

The Binary Self-Dual Codes of Length up to 32: A Revised Enumeration

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This paper presents a revised enumeration of the binary self-dual codes of length up to 32 given by Conway and Pless in 1978–1980. The list of 85 doubly-even self-dual codes of length 32 is essentially correct, but several of their descriptions need amending. The principal change is that there are 731 (not 664) inequivalent self-dual codes of length 30. Furthermore, there are three (not two) $[28, 14, 6]$ and 13 (not eight) $[30, 15, 6]$ self-dual codes. Some additional information is provided about the self-dual codes of length less than 32. © 1992 Academic Press, Inc.

1. THE 85 DOUBLY-EVEN CODES OF LENGTH 32

In the course of preparing Ref. [4] we discovered certain errors in [10], and this led us to recheck the list of 85 doubly-even self-dual codes of length 32 given in [2]. The actual enumeration of these 85 codes in [2] was subject to many computer checks and was correct, but unfortunately several errors and obscurities have crept into their descriptions as printed in [2]. There are also serious errors in the numbers of children of length 30

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(found by hand) for many of these codes. We therefore give (in Table A) an amended version of Table III of [2], omitting the glue vectors. The remainder of this section contains comments on this table and further errata to [2].

Names. The 85 codes are given in the same order as in Table III of [2]. We label them C1, ..., C85 (in the first column of Table A). A star indicates that the code is mentioned in Table C below.

Components. The second column gives the components. Although the component codes d_n, e_n, \dots are described in [2], some additional remarks are appropriate.

The code g_{24-m} ($m = 0, 2, 3, 4, 6, 8$) is obtained by taking the words of the extended binary Golay code g_{24} (see [3, 7]) that vanish on m digits (and then deleting those digits). For the $[16, 5, 8]$ first-order Reed–Muller code g_{16} (wrongly called a second-order code in [2]) the eight digits must be a special octad, while for g_{18} they must be an umbral hexad (see [3 or 5] for terminology). For $0 \leq m \leq 6$, g_{24-m} is a $[24-m, 12-m, 8]$ code.

The $[24, 11, 8]$ half-Golay code h_{24} consists of the Golay codewords that intersect a given tetrad evenly.

Under the action of $\text{Aut}(g_{24})$ there are two distinct ways to select tetrads $t = \{c, d, e, f\}$, $u = \{a, b, e, f\}$, $v = \{a, b, c, d\}$ so that $t + u + v = 0$, depending on whether $\{a, b, \dots, f\}$ is a special hexad or an umbral hexad (see Fig. 1). Correspondingly there are two $[24, 10, 8]$ quarter Golay codes q_{24}^+, q_{24}^- , consisting of the codewords of g_{24} that intersect all of t, u, v

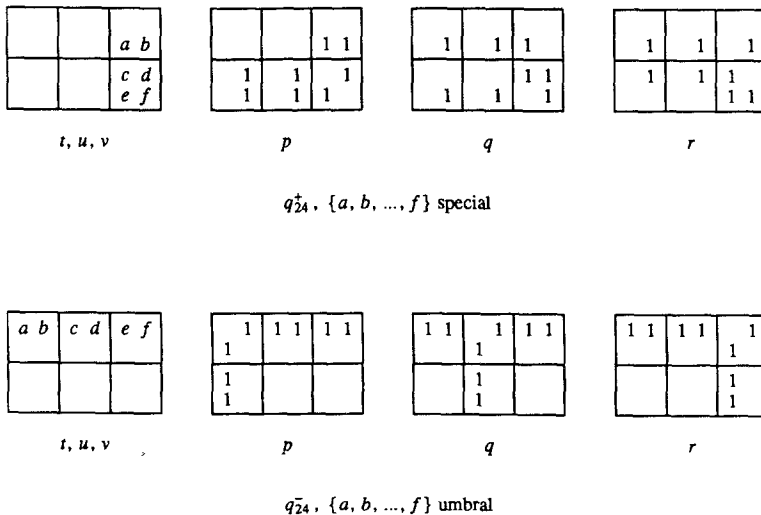


FIG. 1. The codes q_{24}^+, q_{24}^- .

