Orbit and Coset Analysis of the Golay and Related Codes

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Abstract—Let $\mathscr{C}$ be a code of length n over a field $\mathbb{F}$, with automorphism group G; $\mathscr{C}_w$ denotes the subset of codewords of weight w. Our goal is to classify the vectors of $\mathbb{F}^n$ into orbits under G and to determine their distances from the various subcodes $\mathscr{C}_w$. We do this for the first-order Reed–Muller, Nordstrom–Robinson, and Hamming codes of length 16, the Golay and shortened Golay codes of lengths 22, 23, 24, and the ternary Golay code of length 12.

I. Introduction

Let $\mathscr{C}$ be one of the following codes: the first-order Reed–Muller, Nordstrom–Robinson, or Hamming codes of length 16, the Golay and shortened Golay codes of lengths 22, 23, or 24 (all these are binary), or the ternary Golay codes of lengths 11 or 12. The main results of this paper are the graphs in Figs. 1–5, which classify the vectors of $\mathbb{F}^n$ (where n is the length of $\mathscr{C}$ and $\mathbb{F}=\mathbb{F}_2$ or $\mathbb{F}_3$ is the appropriate field) into orbits under the action of the automorphism group of $\mathscr{C}$. The groups considered are M_{11}, $2.M_{12}$, M_{22}, $M_{22}{:}2$, M_{23}, M_{24} (where M_n denotes a Mathieu group [4], [8]), and the subgroups of M_{24} isomorphic to $2^4{:}\ A_7$ and $2^4{:}\ A_8$. Other properties of the orbits are summarized in Tables I, IV, V, VII, VIII, XI, XIII, and Fig. 6.

The circled nodes in the graphs indicate the constant weight subcodes $\mathscr{C}_w$ of each code. Since distances in these graphs (measured by number of edges) coincide with Hamming distances between orbits, these graphs also classify the vectors of $\mathbb{F}^n$ according to their distances from the constant weight subcodes.

Tables II, VI, IX, X, XII, and XIV show how the cosets of these codes are decomposed into orbits under the groups. These tables are expanded versions of the usual coset weight distribution tables. The final table, Table XV, gives the weight distributions of the cosets of the [11, 6, 5] perfect ternary Golay code.

Orbits of binary vectors under M_{24} (the case when $\mathscr{C}$ is the Golay code of length 24) were classified in ([2], [8], Chap. 10). In the present paper we introduce a new parameter, the specification number (or spec), to describe these orbits—see Fig. 1 and Table I. This makes it easy to determine the distance of an orbit from the code and to tell when one orbit is contained in another.

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II. The [24, 12, 8] Golay Code

The automorphism group of the [24, 12, 8] Golay code $\mathscr{G}$ is the Mathieu group M_{24} (see [4], [8]). As described in ([2], [8], Chap. 10), there are 49 orbits of vectors in $\mathbb{F}_2^{24}$ under the action of M_{24}, denoted by $S_w (0 \le w \le 24)$, $T_w (8 \le w \le 16)$, $U_w (6 \le w \le 18)$, P_{12} and X_{12}, where the subscript gives the weight of the vectors. These orbits are displayed in Fig. 1 and their properties are summarized in Table I.

In Fig. 1 two orbits A, B are joined by an edge if a vector in B can be obtained from some vector in A by complementing a single bit. The edge joining A and B is labeled near A with the number of choices for this bit.

The Golay code $\mathscr{G}$ itself consists of the orbits $\mathscr{G}_0 = S_0 = \{\mathbf{0}\}$, $\mathscr{G}_8 = S_8$ (the 759 *special octads*, forming the Steiner system $S(5, 8, 24)$), $\mathscr{G}_{12} = U_{12}$ (the 2576 *umbral dodecads*), $\mathscr{G}_{16} = S_{16}$ (the 759 *special 16-sets*) and $\mathscr{G}_{24} = S_{24} = \{\mathbf{1}\}$. These nodes are circled in Fig. 1. The vectors of S_w for $w < 12$ contain or are contained in a special octad and are called *special w-sets*; the vectors of U_w for $w < 12$ are contained in an umbral dodecad and are called *umbral w-sets*; the vectors of T_w are called *transverse w-sets*; while the vectors of X_{12} (called S_{12}^+ in [1], [2]) and P_{12} (called U_{12}^- in [1], [2]) are the *extraspecial* and *penumbral dodecads*, respectively. (This terminology was introduced in [2], [13].) The vectors in S_w, T_w, U_w are the complements of the vectors in S_{24-w}, T_{24-w}, U_{24-w}, respectively, while the types P_{12} and X_{12} are self-complementary.

Fig. 1 has the convenient property that the minimal Hamming distance between two orbits is given by the minimal number of edges joining the corresponding nodes of the graph. In other words, distance in the graph is the same as Hamming distance.

The orbits in Fig. 1 are positioned according to their weight (increasing downwards) and *specification number* or *spec* (increasing across). For a vector of weight $w \le 12$ not in T_{12} or X_{12}, the specification number is defined to be the number of points in its support that lie in a nearest octad, minus the number of points outside that octad, while for vectors in T_{12} or X_{12} it is 3 and 5, respectively. The specification number of a vector of weight greater

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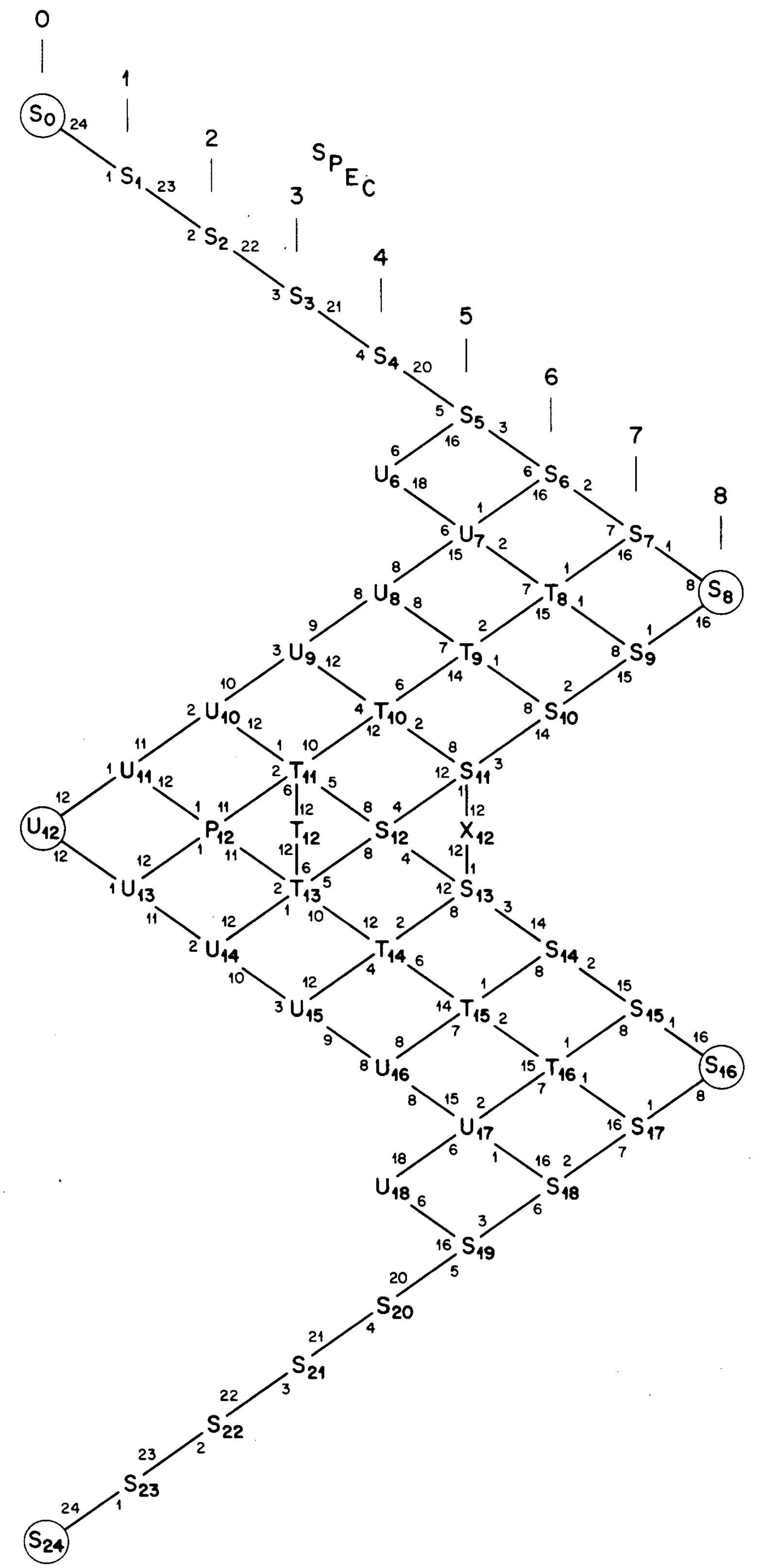


Fig. 1. Orbits of vectors of length 24 under action of M_{24}. Orbits are arranged according to weight (increasing downwards) and specification number (increasing across). Words of Golay code $\mathscr{G}$ are circled.

than 12 is defined to be the same as that of its complement.

