Math.*, vol. 24, pp. 25–28, 1989.

[11] M. R. Best, "$A(11,4,4)=35$ or some new optimal constant weight codes," Math. Centr. Report ZN 71/77, Amsterdam, Feb. 1977.

[12] ____, "Binary codes with a minimum distance of four," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 738–742, 1980.

[13] ____, A. E. Brouwer, F. J. MacWilliams, A. M. Odlyzko, and N. J. A. Sloane, "Bounds for binary codes of length less than 25," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 81–93, 1978.

[14] T. Beth, D. Jungnickel, and H. Lenz, *Design Theory*. New York: Cambridge University Press, 1986.

[15] M. Bláum, "A (16,9,6,5,4) error-correcting dc free block code," *IEEE Trans. Inform. Theory*, vol. 34, no. 1, pp. 138–141, Jan. 1988.

[16] B. Bose and T. R. N. Rao, "Theory of unidirectional error correcting/detecting codes," *IEEE Trans. Comput.*, vol. 31, pp. 521–530, 1982.

[17] R. C. Bose and S. Chowla, "Theorems in the additive theory of numbers," *Comment. Math. Helvet.*, vol. 37, pp. 141–147, 1962–63.

[18] R. C. Bose and W. S. Connor, "Combinatorial properties of group divisible incomplete block designs," *Ann. Math. Stat.*, vol. 23, pp. 367–383, 1952.

[19] D. R. Breach and A. P. Street, "Partitioning sets of quadruples into designs II," *J. Comb. Math. Comb. Computing*, vol. 3, pp. 41–48, 1988.

[20] A. E. Brouwer, "$A(17,6,4)=20$ or the nonexistence of the scarce design $SD(4,1;17,21)$." Math. Centr. Report ZW62, Amsterdam, Dec. 1975.

[21] ____, "On the packing of quadruples without common triples," *Ars. Comb.*, vol. 5, pp. 3–6, 1978.

[22] ____, "Optimal packings of K_4's into a K_n," *J. Comb. Theory*, vol. 26A, pp. 278–297, 1979.

[23] ____, "A few new constant weight codes," *IEEE Trans. Inform. Theory*, vol. 26, p. 366, 1980.

[24] ____, "Some unitals on 28 points and their embeddings in projective planes of order 9," in *Geometries and Groups*, M. Aigner and D. Jungnickel, Eds., Lect. Notes Math., vol. 893. New York: Springer-Verlag, 1981, pp. 183–188.

[25] ____, "Partial linear spaces with one point and one line more than affine planes—Report on results of K. Metsch as communicated by A. Blokhuis." unpublished manuscript, 1989.

[26] ____, "A unital in the Hughes plane of order nine", preprint.

[27] ____, "Block designs," *Discrete Math.*, vol. 77, pp. 55–56, 1989.

[28] A. E. Brouwer, P. Delsarte and P. Piret, "On the (23,14,5) Wagner code," *IEEE Trans. Inform. Theory*, vol. 26, pp. 742–743, 1980.

[29] A. E. Brouwer, A. Schrijver, and H. Hanani, "Group divisible designs with block-size four," *Discrete Math.*, vol. 20, pp. 1–10, 1977.

[30] P. J. Cameron. *Parallelisms of Complete Designs*. New York: Camb. Univ. Press, 1976.

[31] P. J. Cameron and J. H. van Lint, *Graphs, Codes and Designs*. New York: Camb. Univ. Press, 1980.

[32] A. Cayley, "On the triadic arrangements of seven and fifteen things," *Phil. Mag.*, vol. 37, pp. 50–53, 1850; *Collected Mathematical Papers*, vol. I, pp. 481–484.

[33] Y. Chen, "The Steiner system $S(3,6,26)$," *J. Geometry*, vol. 2, pp. 7–22, 1972.

[34] Z. Chen, F. Jin, and P. Fan, "On a new binary (22,13,5) code," *IEEE Trans. Inform. Theory*, vol. 36, no. 1, pp. 228–229, Jan. 1990.

[35] W. R. Cheswick, personal communication.