The specification number has two useful properties.

a) A vector of weight W and spec S contains a vector of weight w and spec s just if $W-w \geq |S-s|$.
b) The distance of a vector of spec s from the Golay code is at least $\min\{s, 8-s\}$, and is equal to this except when the parity is wrong; that is to say, except for the vectors of T_{12} and X_{12}, which are at distance 4 (not 3) from the code.

We also record some other properties of Fig. 1. The sum of the labels on edges upwards from an orbit of weight w is equal to w, while the sum of the labels on downward edges is $n-w$, where n is the length of the code. Furthermore if there is an edge from orbit A to orbit B labeled α (at A) and β (at B), then

$$\alpha|A| = \beta|B|. \tag{1}$$

Before describing Table I we introduce our notation for Golay codewords. We shall write Golay codewords in the 4×6 MOG (or miracle octad generator) array, as described in [3]–[6], [8]–[10]. We follow the version given in [8], Chaps. 10, 11, and first define the *hexacode* to be the [6, 3, 4] code over $\mathbb{F}_4$ with generator matrix

$$\begin{bmatrix} 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & \omega & \bar{\omega} \\ 1 & 0 & 0 & 1 & \bar{\omega} & \omega \end{bmatrix}$$

(see [8], pp. 300–301). Then the [24, 12, 8] Golay code consists of all 4×6 binary arrays with the properties that the weights of the columns and the top row have the same parity, and the six inner products of the columns with the vector $(0,1,\omega,\bar{\omega})$ forms a word of the hexacode ([8], pp. 303–304). We order the coordinates of the MOG's by reading down the columns, from left to right. When the Golay code defined by MOG coordinates is read in this way it coincides with the lexicographic version of this code ([7], [8], p. 327).

Table I begins by giving (in column 2) the number of vectors in each orbit. These numbers are easily calculated from Fig. 1, using (1), and an alternative enumeration is given later in this section. The next column describes the subgroup of M_{24} fixing a vector in the orbit. We use the ATLAS notation (see [4], [8]) for these groups. In particular, $A\times B$ indicates a direct product, $A.B$ or AB is a group with a normal subgroup isomorphic to A for which the corresponding quotient group is isomorphic to B, $A:B$ denotes the case of $A.B$ which is a split extension (or semidirect product), and $\frac{1}{2}(S_m \times S_n)$ indicates the even permutations of the group $S_m \times S_n$ acting on $m+n$ objects.

The fourth column gives the action of this group on the 24 coordinates, with the action on the 1-coordinates and on the 0-coordinates separated by a vertical bar. Orbits are separated by commas, so for example 6, 5, 2 indicates three orbits of sizes 6, 5, and 2. A symbol such as 2^7 indicates an orbit of 14 points having an invariant partition (or system of imprimitivity) into seven sets of 2, while 4×3 indicates an orbit of 12 points having invariant partitions into four sets of 3 and three sets of 4.

TABLE I
ORBITS UNDER M_{24}

Orbit	Size	Stabilizer	Action	Spec	Error Pattern
S_0	1	M_{24}	$0\mid 24$	0	0_0
S_1	24	M_{23}	$1\mid 23$	1	1_1
S_2	276	$M_{22}:2$	$2\mid 22$	2	2_2
S_3	2024	$M_{21}:S_3$	$3\mid 21$	3	3_3
S_4	10626	$2^6:\frac{1}{2}(S_3\times S_5)$	$4\mid 4^5$	4	4_{400000}
S_5	42504	$2^4:\frac{1}{2}(S_3\times S_5)$	$5\mid 16,3$	5	3_0
S_6	21252	$2^4:S_6$	$6\mid 16,2$	6	2_0
U_6	113344	$3S_6$	$6\mid 3^6$	4	4_{111111}
S_7	6072	$2^4:A_7$	$7\mid 16,1$	7	1_0
U_7	340032	S_6	$6,1\mid 15,2$	5	3_1
S_8	759	$2^4:A_8$	$8\mid 16$	8	0_0
T_8	97152	A_7	$7,1\mid 15,1$	6	2_1
U_8	637560	$2^4.S_4$	$2^4\mid 4^2,2^4$	4	4_{222200}
S_9	12144	A_8	$8,1\mid 15$	7	1_1
T_9	728640	$L_2(7).2$	$7,2\mid 2^7,1$	5	3_2
U_9	566720	$3^2:2S_4$	$9\mid 3^4,3$	3	3_0
S_{10}	91080	$2^3:L_3(2).2$	$8,2\mid 2^7$	6	2_2
T_{10}	1700160	$S_3\times S_4$	$3^2,4\mid 4\times 3,2$	4	4_{331111}
U_{10}	170016	$S_6.2$	$10\mid 6^2,2$	2	2_0
S_{11}	425040	$\frac{1}{2}(S_4\times S_4).2$	$4^2,3\mid 4^3,1$	5	3_3
T_{11}	2040192	S_5	$10,1\mid 6,5,2$	3	3_1
U_{11}	30912	M_{11}	$11\mid 12,1$	1	1_0
X_{12}	35420	$2^6.3.S_3^2.2$	$4^3\mid 4^3$	5	4_{444000}
S_{12}	1275120	$2^3.S_4$	$2^4,4\mid 2^4,4$	4	4_{422220}
T_{12}	1020096	$(2\times A_5).2$	$2^6\mid 2^6$	3	4_{222222}
P_{12}	370944	$L_2(11)$	$11,1\mid 11,1$	2	2_1
U_{12}	2576	M_{12}	$12\mid 12$	0	0_0

The fifth column gives the specification number (defined earlier).

The last column gives the distance d from the code, with a subscript describing the minimal error pattern(s). If v is a vector in the orbit, and d is at most 3, there is a unique closest codeword $c \in \mathcal{G}$. Then $e = v + c$ is the error pattern and the entry in the last column is d_i, where $i = wt(v \cap e)$. On the other hand if v is at distance 4 from the code then there are six codewords $c_0,\cdots,c_5$ (say) all at distance 4 form v, and six equally likely minimal error patterns, $e_r = v + c_r$ $(0 \leq r \leq 5)$. In this case the entry is $4_{i_0 i_1 \cdots i_r}$, where $i_r = wt(v \cap e_r)$.

The six vectors $e_0,\cdots,e_5$ all have weight 4, with their 1's in disjoint sets of coordinates, and any sum $e_r + e_s (r \neq s)$ is a codeword of weight 8. In this situation the individual 4-sets are called *tetrads* and the set of six tetrads is called a *sextet* ([8], Chap. 10). Any 4-set belongs to exactly one sextet, and there are $\frac{1}{6}\binom{24}{4} = 1771$ distinct sextets. The six columns of the MOG form a sextet, and we shall usually take this as our typical example. We see that the vectors in S_4, U_6, U_8, T_{10}, E_{12}, S_{12}, T_{12}, T_{14}, U_{16}, U_{18}, and S_{20} (the "deep holes" in the Golay code) are at distance 4 from the code and reduce modulo the code to any of the six tetrads of some sextet.