[36] L. G. Chouinard II, "Partitions of the 4-subsets of a 13-set into disjoint projective planes," *Discrete Math.*, vol. 45, pp. 297–300, 1983.

[37] H. Chung and P. V. Kumar, "Optical orthogonal codes—New bounds and an optimal construction," *IEEE Trans. Inform. Theory*, vol. 36, no. 4, July 1990.

[38] F. N. Cole, L. D. Cummings, and H. S White, "The complete enumeration of triad systems in 15 elements," *Proc. Nat. Acad. Sci. USA*, vol. 3, 1917, pp. 197–199.

[39] S. D. Constantin and T. R. N. Rao, "On the theory of binary asymmetric error correcting codes," *Inform. Contr.*, vol. 40, pp. 20–36, 1979.

[40] J. H. Conway, "The miracle octad generator," in Topics in Group Theory and Computation, M. P. J. Curran, Ed. New York: Academic Press, 1977, pp. 62–68.

[41] J. H. Conway, R. T. Curtis. S. P. Norton, R. A. Parker, and R. A. Wilson, *ATLAS of Finite Groups*. New York: Oxford Univ. Press, 1985.

[42] J. H. Conway, R. A. Parker, and N. J. A. Sloane, "The covering radius of the Leech lattice," *Proc. Roy. Soc. Lon.*, vol. 380A, pp. 261–291, 1982.

[43] J. H. Conway and N. J. A. Sloane, "Laminated lattices," *Ann. Math.*, vol. 116, pp. 593–620, 1982.

[44] ____, "Lexicographic codes: Error-correcting codes from game theory," *IEEE Trans. Inform. Theory*, vol. 32, pp. 337–348, 1986.

[45] ____, *Sphere Packings, Lattices and Groups*. New York: Springer-Verlag, 1988.

[46] ____, "Orbit and coset analysis of the Golay and related codes," *IEEE Trans. Inform. Theory*, vol. 36, no. 5, pp. 1038–1050, Sept. 1990.

[47] R. T. Curtis, "On subgroups of ·0. I. Lattice stabilizers," *J. Alg.*, vol. 27, pp. 549–573, 1973.

[48] ____, "A new combinatorial approach to M_{24}," *Math. Proc. Camb. Phil. Soc.*, vol. 79, pp. 25–42, 1976.

[49] N. Darwish and B. Bose, "New lower bounds for constant weight and asymmetric error correcting codes," preprint.

[50] P. Delsarte and P. Piret, "Spectral enumerators for certain additive-error-correcting codes over integer alphabets," *Inform. Contr.*, vol. 48, pp. 193–210, 1981.

[51] ____, "Bounds and constructions for binary asymmetric error-correcting codes," *IEEE Trans. Inform. Theory*, vol. 27, pp. 125–128, 1981; correction, vol. 36, no. 4, p. 954, July 1990.

[52] P. Dembowski, *Finite Geometries*. Berlin: Springer-Verlag, 1968.

[53] R. H. F. Denniston, "Some maximal arcs in finite projective planes," *J. Combin. Theory*, vol. 6, pp. 317–319, 1969.

[54] ____, "Some packings with Steiner triple systems," *Discrete Math.*, vol. 9, pp. 213–227, 1974.

[55] ____, "Sylvester's problem of the 15 schoolgirls," *Discrete Math.*, vol. 9, pp. 229–233, 1974.

[56] ____, "Some new 5-designs," *Bull. Lond. Math. Soc.*, vol. 8, pp. 263–267, 1976.

[57] ____, "Four doubly resolvable complete 3-designs," *Ars. Comb.*, vol. 7, pp. 265–272, 1979.

[58] ____, "Enumeration of symmetric designs (25,9,3)," *Ann. Discrete Math.*, vol. 15, pp. 111–127, 1982.

[59] I. Diener, "On cyclic Steiner systems $S(3,4,22)$," pp. 301–313 of [128].

[60] J. Doyen, "Constructions of disjoint Steiner triple systems," *Proc. Amer. Math. Soc.*, vol. 32, pp. 409–416, 1972.