Table I describes only orbits of weight $w \leq 12$. The entries for S_{24-w}, T_{24-w}, U_{24-w} $(w \leq 11)$ are the same as those for S_w, T_w, U_w, respectively, except that the "Action" column is reversed, and in the final column d_a

TABLE II
COSETS OF [24, 12, 8] GOLAY CODE $\mathscr{G}$

No.	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1 S_0								759 S_8				2576 U_{12}
24		1 S_1						253 S_7		506 S_9		1288 U_{11}	
276			1 S_2				77 S_6		352 T_8		330+616 $S_{10}\cup T_{10}$		1344 P_{12}
2024				1 S_3		21 S_5		168 U_7		360+280 $T_9\cup U_9$		210+1008 $S_{11}\cup T_{11}$	
1771					6 S_4		64 U_6		360 U_8		960 T_{10}		20+720+576 $X_{12}\cup S_{12}\cup T_{12}$

TABLE III-A
HOW MANY SPECIAL OCTADS?

759								
506	253							
330	176	77						
210	120	56	21					
130	80	40	16	5				
78	52	28	12	4	1			
46	32	20	8	4	0	1		
30	16	16	4	4	0	0	1	
30	0	16	0	4	0	0	0	1

TABLE III-B
HOW MANY UMBRAL DODECADS?

2576								
1288	1288							
616	672	616						
280	336	336	280					
120	160	176	160	120				
48	72	88	88	72	48			
16	32	40	48	40	32	16		
0	16	16	24	24	16	16	0	
0	0	16	0	24	0	16	0	0

becomes d_{d-a} and $4_{i_0 \cdots i_5}$ becomes $4_{j_5 \cdots j_0}$ where $j_r = 4 - i_r$. For example, for T_{14} and T_{15} the actions are 4×3, $2\,|\,3^2$, 4 and 2^7, $1\,|\,7$, 2, respectively, and the minimal error patterns are described by 4_{333311} and 3_1, respectively.

From Fig. 1 and Table I we may obtain a complete analysis of the cosets of the Golay code, as displayed in Table II. This is an expanded version of the usual coset weight distribution table (as found for example on p. 69 of [11]), and is more-or-less obtained by folding Fig. 1 about a vertical line through its center (and transposing).

We next show how to construct and enumerate the vectors in each orbit. For orbits at distance ≤ 3 from the code (belonging to the first four rows of Table II), there is a unique description that can be read off Fig. 1. For example, any vector of type T_9 is obtained by adding two points to a special octad and deleting one point from that octad. To count such vectors we make use of the familiar "Leech triangles" of numbers shown in Tables III-A, III-B (cf. [8], p. 278, [11], p. 68).

If $\{a_1, a_2, \cdots, a_8\}$ is the (support of) a special octad, then the number of special octads intersecting $\{a_1, \cdots, a_i\}$ in exactly $\{a_1, \cdots, a_j\}$ is the $(j+1)$th entry in the $(i+1)$th row of Table III-A. Similarly Table III-B gives the number of umbral dodecads meeting $\{a_1, \cdots, a_i\}$ in exactly $\{a_1, \cdots, a_j\}$.

It then follows that the numbers in the ith row of Table II for $i \leq 3$ are found by multiplying the ith row of each Leech triangle by the ith row of Pascal's triangle! For example the numbers

$$\begin{array}{cccccc} 77 & 352 & 330 & & \\ & & +616 & 1344 & 616 \end{array}$$

in row 3 of Table II are obtained from row 3 of Tables III-A, III-B:

$$\begin{array}{cccccc} 77\times 1 & 176\times 2 & 330\times 1 & & \\ & & +616\times 1 & 672\times 2 & 616\times 1. \end{array}$$

Similarly the fourth row

$$\begin{array}{cccccc} 21 & 168 & 360 & 210 & & \\ & & +280 & 1008 & 1008 & 280 \end{array}$$

follows from

$$\begin{array}{cccccc} 21\times 1 & 56\times 3 & 120\times 3 & 210\times 1 & & \\ & & +280\times 1 & 336\times 3 & 336\times 3 & 280\times 1. \end{array}$$

The vectors in the final row of Table II, the deep holes in $\mathscr{G}$, may also be enumerated in this way, but (because

TABLE IV
DEEP HOLES IN THE [24, 12, 8] GOLAY CODE

Name	Error pattern	Example	Number ÷ 1771
S_4	4_{400000}	Column of MOG	$1\cdot 6=6$
U_6	4_{111111}	H(word)	$64\cdot 1=64$
U_8	4_{222200}	H(weight 4 word) + top row	$45\cdot 2^3=360$
T_{10}	4_{331111}	H(word) + 2 columns	$64\cdot\binom{6}{2}=960$
X_{12}	4_{444000}	3 columns of MOG	$1\cdot\binom{6}{3}=20$
S_{12}	4_{422220}	H(weight 4 word) + top row + column	$45\cdot(2^3\cdot 2)=720$
T_{12}	4_{222222}	H(weight 6 word) + top row	$18\cdot 2^5=576$

the representatives modulo $\mathscr{G}$ are no longer unique), it is simpler to enumerate them from their error patterns (given in the last column of Table I). The results are shown in Table IV.

Consider for example a vector of type S_4, which, since its error pattern is described by 4_{400000}, consists of one tetrad from a sextet. Since there are 1771 sextets, each containing six tetrads, the number of S_4 vectors is $1771\times 6=10626$. As an example we may take any of the six columns of the MOG.

Vectors of type U_6 have error pattern 4_{111111}, and typical examples consist of 4×6 MOG arrays with a single 1 in each column, chosen so that the positions of the 1's (when the rows of the array are labeled $0,1,\omega,\overline{\omega}$) form a word w in the hexacode. We call this vector $H(w)$. The number of such vectors is 1771 (for the choice of sextet) times 64 (for the choice of a hexacodeword). In the column headed "Number" in Table IV, the first factor is the appropriate number of hexacodewords, and the second factor gives the number of other choices that must be made.

We omit details of the remaining entries in Table IV. (Readers familiar with Chap. 11 of [8] will have no difficulty in verifying these enumerations, and the numbers are in any case available in Table I.)

Finally, Fig. 1 makes it easy to find the vectors at a specified distance from the code. For example, in constructing constant weight codes in [1] it was necessary to determine the vectors of length 24, weight 12 and having distance 6 from the 2576 words of $\mathscr{G}_{12}=U_{12}$. From Fig. 1 and Table I we see that there are exactly 35 420 such vectors, those of the orbit X_{12}.

III. THE [23, 12, 7] GOLAY CODE

The [23, 12, 7] perfect Golay code $\mathscr{G}'$ is obtained by deleting one fixed coordinate (which we label ∞) from every word of $\mathscr{G}$, and Aut$(\mathscr{G}')$ is the Mathieu group M_{23}. Of course the dual code to $\mathscr{G}'$, the [23, 11, 8] even weight subcode of $\mathscr{G}'$, has the same group.

Let v be a vector of length 23 and weight w, and let x and y be the vectors of length 24 obtained from v by adjoining a 0 or 1 respectively in the ∞ coordinate. If x belongs to the orbit A_w of Fig. 1, and y to the orbit B_{w+1}, then v corresponds to the *edge* in Fig. 1 from A_w to B_{w+1}. We describe v by saying it is of type A_{wB}. Its complement $\bar{v}$ is of type $B_{w'A}$, where $w'=23-w$.

It is not difficult to verify (we omit the details) that M_{23} is transitive on vectors of each type. We conclude that orbits of vectors in $\mathbb{F}_2^{23}$ under M_{23} are in one-to-one correspondence with the edges of Fig. 1. There are therefore 72 orbits.

These orbits are shown in Fig. 2, which uses the same conventions—except for specification number—as Fig. 1. The edge labels and the sizes of the orbits (given in Table V) can be determined from the information in Fig. 1 and Table 1, as we now demonstrate.

TABLE V
SIZES OF ORBITS UNDER M_{23}

S_{0S}	1	U_{7T}	28336	S_{10S}	53130
S_{1S}	23	U_{7U}	212520	T_{10S}	141680
S_{2S}	253	S_{8S}	506	T_{10T}	850080
S_{3S}	1771	T_{8S}	4048	U_{10T}	85008
S_{4S}	8855	T_{8T}	60720	U_{10U}	14168
S_{5S}	5313	U_{8T}	212520	S_{11X}	17710
S_{5U}	28336	U_{8U}	212520	S_{11S}	212520
S_{6S}	1771	S_{9S}	7590	T_{11S}	425040
S_{6U}	14168	T_{9S}	30360	T_{11T}	510048
U_{6U}	85008	T_{9T}	425040	T_{11P}	170016
S_{7S}	253	U_{9T}	283360	U_{11P}	15456
S_{7T}	4048	U_{9U}	70840	U_{11U}	1288

Consider for example the edges in Fig. 1 at the node T_9. There is an edge from T_9 to T_{10} (labeled 14 at T_9), and an edge from T_9 to S_{10} (labeled 1). Since there are 728 640 vectors of type T_9 (from Table I), there are

$$\frac{14}{24}\times 728640=425040$$

vectors of type T_{9T}, and

$$\frac{1}{24}\times 728640=30360$$

vectors of type T_{9S}.

The calculation of the edge labels in Fig. 2 is only slightly more complicated. Consider for example a vector $v\in\mathbb{F}_2^{23}$ of type T_{9T}, so that x (v with a 0 adjoined) is of type T_9 and y (v with a 1 adjoined) is of type T_{10}. From the edge labels in Fig. 1 we see that complementing a 0 in x leads in one way to a vector of S_{10} and in 14 ways to a vector of T_{10} (one of which is y). In Fig. 2, therefore, there is one edge from T_{9T} to a node of type S_{10*} and 13 edges to nodes of type T_{10*} (where the stars indicate unknown letters). On the other hand, complementing a 1 in y leads in two ways to a vector of S_{11} and in 12 ways to a vector of T_{11}. This tells us that in Fig. 2 there are two edges to nodes of type ${}^*{}_{10S}$ and 12 edges to nodes of type ${}^*{}_{10T}$.

The possible nodes that T_{9T} can be joined to are therefore S_{10S}, S_{10T}, T_{10S} and T_{10T}. However, from Fig. 1 we see that S_{10} is not joined to T_{11}, so a node of type S_{10T}

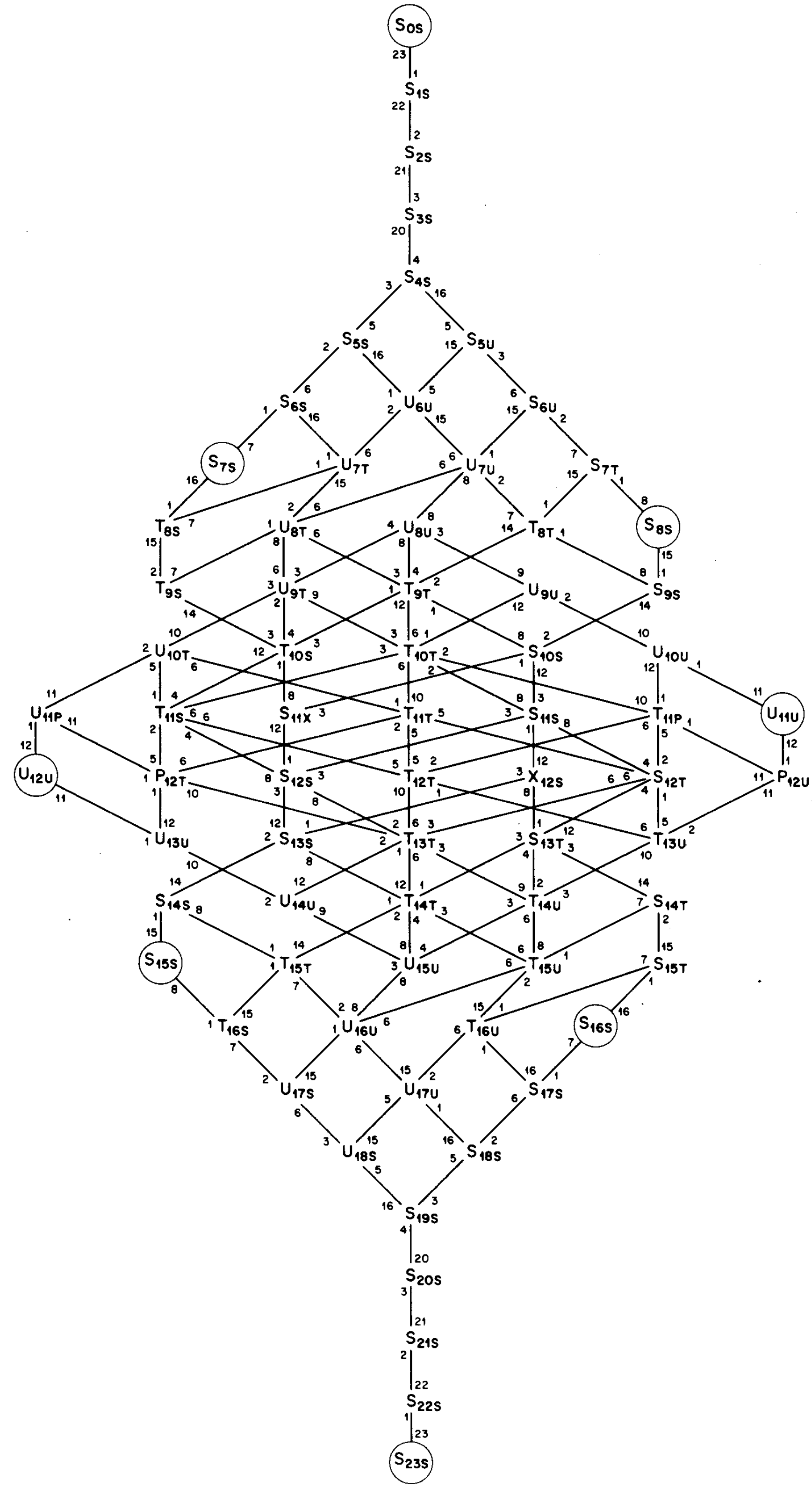


Fig. 2. Orbits of vectors of length 23 under action of M_{23}. Orbit A_{wB} consists of vectors which, when 0 (1) is adjoined, belong to orbit A_w (B_{w+1}) of Fig. 1. Words of Golay code $\mathscr{G}'$ are circled.

TABLE VI
COSETS OF [23, 12, 7] GOLAY CODE $\mathscr{G}'$

No.	0	1	2	3	4	5	6	7	8	9	10	11
1	1							253	506			1288
	S_{0S}							S_{1S}	S_{8S}			U_{11U}
23		1					77	176	176	330	616	672
		S_{1S}					S_{6S}	S_{7T}	T_{8S}	S_{9S}	U_{10U}	U_{11P}
253			1			21	56	112	240	120+280	210+336	672
			S_{2S}			S_{5S}	S_{6U}	U_{7T}	T_{8T}	$T_{9S}\cup U_{9U}$	$S_{10S}\cup U_{10T}$	T_{11P}
1771				1	5	16	48	120	120+120	240+160	80+480	**
				S_{3S}	S_{4S}	S_{5U}	U_{6U}	U_{7U}	$U_{8T}\cup U_{8U}$	$T_{9T}\cup U_{9T}$	$T_{10S}\cup T_{10T}$	

**: 10+120+240+288 corresponding to $S_{11X}\cup S_{11S}\cup T_{11S}\cup T_{11T}$.

is impossible. We conclude that a vector of type T_{9T} transforms in one way to type S_{10S}, in 12 ways to type T_{10T} and in one way to type T_{10S}. The labels at the bottom ends of these edges are then found from (1) and Table V.

From Fig. 2 and Table V we obtain a complete analysis of the cosets of $\mathscr{G}'$, as shown in Table VI.

IV. THE SHORTENED GOLAY CODES OF LENGTH 22

By shortening $\mathscr{G}$ to length 22 we obtain [22, 10, 8], [22, 11, 7], and [22, 12, 6] codes. The automorphism group of the first and third of these is $M_{22}:2$, while the automorphism group of the [22, 11, 7] code (obtained from the words of $\mathscr{G}$ that begin 00 or 01) is M_{22}.

Without giving any details we mention that the orbits of M_{22} are in one-to-one correspondence with the edges of Fig. 2. There are therefore 130 orbits, which can be named in the following way. An edge in Fig. 2 directed from A_{wB} to $C_{w+1,D}$ indicates that there is a vector $v\in\mathbb{F}_2^{22}$ of weight w such that $v00\in A_w$, $v01\in B_{w+1}$, $v10\in C_{w+1}$, $v11\in D_{w+2}$. The appropriate name for the orbit of v under M_{22} is then A_{wBCD}.