[61] J. Doyen and A. Rosa, "An updated bibliography and survey of Steiner systems," *Ann. Discrete Math.*, vol. 7, pp. 317–349, 1980.

[62] ____, "A bibliography and survey of Steiner systems," preprint, Feb. 1989.

[63] G. Dueck and T. Scheuer, "Threshold accepting: A general purpose optimization algorithm appearing superior to simulated annealing," preprint.

[64] T. Etzion, "Constructions of error-correcting DC-free block codes," *IEEE Trans. Inform. Theory*, vol. 36, no. 4, pp. 889–905, July 1990.

[65] ____, "Optimal partitions for triples," *J. Comb. Theory*, to appear.

[66] ____, personal communication.

[67] T. Etzion and A. Hartman, "Towards a large set of Steiner quadruple systems," *SIAM J. Discrete Math.*, to appear.

[68] T. Etzion and C. L. M. van Pul, "New lower bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. 35, no. 6, pp. 1324–1329, Nov. 1989.

[69] A. A. El Gamal, L. A. Hemachandra, I. Shperling, and V. K. Wei, "Using simulated annealing to design good codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 116–123, 1987.

[70] L. Gerhards and W. Lindenberg, "Clique detection for nondirected graphs: Two new algorithms," *Computing*, vol. 21, pp. 295–322, 1979.

[71] M. Goldberg and T. Spencer, "A new parallel algorithm for the maximal independent set problem," *SIAM J. Comput.*, vol. 18, pp. 419–427, 1989.

[72] R. L. Graham and N. J. A. Sloane, "Lower bounds for constant weight codes," *IEEE Trans. Inform. Theory*, vol. IT-26, pp. 37–43, 1980.

[73] ____, "On additive bases and harmonious graphs," *SIAM J. Alg. Discrete Methods*, vol. 1, pp. 382–404, 1980.

[74] M. J. Grannell and T. J. Griggs, "A note on the Steiner systems $S(5,6,24)$," *Ars. Comb.*, vol. 8, pp. 45–48, 1979.

[75] R. K. Guy. *Unsolved Problems in Number Theory*. New York: Springer-Verlag, 1981.

[76] H. Halberstam and K. F. Roth, *Sequences*. New York:

Springer-Verlag, 1966.
[77] M. Hall, Jr., *Combinatorial Theory*. Watham, MA: Blaisdell, 1967.
[78] M. Hall, Jr. and J. D. Swift, "Determination of Steiner triple systems of order 15," *Math. Tables Aids Comput.*, vol. 9, pp. 146–152, 1955.
[79] H. Hämäläinen, "Two new binary codes with minimum distance three," *IEEE Trans. Inform. Theory*, vol. 34, p. 875, 1988.
[80] ____, personal communication.
[81] H. Hanani, "On quadruple systems," *Canad. J. Math.*, vol. 12, pp. 145–157, 1960.
[82] ____, "The existence and construction of balanced incomplete block designs," *Ann. Math. Stat.*, vol. 32, pp. 361–386, 1961.
[83] ____, "A balanced incomplete block design," *Ann. Math. Stat.*, vol. 36, p. 711, 1965.
[84] ____, "Balanced incomplete block designs and related designs," *Discrete Math.*, vol. 11, pp. 255–369, 1975.
[85] A. Hartman, "The existence of resolvable Steiner quadruple systems," *J. Comb. Theory*, vol. 44A, pp. 182–206, 1987.
[86] A. Hartman, W. H. Mills, and R. C. Mullin, "Covering triples by quadruples: An asymptotic solution," *J. Combin. Theory*, vol. 41A, pp. 117–138, 1986.
[87] A. A. Hashim and A. G. Constantinides, "Some new results on binary linear block codes," *Electron. Lett.*, vol. 10, pp. 31–33, 1974.
[88] A. A. Hashim and V. S. Pozdniakov, "Computerized search for binary linear codes," *Electron. Lett.*, vol. 12, pp. 350–351, 1976.
[89] H. J. Helgert and R. D. Stinaff, "Minimum-distance bounds for binary linear codes," *IEEE Trans. Inform. Theory*. vol. 19, pp. 344–356, 1973.
[90] T. Helleseth and T. Kløve, "On group theoretic codes for asymmetric channels," *Inform. Contr.*, vol. 49, pp. 1–9, 1981.
[91] I. Honkala, "Lower bounds for constant weight codes," (in Finnish), Master's dissertation. Mathematics Dept., Univ. Turku, Turku, Finland, Jan. 1985.
[92] ____, "Some optimal constant weight codes," *Ars. Comb.*, vol. 20B, pp. 43–47, 1985.
[93] ____, personal communication.
[94] I. Honkala, H. Hämäläinen, and M. K. Kaikkonen, "A modification of the Zinoviev lower bound for constant weight codes," *Discrete Appl. Math.*, vol. 11, pp. 307–310, 1985.
[95] ____, "Some lower bounds for constant weight codes," *Discrete Appl. Math.*, vol. 18, pp. 95–98, 1987.
[96] D. R. Hughes and F. C. Piper, *Design Theory*. Cambridge: Cambridge Univ. Press, 1985.
[97] B. Hupert, *Endliche Gruppen I*. New York: Springer-Verlag, 1967.
[98] D. S. Johnson, C. R. Aragon, L. A. McGeoch, and C. Schevon, "Optimization by simulated annealing: An experimental evaluation. Part 1," *Oper. Res.*, vol. 37, pp. 865–892, 1989.
[99] S. M. Johnson, "A new upper bound for error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-8, pp. 203–207, 1962.
[100] ____, "Improved asymptotic bounds for error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-9, pp. 198–205, 1963.
[101] ____, "On upper bounds for unrestricted binary error-correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-17, pp. 466–478, 1971.
[102] ____, "Upper bounds for constant weight error correcting codes," *Discrete Math.*, vol. 3, pp. 109–124, 1972.
[103] G. A. Kabatiansky and V. I. Panchenko, "Packings and coverings of Hamming space with unit spheres," (in Russian), Probl. Peredach. Inform., vol. 24, no. 4, pp. 3–16, 1988.
[104] M. K. Kaikkonen, "A new four-error correcting code of length 20," *IEEE Trans. Inform. Theory*, vol. 35, no. 6, p. 1344, Nov. 1989.
[105] ____, personal communication.
[106] J. G. Kalbfleisch and R. G. Stanton, "Maximal and minimal coverings of $(k-1)$-tuples by k-tuples," *Pac. J. Math.*, vol. 26, pp. 131–140, 1968.
[107] W. M. Kantor, "On the inequivalence of generalized Preparata codes," *IEEE Trans. Inform. Theory*, vol. IT-29, pp. 345–348, 1983.
[108] R. E. Kibler, "Some new constant weight codes," *IEEE Trans. Inform. Theory*, vol. 26, pp. 364–365, 1980.
[109] T. P. Kirkman, "On a problem in combinations," *Cambridge and Dublin Math. J.*, vol. 2, pp. 191–204, 1847.
[110] T. Kløve, "A lower bound for $A(n,4,w)$," *IEEE Trans. Inform. Theory*, vol. 27, pp. 257–258, 1981.
[111] K.-U. Koschnick, "On the construction of some new constant weight codes," preprint.