Under the action of $M_{22}:2$, however, the orbits A_{wBCD} and A_{wCBD} fuse, and the composite orbit should be named $A_{w\{BC\}D}$. For example the M_{22} orbits U_{8TUT} and U_{8UTT} fuse under $M_{22}:2$ to give the orbit $U_{8\{TU\}T}$. There are 105 distinct orbits under $M_{22}\cdot 2$.

V. THE FIRST-ORDER REED–MULLER AND HAMMING CODES OF LENGTH 16

The [16, 5, 8] first-order Reed–Muller code $\mathscr{R}$ and the [16, 11, 4] Hamming code $\mathscr{H}$ are duals and both have automorphism group $G\cong 2^4\colon A_8$, where A_8 is the alternating group of order 8 ([8], p. 277). To define these codes and the Nordstrom–Robinson code of Section VI we divide the coordinates of the MOG into three "bricks" of eight coordinates each, and label the left-hand brick as follows:

∞	0		
3	2		
5	1		
6	4		

(cf. [8], p. 316).

Then $\mathscr{R}$ consists of the codewords of the [24, 12, 8] Golay code $\mathscr{G}$ that vanish on the left-hand brick (with this brick deleted), while $\mathscr{H}$ is the projection of $\mathscr{G}$ onto the last two bricks.

To study how vectors $\hat{v}\in\mathbb{F}_2^{16}$ of weight $w\le 8$ fall into orbits under G we shall adjoin the left-hand brick (a special octad) to $\hat{v}$, obtaining a vector v of weight $8+w$, belonging to one of the orbits of Fig. 1. Conversely, each orbit in Fig. 1 that contains a special octad arises in this way. To classify vectors of $\mathbb{F}_2^{16}$ under G we must therefore take the orbits in Fig. 1 that contain a special octad and study them according to the special octads they contain. We denote by $\hat{X}_w$ the type of vector formed by removing a special octad from a vector of type X_w. It turns out (as usual we omit the details) that G is transitive on vectors of each of these types, except for $\hat{U}_{16}$, which splits into two orbits $\hat{U}_{16}^0$ and $\hat{U}_{16}^1$. So there are 32 orbits under G, as displayed in Fig. 3, whose properties are summarized in Tables VII and VIII.

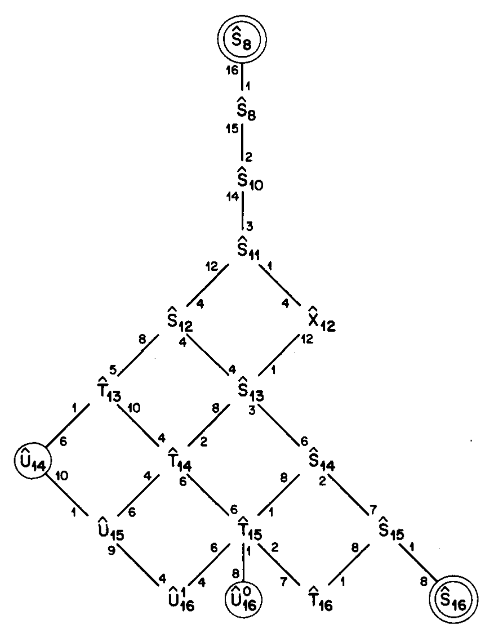


Fig. 3. Orbits of vectors of length 16 under action of automorphism group ($2^4\colon A_8$) of Reed–Muller code $\mathscr{R}$ and Hamming code $\mathscr{H}$. Words in $\mathscr{R}$ have two circles, words in $\mathscr{H}$ have one or two circles. Weight is 8 less than subscript. Omitted lower half of graph can be obtained by taking mirror image of top half.

TABLE VII
PROPERTIES OF ORBITS UNDER AUT($\mathscr{R}$) = AUT($\mathscr{H}$)

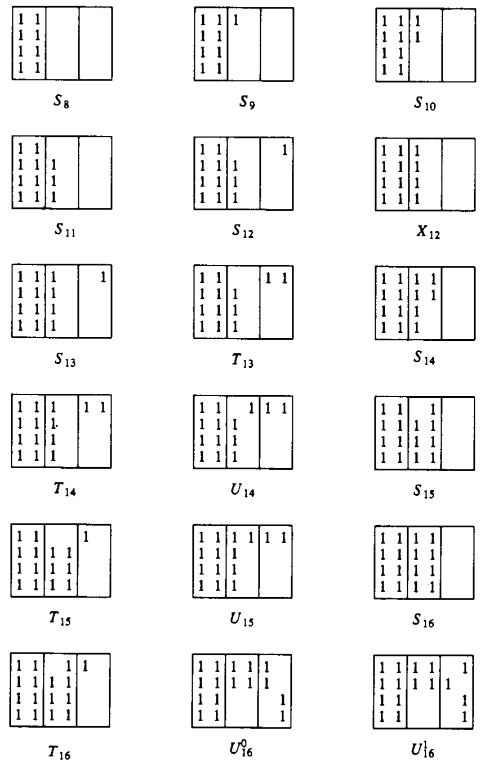
Weight	Name	Size	n_8	n_{12}	n_{16}	Orbits
0	$\hat{S}_8$	1	1	0	0	8
1	$\hat{S}_9$	16	1	0	0	8
2	$\hat{S}_{10}$	120	1	0	0	8
3	$\hat{S}_{11}$	560	1	0	0	8
4	$\hat{S}_{12}$	1680	3	0	0	8
4	$\hat{X}_{12}$	140	1	0	0	8
5	$\hat{S}_{13}$	1680	3	0	0	8
5	$\hat{T}_{13}$	2688	1	0	0	8
6	$\hat{S}_{14}$	840	7	0	0	8
6	$\hat{T}_{14}$	6720	3	0	0	8
6	$\hat{U}_{14}$	448	2	1	0	2+6
7	$\hat{S}_{15}$	240	15	0	0	8
7	$\hat{T}_{15}$	6720	7	0	0	8
7	$\hat{U}_{15}$	4480	6	1	0	2+6
8	$\hat{S}_{16}$	30	30	0	1	8
8	$\hat{T}_{16}$	1920	15	0	0	8
8	$\hat{U}_{16}^0$	840	1+12	2	0	8
8	$\hat{U}_{16}^1$	10080	12+1	2	0	4+4

Note that now the weight of any type of vector is 8 less than the subscript on its symbol. The vectors of $\mathscr{R}$ are marked with double circles, the remaining vectors of $\mathscr{H}$ with single circles. The omitted lower half of the graph in Fig. 3 can be obtained by taking the mirror image of the top half. The types $\hat{S}_{16}$, $\hat{T}_{16}$, $\hat{U}_{16}^0$, $\hat{U}_{16}^1$ of weight 8 vectors are self-complementary.

In Table VII, the columns headed n_8, n_{12}, and n_{16} give the numbers of special octads, umbral dodecads and special 16-ads contained in v, while the last column shows how the stabilizer of v acts on the 8 coordinates of the left-hand brick. To explain the last two rows of Table VII, we note that if v is of type U_{16} the it contains 13 special octads, which fall into orbits of sizes 1 and 12 under the stabilizer of v. Thus the left-hand brick can be chosen in two essentially different ways, producing the orbits $\hat{U}_{16}^0$ and $\hat{U}_{16}^1$. Table VIII contains samples of the vectors v; orbit representatives $\hat{v}$ for Aut($\mathscr{R}$) = Aut($\mathscr{H}$) are obtained by omitting the left-hand brick.

The cosets of $\mathscr{R}$ and $\mathscr{H}$ are analyzed in Tables IX and X, respectively. (The weight distributions of the cosets of $\mathscr{R}$ were originally given in [12].)

TABLE VIII*



*Omitting the left-hand 8 coordinates from these pictures produces samples from the orbits of Aut($\mathscr{R}$) = Aut($\mathscr{H}$).