[112] A. Kotzig and J. Turgeon, "On certain constructions for Latin squares with no subsquares of order two," *Discrete Math.*, vol. 16, pp. 263–270, 1976.
[113] E. S. Kramer, "Some triple system partitions for prime powers," *Utilitas Math.*, vol. 12, pp. 113–116, 1977.
[114] E. S. Kramer and S. S. Magliveras, "Some mutually disjoint Steiner systems," *J. Combin. Theory*, vol. 17A, pp. 39–43, 1974.
[115] E. S. Kramer and D. M. Mesner, "Intersections among Steiner systems," *J. Combin. Theory*, vol. 16A, pp. 273–285, 1974.
[116] ____, "The possible (impossible) systems of 11 disjoint $S(2,3,13)$'s ($S(3,4,14)$'s)," *Utilitas Math.*, vol. 7, pp. 55–58, 1975.
[117] ____, "Admissible parameters for Steiner systems $S(t,k,v)$ with a table for all $(v-t) \leq 498$," *Utilitas Math.*, vol. 7, pp. 211–222, 1975.
[118] ____, "t-Designs on hypergraphs," *Discrete Math.*, vol. 15, pp. 263–296, 1976.
[119] D. L. Kreher and S. P. Radziszowski, "The existence of simple 6-(14,7,4) designs," *J. Combin. Theory*, vol. 43A, pp. 237–243, 1986.
[120] C. W. H. Lam, L. Thiel, and S. Swiercz, "The non-existence of finite projective planes of order 10," *Canad. J. Math.*, vol. 41, pp. 1117–1123, 1989.
[121] H. Lenz, "Tripling Steiner quadruple systems," *Ars. Combin.*, vol. 20, pp. 193–202, 1985.
[122] V. I. Levenshtein, "The application of Hadamard matrices to a problem in coding," *Probl. Kibernetiki*, vol. 5, pp. 123–136, 1961. English translation in *Probl. Cybern.*, vol. 5, pp. 166–184, 1961.
[123] Shen Lin, personal communication.
[124] C. C. Lindner, "A note on disjoint Steiner quadruple systems," *Ars. Combin.*, vol. 3, pp. 271–276, 1977.
[125] ____, "On the construction of pairwise disjoint Steiner quadruple systems," *Ars. Combin.*, vol. 19, pp. 153–156, 1985.
[126] C. C. Lindner and A. Rosa, "There are at least 31,021 non-isomorphic Steiner quadruple systems of order 16," *Utilitas Math.*, vol. 10, pp. 61–64, 1976.
[127] ____, "Steiner quadruple systems—A survey," *Discrete Math.*, vol. 22, pp. 147–181, 1978.
[128] C. C. Lindner and A. Rosa, Eds., "Topics on Steiner systems," *Annals of Discrete Mathmatics*, vol. 7. Amsterdam: North-Holland, 1980.
[129] J. H. van Lint, "A new description of the Nadler code," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 825–826, 1972.
[130] J. X. Lu, "On large sets of disjoint Steiner triple systems I–VI," *J. Combin. Theory*, vol. 34A, pp. 140–183, 1983; vol. 37A, pp. 136–192, 1984.
[131] ____, "On large sets of disjoint Steiner triple systems VII," unfinished manuscript.
[132] F. J. MacWilliams and N. J. A. Sloane, *The Theory of Error-Correcting Codes*. Amsterdam: North-Holland, 1979.
[133] R. Mathon and A. Rosa, "Tables of parameters of BIBDs with $r \leq 41$ including existence, enumeration, and resolvability results," *Ann. Discrete Math.*, vol. 26, pp. 275–308, 1985.
[134] B. D. McKay, "NAUTY user's guide (version 1.2)," Dept. Comput. Sci. Tech. Rep. TR-CS-87-03, Australian Nat. Univ., Canberra, Australia, 1987.
[135] ____, "NAUTY notes," preprint, Feb. 2, 1989.
[136] E. Mendelsohn and A. Rosa, "One-factorizations of the complete graph—A survey," *J. Graph Theory*, vol. 9, pp. 43–65, 1985.
[137] N. S. Mendelsohn and S. H. Y. Hung, "On the Steiner systems $S(3,4,14)$ and $S(4,5,15)$," *Utilitas Math.*, vol. 1, pp. 5–95, 1972.
[138] W. H. Mills, "A new 5-design," *Ars. Combin.*, vol. 6, pp. 193–195, 1978.
[139] R. C. Mullin and A. Rosa, "Orthogonal Steiner systems and generalized Room squares," in *Proc. 6th Manitoba Conf. Num. Math.*, Winnipeg, 1976. Congr. Numer. XVIII, Util. Math. Publ. Inc., Winnipeg, 1977, pp. 319–323.
[140] P. M. Pardalos and G. P. Rodgers, "A branch and bound algorithm for the maximum clique problem," preprint.
[141] I. Peterson, Search yields math proof no one can check," *Science*, vol. 134, p. 406 Dec. 1988.
[142] K. T. Phelps, "On cyclic Steiner quadruple systems $S(3,4,20)$," pp. 277–300 of [128].
[143] ____, "A class of large sets of Steiner triple systems of order 15," preprint.
[144] ____, "A class of 2-chromatic SQS(22)," preprint.
[145] ____, personal communication.

[146] K. T. Phelps and A. Rosa, "2-chromatic Steiner quadruple systems," *Europ. J. Combin.*, vol. 1, pp. 253–258, 1980.

[147] P. Piret, "Good linear codes of length 27 and 28," *IEEE Trans. Inform. Theory*, vol. IT-26, p. 227, 1980.

[148] V. Pless, "The children of the (32, 16) doubly even codes," *IEEE Trans. Inform. Theory*, vol. IT-24, pp. 738–746, 1982.

[149] C. L. M. van Pul, "On bounds on codes," M.Sc. dissertation, Dept. of Math. and Comput. Sci., Eindhoven Univ. of Technol., Aug. 1982.

[150] ____, "Some distance problems in coding theory," Ph.D. dissertation, Dept. of Math. and Comput. Sci., Eindhoven Univ. of Technol., 1987.

[151] S. Rankinen, personal communication.

[152] D. K. Ray-Chaudhuri and R. M. Wilson, "Solution of Kirkman's schoolgirl problem," in *Combinatorics*, T. S. Motzkin, Ed., *Proc. Symp. Pure Math.*, vol. 19, pp. 187–203, 1971.

[153] M. Reiss, "Uber eine Steinersche Combinatorische Aufgabe welche im 45sten Bande dieses Journals, Seite 181, gestellt worden ist," *J. Reine Angew. Math.*, vol. 56, pp. 326–344, 1859.

[154] V. Rödl, "On a packing and covering problem," *Europ. J. Combin.*, vol. 6, pp. 69–78, 1985.

[155] A. M. Romanov, "New binary codes with minimal distance three," (in Russian), *Probl. Peredach. Inform.*, vol. 19, no. 3, pp. 101–102, 1983.

[156] A. Rosa, R. A. Mathon, and K. T. Phelps, "Small Steiner triple systems and their properties," *Ars. Combin.*, vol. 15, pp. 3–110, 1983.

[157] J. Schönheim, "On maximal systems of k-tuples," *Studia Scient. Math. Hungar.*, vol. 1, pp. 363–368, 1966.

[158] S. Schreiber, "Covering all triples on n marks by disjoint Steiner systems," *J. Combin. Theory*, vol. 15A, pp. 347–350, 1973.

[159] N. V. Semakov and V. A. Zinoviev, "Complete and quasi-complete balanced codes," (in Russian), *Probl. Peredach. Inform.*, vol. 5, no. 2, pp. 14–18, 1969. English translation in *Probl. Inform. Transm.*, vol. 5, no. 2, pp. 11–13, 1969.

[160] ____, "Balanced codes and tactical configurations," (in Russian), *Probl. Peredach. Inform.*, vol. 5, no. 3, pp. 28–37, 1969. English translation in *Probl. Inform. Transm.*, vol. 5, no. 3, pp. 22–28, 1969.

[161] M. J. Sharry and A. P. Street, "Partitioning sets of triples into designs I," *Ars. Combin.*, vol. 26B, pp. 51–66, 1988.

[162] ____, "Partitioning sets of triples into designs II," *J. Combin. Math. Combin. Computing*, vol. 4, pp. 53–68, 1988.

[163] S. S. Shrikhande, "Affine resolvable balanced incomplete block designs: A survey," *Aequat. Math.*, vol. 14, pp. 251–269, 1976.

[164] N. J. A. Sloane, S. M. Reddy, and C. L. Chen, "New binary codes," *IEEE Trans. Inform. Theory*, vol. IT-18, pp. 503–510, 1972.