TABLE IX
COSETS OF [16, 5, 8] REED–MULLER CODE $\mathscr{R}$

No.	0	1	2	3	4	5	6	7	8
1	1($\hat{S}_8$)								30($\hat{S}_{16}$)
16		1($\hat{S}_9$)						15($\hat{S}_{15}$)	
120			1($\hat{S}_{10}$)				7($\hat{S}_{14}$)		16($\hat{T}_{16}$)
560				1($\hat{S}_{11}$)		3($\hat{S}_{13}$)		12($\hat{T}_{15}$)	
840					2($\hat{S}_{12}$)		8($\hat{T}_{14}$)		12($\hat{U}_{16}^1$)
35					4($\hat{X}_{12}$)				24($\hat{U}_{16}^0$)
448						6($\hat{T}_{13}$)		10($\hat{U}_{15}$)	
28							16($\hat{U}_{14}$)		

TABLE X
COSETS OF [16, 11, 4] HAMMING CODE $\mathscr{H}$

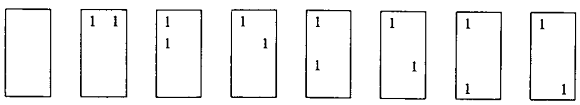
No.	0	1	2	3	4	5	6	7	8
1	1 $\hat{S}_8$				140 $\hat{X}_{12}$		448 $\hat{U}_{14}$		30+840 $\hat{S}_{16} \cup \hat{U}_{16}^0$
16		1 $\hat{S}_9$		35 $\hat{S}_{11}$		105+168 $\hat{S}_{13} \cup \hat{T}_{13}$		15+420+280 $\hat{S}_{15} \cup \hat{T}_{15} \cup \hat{U}_{15}$	
15			8 $\hat{S}_{10}$		112 $\hat{S}_{12}$		56+448 $\hat{S}_{14} \cup \hat{T}_{14}$		128+672 $\hat{T}_{16} \cup \hat{U}_{16}^1$

VI. The Nordstrom–Robinson Code of Length 16

We use the notation of the previous section. Let $\mathscr{R}_i$ $(0 \le i \le 6)$ denote the words of the Golay code $\mathscr{G}$ that have 1's in coordinates ∞ and i, and 0's elsewhere in the first 8 coordinates, with the first 8 coordinates deleted. Each $\mathscr{R}_i$ is a translate of $\mathscr{R}$ containing 16 words of weight 6 and 16 of weight 10, and

$$\mathscr{N} = \mathscr{R} \cup \mathscr{R}_0 \cup \mathscr{R}_1 \cup \cdots \cup \mathscr{R}_6$$

is the Nordstrom–Robinson code. Thus $\mathscr{N}$ consists of the words of $\mathscr{G}$ that begin with one of



with these first 8 coordinates deleted, and $\mathrm{Aut}(\mathscr{N}) \cong 2^4 : A_7$.

Again we study vectors $\hat{v} \in \mathbb{F}_2^{16}$ by adjoining the left-hand octad (consisting of 8 "ghostly" points), one of which (∞, or the "focus") is special, obtaining a vector $v \in \mathbb{F}_2^{24}$. We classify $\hat{v}$ by saying what v reduces to modulo $\mathscr{G}$, i.e., its minimal error pattern. This is either a vector e of weight at most 3, or six vectors $e_0, \cdots, e_5$ of weight 4, all mutually congruent modulo $\mathscr{G}$, i.e., a *sextet* (see Section II). These minimal error patterns (e or $\{e_0, \cdots, e_5\}$) are described in the fourth column of Table XI, using the symbols F for the "focus" (or ∞ coordinate), G for a "ghostly" point (one of the other seven points in the left-hand brick), 0 for a coordinate out of the last 16 where v is 0, and 1 for a coordinate where v is 1.

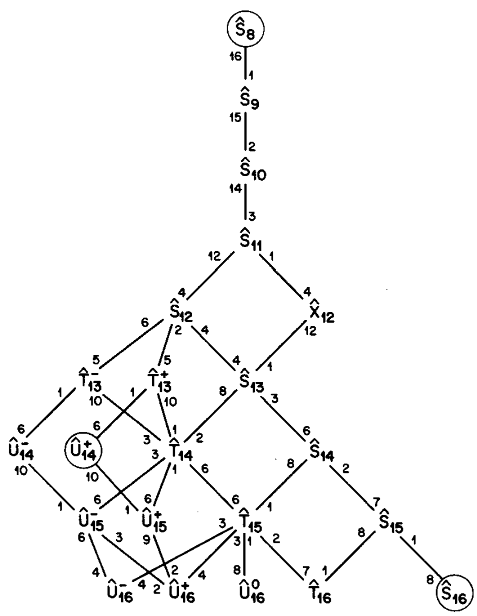


Fig. 4. Orbits of vectors of length 16 under action of automorphism group ($2^4 : A_7$) of Nordstrom–Robinson code $\mathscr{N}$. Vectors in $\mathscr{N}$ are circled. Weight is 8 less than subscript.

It turns out that the minimal error pattern is enough to distinguish the orbits of $\mathbb{F}_2^{16}$ under $\mathrm{Aut}(\mathscr{N})$, and furthermore that $\mathrm{Aut}(\mathscr{N})$ is transitive on vectors of each type. Once again we omit the proof. There are therefore 39 orbits under $\mathrm{Aut}(\mathscr{N})$, those of weight at most 8 being shown in Fig. 4 and Table XI.

In Fig. 4, as in Fig. 3, the weight is 8 less than the subscript. Again the bottom half of the graph has been omitted. The types $\hat{S}_{16}, \hat{T}_{16}, \hat{U}_{16}^0$ are self-complementary, while $\hat{U}_{16}^+$ complements to $\hat{U}_{16}^-$. Fig. 4 closely resembles Fig. 3, except that certain nodes and edges have been split.

The sizes and error patterns for the orbits are given in Table XI.

TABLE XI
Properties of Orbits under Aut($\mathscr{N}$)

Weight	Name	Size	Error Patterns under $\mathscr{G}$
0	$\hat{S}_8$	1	—
1	$\hat{S}_9$	16	1
2	$\hat{S}_{10}$	120	1^2
3	$\hat{S}_{11}$	560	1^3
4	$\hat{S}_{12}$	1680	$\{FG0^2, G^20^2, G^20^2, G^20^2, 1^4, 0^4\}$
4	$\hat{X}_{12}$	140	$\{FG^3, G^4, 1^4, 0^4, 0^4, 0^4\}$
5	$\hat{S}_{13}$	1680	0^3
5	$\hat{T}_{13}^+$	672	$FG0$
5	$\hat{T}_{13}^-$	2016	G^20
6	$\hat{S}_{14}$	840	0^2
6	$\hat{T}_{14}$	6720	$\{FG10, G^210, G^210, G^210, 10^3, 10^3\}$
6	$\hat{U}_{14}^+$	112	FG
6	$\hat{U}_{14}^-$	336	G^2
7	$\hat{S}_{15}$	240	0
7	$\hat{T}_{15}$	6720	10^2
7	$\hat{U}_{15}^+$	1120	$FG1$
7	$\hat{U}_{15}^-$	3360	G^21
8	$\hat{S}_{16}$	30	—
8	$\hat{T}_{16}$	1920	10
8	$\hat{U}_{16}^0$	840	$\{FG^3, G^4, 1^20^2, 1^20^2, 1^20^2, 1^20^2\}$
8	$\hat{U}_{16}^+$	5040	$\{FG1^2, G^21^2, G^20^2, G^20^2, 1^20^2, 1^20^2\}$
8	$\hat{U}_{16}^-$	5040	$\{G^21^2, G^21^2, FG0^2, G^20^2, 1^20^2, 1^20^2\}$

Although the Nordstrom–Robinson code $\mathscr{N}$ is nonlinear, it has the property that certain of its translates partition the whole space (see Table XII). The union of $\mathscr{N}$ and the seven translates described by the last row of Table XII is the Hamming code $\mathscr{H}$.

VII. The Ternary Golay Codes of Length 11 and 12

The automorphism group of the [12, 6, 6] ternary Golay code $\mathscr{T}$ is the group $2.M_{12}$ (see [4], [8]). In this section we classify orbits of $\mathbb{F}_3^{12}$ under the action of this group.