[165] N. J. A. Sloane and J. J. Seidel, "A new family of codes obtained from conference matrices," *Ann. NY Acad. Sci.*, vol. 175 Article 1, pp. 363–365, 1970.

[166] N. J. A. Sloane and D. S. Whitehead, "A new family of single-error correcting codes," *IEEE Trans. Inform. Theory*, vol. IT-16, pp. 717–719, 1970.

[167] Warren D. Smith and N. J. A. Sloane, "Bounds for spherical codes," preprint.

[168] R. G. Stanton and I. P. Goulden, "Graph factorization, general triple systems, and cyclic triple systems," *Aequat. Math.*, vol. 22, pp. 1–28, 1981.

[169] R. G. Stanton and M. J. Rogers, "Packings and coverings by triples," *Ars. Combin.*, vol. 13, pp. 61–69, 1982.

[170] D. Stinson, "Determination of a packing number," *Ars. Combin.*, vol. 3, pp. 89–114, 1977.

[171] L. Teirlinck, "On the maximum number of disjoint Steiner triple systems," *Discrete Math.*, vol. 6, pp. 299–300, 1973.

[172] ____, "On making two Steiner triple systems disjoint," *J. Combin. Theory*, vol. 23A, pp. 349–350, 1977.

[173] ____, "On large sets of disjoint ordered designs," *Ars. Combin.*, vol. 25, pp. 31–37, 1988.

[174] ____, "A completion of Lu's determination of the spectrum for large sets of disjoint Steiner triple systems," preprint.

[175] H. C. A. van Tilborg and M. Blaum, "On error-correcting balanced codes," *IEEE Trans. Inform. Theory*, vol. 35, no. 5, pp. 1091–1095, Sept. 1989.

[176] V. D. Tonchev, *Combinatorial Configurations, Designs, Codes, Graphs.* New York: Longman, Harlow, 1988.

[177] T. Verhoeff, "An updated table of minimum-distance bounds for binary linear codes," *IEEE Trans. Inform. Theory*, vol. IT-33, pp. 665–680, 1987.

[178] ____, "An updated table of minimum-distance bounds for binary linear codes," (revised), preprint, Jan. 1989.

[179] T. J. Wagner, "A search technique for quasi-perfect codes," *Inform. Contr.*, vol. 9, pp. 94–99, 1966.

[180] J. H. Weber, C. de Vroedt, and D. E. Boekee, "Bounds and constructions for binary codes of length less than 24 and asymmetric distance less than 6," *IEEE Trans. Inform. Theory*, vol. 34, pp. 1321–1331, 1988.

[181] R. M. Wilson, "Some partitions of all triples into Steiner triple systems," in *Hypergraph Seminar*, Ohio State Univ., 1972: *Lecture Notes Math.*, vol. 411. New York: Springer-Verlag, 1974, pp. 267–277.

[182] E. Witt, "Die 5-fach transitiven Gruppen von Mathieu," Abh. Math. Sem. Univ. Hamburg, vol. 12, pp. 256–264, 1938.

[183] ____, "Ueber Steinersche Systeme," loc. cit., vol. 12, pp. 265–275, 1938.

[184] A. Zaptcioglu, "Entwicklung störsicherer Binärcodes," Diplomarbeit, Lehrstuhl für Informatik, Technische Universität Braunschweig, Germany, Aug. 6, 1981.

[185] L. Zhu, "A pair of orthogonal Steiner systems $S(3,5,17)$," *Discrete Math.*, vol. 31, pp. 111–113, 1980.

[186] V. A. Zinoviev, "Cascade equal-weight codes and maximal packings," *Probl. Contr. and Inform. Theory*, vol. 12, no 1, pp. 3–10, 1983.

[187] ____, "On a generalization of the Johnson bound for constant-weight codes," (in Russian), *Probl. Peredach. Inform.*, vol. 20, no. 3, pp. 105–108, 1984.

[188] V. A. Zinoviev and S. N. Litsyn, "On recurrent relations for the cardinality of equal-weight codes," in *Proc. Internat. Workshop Algebraic and Combin. Coding Theory*, Varna, Bulgaria, 1988, pp. 181–183.