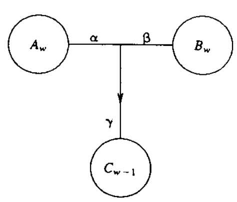
There is an essential difference between the binary and ternary classifications. In the binary case there is only one way to change a bit, so edges in the graphs of Figs. 1–4 link *pairs* of orbits. An edge linking orbits A_w and B_{w-1} indicates that any vector in B_{w-1} can be obtained by changing a 1 in some vector of A_w to a 0.

TABLE XII
TRANSLATES OF LENGTH 16 NORDSTROM–ROBINSON CODE THAT PARTITION THE WHOLE SPACE

No.	0	1	2	3	4	5	6	7	8
1	1 $\hat{S}_8$						112 $\hat{U}_{14}^+$		30 $\hat{S}_{16}$
16		1 $\hat{S}_9$				42 $\hat{T}_{13}^+$		15+70 $\hat{S}_{15} \cup \hat{U}_{15}^+$	
120			1 $\hat{S}_{10}$		14 $\hat{S}_{12}$		7+56 $\hat{S}_{14} \cup \hat{T}_{14}$		16+42+42 $\hat{T}_{16} \cup \hat{U}_{16}^+ \cup \hat{U}_{16}^-$
112				5 $\hat{S}_{11}$		15+18 $\hat{S}_{13} \cup \hat{T}_{13}^-$		60+30 $\hat{T}_{15} \cup \hat{U}_{15}^-$	
7					20 $\hat{X}_{12}$		48 $\hat{U}_{14}^-$		120 $\hat{U}_{16}^0$

In the ternary case we take the components of the vectors $u \in \mathbb{F}_3^n$ to be 0's, +'s (or +1's) and −'s (−1's). Consider the pair of vectors v, v' at Hamming distance 1 from u that are obtained by changing a particular nonzero component of u. One (v say), obtained by changing the sign of this component, has the same weight as u; the other (v' say), obtained by changing this component to a 0, has weight one less. This process links the words of $\mathbb{F}_3^n$ in *triples*.

If u, v, v' belong to different orbits A_w, B_w, C_{w-1}, respectively, we indicate this by a "trident":

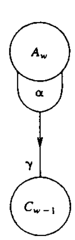


It turns out that two different v's obtained from u in this way are in the same orbit under $2.M_{12}$ just if the corresponding v''s are. We may therefore label the trident with the numbers α, β, γ, where α is the number of ways to choose the nonzero component of $u \in A_w$ that leads to a $v \in B_w$ when its sign is changed and to a $v' \in C_{w-1}$ when it is replaced by a 0.

Similarly γ is the number of zero components of $v' \in C_{w-1}$ that when replaced by one sign lead to a $u \in A_w$ and when replaced by the other sign to a $v \in B_w$. We then have

$$\alpha |A_w| = \beta |B_w| = \gamma |C_{w-1}|. \tag{2}$$

Of course it may happen that u and v are in the same orbit, in which case we make the top arms of the trident coincide:



Now

$$\alpha |A_w| = 2\gamma |C_{w-1}|. \tag{3}$$

There are 48 orbits in $\mathbb{F}_3^{12}$ under $2.M_{12}$, displayed in Figs. 5 and 6, and Table XIII. Unfortunately the graph in Fig. 5 (strictly speaking a hypergraph, since the nodes are linked in triples) is too complicated to be conveniently drawn in one piece. We have therefore broken it up into five sections, giving the orbits of weights 12-10, 9, 8, 7, and 6-0 separately. As in the binary case, Hamming distance between orbits is measured by the distance in the graph, only now one must remember that following two of the three arms of a trident takes one unit of Hamming distance. The Golay code itself is indicated by double circles.

We shall write words in the ternary Golay code $\mathscr{T}$ in 3×4 MINIMOG arrays; the reader is referred to [4] and [8] for the definition. (Note the erratum at the end of this section.)

The second column in Table XIII gives the number of vectors in each orbit. The third column gives the distance d from the code, with a subscript describing the minimal error pattern(s). Fig. 6 gives an example of a vector from each orbit. If v is a vector in the orbit and d is at most 2, there is a unique closest codeword $c \in \mathscr{T}$. Then the error pattern $e = v - c$ is given (for the particular v of the example) in Fig. 6, and the third column in Table XIII gives d_i, where i is the number of coordinates where v and e are both nonzero. (In Fig. 6 we give only the left-hand one or two columns of the MINIMOG array for e. The rest of this array is zero.)

On the other hand if v is at distance 3 from $\mathscr{T}$ then there are four codewords $c_0, \cdots, c_4$ all at distance 3 from v, and four equally likely minimal error patterns $e_r = v - c_r$ $(0 \le r \le 3)$. The four vectors $e_0, \cdots, e_3$ all have weight 3 and have disjoint supports, and any difference $e_r - e_s$ $(r \ne s)$ is a codeword of weight 6 in $\mathscr{T}$. In this situation the four e_r's are called a *quartering* (analogous to a *sextet* in the binary case). Modulo the code, v is congruent to any of $e_0, \cdots, e_3$. The simplest example of a quartering occurs when $e_0, \cdots, e_3$ are the successive columns of

$$\boxed{\begin{matrix} + & + & + & + \\ + & + & + & + \\ + & + & + & + \end{matrix}}\,. \tag{4}$$

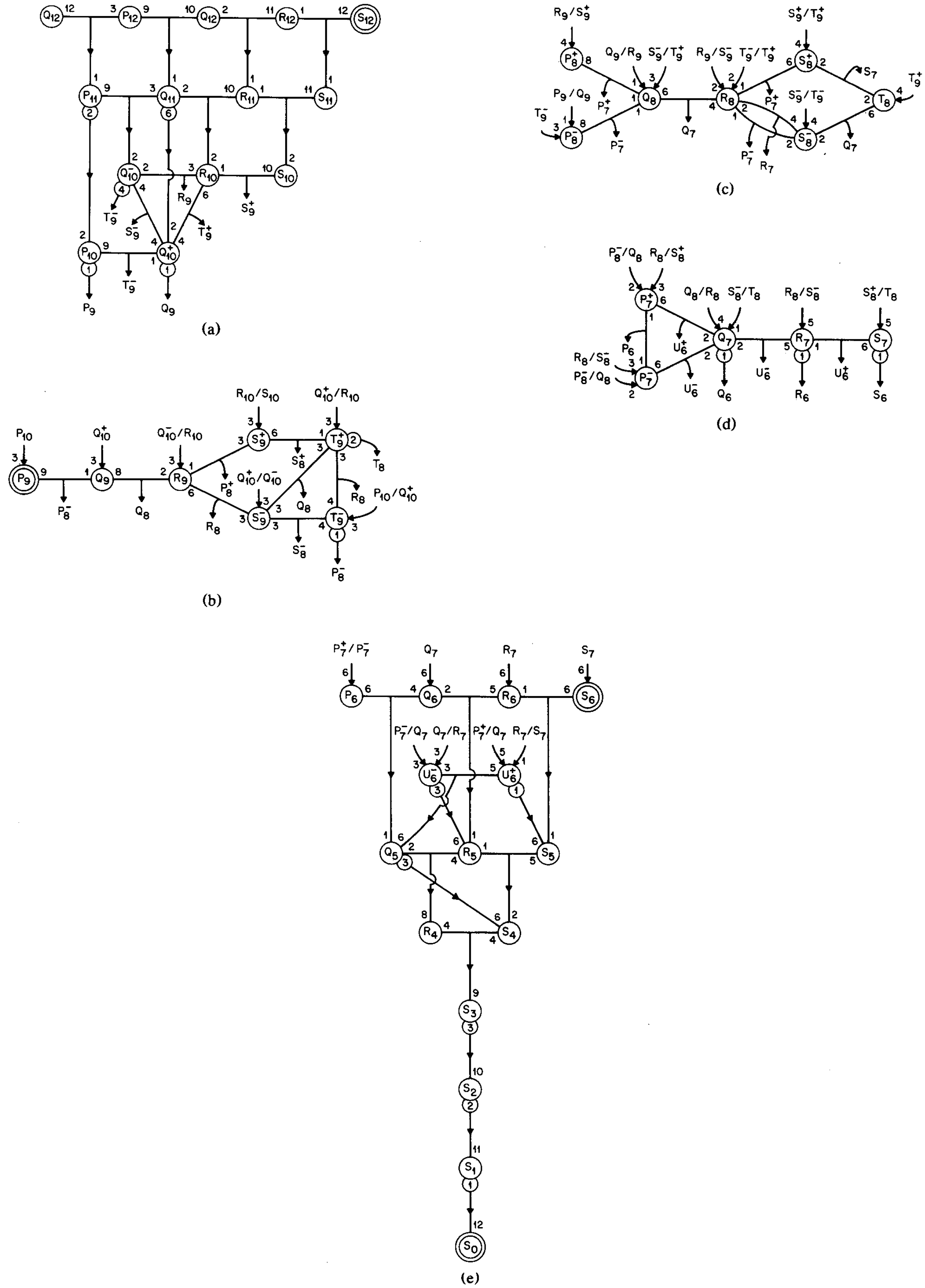


Fig. 5. Orbits under $2.M_{12}$, separated in five pieces. (a) Weights 12–10, (b) 9, (c) 8, (d) 7, and (e) 6–0.

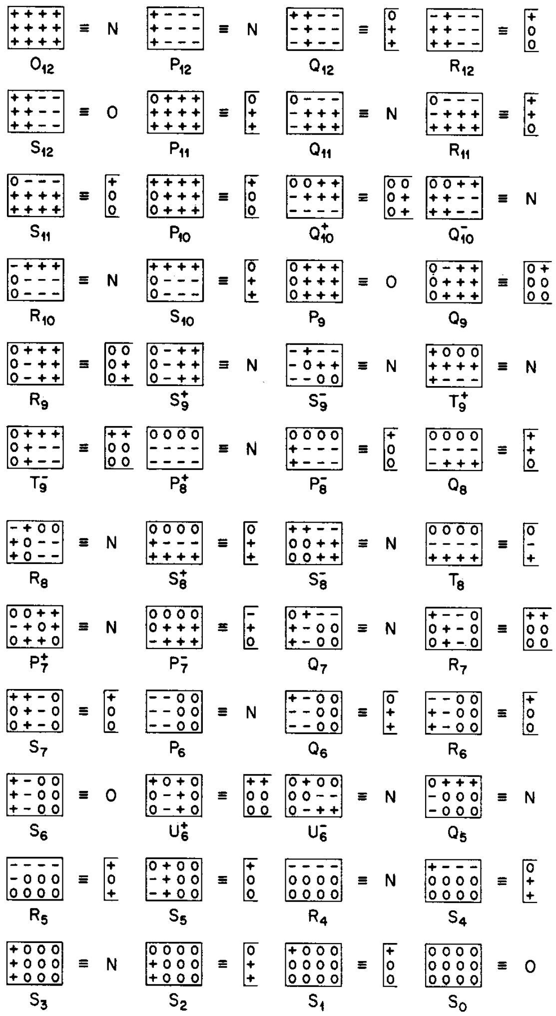


Fig. 6. Example of vector from each orbit of $2.M_{12}$, and minimal error pattern(s) modulo ternary Golay code. N indicates any column of array (4).

TABLE XIII
ORBITS UNDER $2.M_{12}$

Orbit	Size	Error Pattern	Orbit	Size	Error Pattern
O_{12}	440	3_{3333}	R_8	47520	3_{3221}
P_{12}	1760	$3_{\overline{3333}}$	S_8^+	7920	2_2
Q_{12}	1584	2_2	S_8^-	23760	3_{3311}
R_{12}	288	1_1	T_8	7920	2_2
S_{12}	24	0_0	P_7^+	15840	3_{2221}
P_{11}	5280	2_2	P_7^-	15840	2_0
Q_{11}	15840	3_{3332}	Q_7	47520	3_{3211}
R_{11}	3168	2_1	R_7	19008	2_2
S_{11}	288	1_0	S_7	3168	1_1
P_{10}	2640	1_1	P_6	2640	$3_{\bar{3}300}$
Q_{10}^+	23760	2_2	Q_6	3960	2_2
Q_{10}^-	23760	3_{3322}	R_6	1584	1_1
R_{10}	15840	3_{3331}	S_6	264	0_0
S_{10}	1584	2_0	U_6^+	19008	2_1
P_9	440	0_0	U_6^-	31680	3_{2220}
Q_9	3960	1_1	Q_5	15840	3_{2111}
R_9	15840	2_2	R_5	7920	2_1
S_9^+	5280	$3_{33\bar{3}0}$	S_5	1584	1_0
S_9^-	31680	$3_{\bar{3}222}$	R_4	3960	3_{1111}
T_9^+	31680	3_{3222}	S_4	3960	2_0
T_9^-	23760	2_1	S_3	1760	3_{3000}
P_8^+	3960	3_{2222}	S_2	264	2_2
P_8^-	3960	1_0	S_1	24	1_1
Q_8	31680	2_1	S_0	1	0_0

The symbol N in Fig. 6 stands for any of the columns of this array. If v is at distance 3 from $\mathscr{T}$ the entry in the third column of Table XIII is $3_{i_0 i_1 i_2 i_3}$, where i_r is the number of coordinates where v and e_r are both nonzero $(0 \le r \le 3)$. However, if $i_r = 3$ and v and e_r have the opposite sign on each of these three coordinates, then we put a bar over i_r.

This information is sufficient to determine the signs in $e_0, \cdots, e_3$. For each column of v adds up to the same number (σ say) modulo 3, and $\sigma \equiv -wt(v) \pmod 3$. So we can determine the signs of the coordinates where v and e_r intersect, except that three agreements in sign are indistinguishable from three disagreements. The bar then enables us to distinguish these two cases.

The cosets of $\mathscr{T}$ are analyzed in Table XIV.

TABLE XIV
COSETS OF [12, 6, 6] TERNARY GOLAY CODE $\mathscr{T}$

No.	0	1	2	3	4	5	6	7	8	9	10	11	12
1	1						264			440			24
	S_0						S_6			P_9			S_{12}
24		1				66	66	132	165	165	110	12	12
		S_1				S_5	R_6	S_7	P_8^-	Q_9	P_{10}	S_{11}	R_{12}
264			1		15	30	15+72	60+72	120+30+30	60+90	90+6	20+12	6
			S_2		S_4	R_5	$Q_6 \cup U_6^+$	$P_7^- \cup R_1$	$Q_8 \cup S_8^+ \cup T_8$	$R_9 \cup T_9^-$	$Q_{10}^+ \cup S_{10}$	$P_{11} \cup R_{11}$	Q_{12}
440				4	9	36	6+72	36+108	9+108+54	12+72+72	54+36	36	1+4
				S_3	R_4	Q_5	$P_6 \cup U_6^-$	$P_7^+ \cup Q_7$	$P_8^+ \cup R_8 \cup S_8^-$	$S_9^+ \cup S_9^- \cup T_9^+$	$Q_{10}^- \cup R_{10}$	Q_{11}	$O_{12} \cup P_{12}$

TABLE XV
WEIGHT DISTRIBUTION OF COSETS OF [11, 6, 5] GOLAY CODE

No.	0	1	2	3	4	5	6	7	8	9	10	11
1	1	0	0	0	0	132	132	0	330	110	0	24
22	0	1	0	0	30	66	108	180	165	135	32	12
220	0	0	1	6	21	60	123	174	174	114	48	8

Finally, we briefly mention the [11, 6, 5] perfect Golay code, whose automorphism group is $2\times M_{11}$. Each trident in Fig. 5 yields just one orbit under $2\times M_{11}$; there are therefore 56 orbits. Table XV gives the weight distribution of the translates of this code; here we have not separated the entries into orbits.

Erratum to "Sphere Packings, Lattices and Groups"

There is an extensive list (available from the authors) of corrections to [8]. One correction is relevant here. In [8], p. 328, lines 5 and 6 should read

modulo 11:	∞	1	9	3	4	5	0	8	6	2	X	7
mnemonic:	∞	+1	−2	+3	+4	+5	0	−3	+6	−9	−12	−15

(∞ and 0 were accidentally interchanged). We thank Robert Calderbank and Amanda Heaton for pointing this out.

References

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[2] J. H. Conway, "Three lectures on exceptional groups," in *Finite Simple Groups*, M. B. Powell and G. Higman, Eds. New York: Academic Press, 1971, pp. 215–247.

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