TABLE A

Doubly-Even Self-Dual Codes of Length 32

Code	Components	$ G_1 $	$ G_2 $	$ G $	u_4	n_{30}	n_{28}	n_{26}	n_{24}
C1	d_{32}	1	1	$2^{30}3^65^37^{21}11.13$	120	2	2	1	1
C2	$d_{24}e_8$	1	1	$2^{27}3^65^27^211$	80	4	3	2	2
C3	$d_{20}d_{12}$	1	1	$2^{26}3^65^37$	60	5	4	2	2
C4	$d_{18}e_7^2$	1	2	$2^{22}3^65.7^3$	50	5	3	2	1
C5	d_{16}^2	1	2	$2^{29}3^45^27^2$	56	3	2	1	1
C6	$d_{16}e_8^2$	1	2	$2^{27}3^45.7^3$	56	5	3	2	2
C7	$d_{16}d_8^2$	1	2	$2^{27}3^45.7$	40	6	4	2	2
C8	$d_{14}d_{10}e_7f_1$	1	1	$2^{20}3^45^27^2$	38	11	5	3	2
C9	$d_{14}d_6^3$	1	6	$2^{20}3^65.7$	30	6	4	2	1
C10*	$d_{12}^2e_8$	1	2	$2^{25}3^55^27$	44	5	3	2	2
C11	$d_{12}^2d_8$	1	2	$2^{25}3^55^2$	36	6	4	2	2
C12	$d_{12}d_8^2d_4$	1	2	$2^{24}3^45$	28	11	7	2	2
C13	$d_{12}e_7^2d_6$	1	2	$2^{19}3^55.7^2$	32	9	5	3	1
C14	$d_{12}d_6^3f_2$	1	6	$2^{19}3^65$	24	9	4	2	1
C15	$d_{12}d_4^5$	1	120	$2^{22}3^35^2$	20	5	3	1	1
C16	$d_{10}^3f_2$	1	6	$2^{22}3^45^3$	30	5	2	1	1
C17	$d_{10}^2d_6^2$	1	4	$2^{22}3^45^2$	26	7	4	2	1
C18	$d_{10}e_8e_7^2$	1	2	$2^{20}3^45.7^3$	38	8	4	3	2
C19	$d_{10}d_8e_7d_6f_1$	1	1	$2^{19}3^45.7$	26	17	7	4	2
C20	$d_{10}d_8d_6^2f_2$	1	2	$2^{20}3^45$	22	15	6	3	2
C21	$d_{10}d_6^2d_4^2f_2$	1	4	$2^{19}3^35$	18	16	6	2	1
C22*	$d_{10}d_4^4f_6$	6	24	$2^{19}3^35$	14	9	3	1	1
C23	$d_{10}g_{22}$	2	1	$2^{15}3^35^27.11$	10	4	2	1	1
C24	e_8^4	1	24	$2^{27}3^57^4$	56	2	1	1	1
C25	$e_8d_8^3$	1	6	$2^{25}3^57$	32	5	3	2	2
C26	$e_8d_6^4$	1	24	$2^{21}3^67$	26	5	3	2	1
C27	$e_8d_4^9$	3	720	$2^{22}3^45.7$	20	4	2	1	1
C28	e_8g_{24}	1	1	$2^{16}3^45.7^211.23$	14	3	1	1	1
C29	d_8^4	1	24	$2^{27}3^5$	24	3	2	1	1
C30	d_8^4	1	8	$2^{27}3^4$	24	4	2	1	1
C31	$d_8^3d_4^2$	1	6	$2^{23}3^4$	20	6	3	1	1
C32	$d_8^2e_7^2f_2$	1	4	$2^{20}3^47^2$	26	10	3	2	1
C33	$d_8^2d_6^2f_4$	1	4	$2^{20}3^4$	18	14	4	2	1
C34	$d_8^2d_4^2$	1	16	$2^{24}3^2$	16	8	4	1	1
C35	$d_8e_7d_6^2d_4f_1$	1	2	$2^{18}3^47$	20	18	7	3	1
C36	$d_8d_6^4$	1	8	$2^{21}3^5$	18	7	4	2	1
C37	$d_8d_6^2d_6d_4f_2$	1	2	$2^{18}3^4$	16	22	9	3	1
C38	$d_8d_6^2d_4^2f_4$	1	4	$2^{18}3^3$	14	20	7	2	1
C39*	$d_8d_6d_4^3f_6$	2	6	$2^{17}3^3$	12	17	6	2	1
C40	$d_8d_4^6$	1	48	$2^{22}3^2$	12	7	4	1	1
C41*	$d_8d_4^4f_8$	2	24	$2^{18}3^2$	10	12	4	1	1
C42	$d_8d_4^2g_{16}$	36	2	$2^{17}3^3$	8	9	3	1	1
C43	d_8h_{24}	1	1	$2^{16}3^45$	6	5	2	1	1
C44	$e_7^4d_4$	1	24	$2^{17}3^57^4$	29	4	2	1	0

TABLE A—Continued

Code	Components	$ G_1 $	$ G_2 $	$ G $	u_4	n_{30}	n_{28}	n_{26}	n_{24}
C45	$e_7^2 d_6^3$	1	6	$2^{16} 3^6 7^2$	23	6	3	2	0
C46	$e_7 d_6^3 d_4 f_3$	1	6	$2^{15} 3^5 7$	17	13	4	2	0
C47	$e_7 d_6 d_4^4 f_3$	1	24	$2^{17} 3^3 7$	14	12	4	2	0
C48	$e_7 d_4^4 f_9$	18	24	$2^{15} 3^4 7$	11	7	2	1	0
C49	$e_7 d_4 g_{21}$	6	1	$2^{12} 3^4 5 \cdot 7^2$	8	6	2	1	0
C50	$d_6^5 f_2$	1	10	$2^{16} 3^5 5$	15	6	2	1	0
C51	$d_6^4 d_4^2$	1	48	$2^{20} 3^5$	14	6	3	1	0
C52	$d_6^3 d_4 f_2^2$	1	8	$2^{17} 3^4$	13	13	4	1	0
C53*	$d_6^4 f_8$	2	24	$2^{16} 3^5$	12	8	2	1	0
C54	$d_6^3 d_4^3 f_2$	1	6	$2^{16} 3^4$	12	12	5	1	0
C55	$d_6^3 d_4^2 f_6$	1	6	$2^{14} 3^4$	11	15	3	1	0
C56	$d_6^2 d_4^3 f_2^2$	1	8	$2^{17} 3^2$	10	16	4	1	0
C57	$d_6^2 d_4^4 f_4$	2	16	$2^{19} 3^2$	10	13	4	1	0
C58	$d_6^2 d_4^3 f_2 f_6$	1	12	$2^{14} 3^3$	9	18	4	1	0
C59	$d_6^2 d_4^2 f_{12}$	12	4	$2^{14} 3^3$	8	14	3	1	0
C60	$d_6^2 g_{20}$	4	2	$2^{15} 3^3 5$	6	6	2	1	0
C61*	$d_6 d_4^2 d_4^3 f_3^2$	1	12	$2^{15} 3^2$	8	19	6	1	0
C62	$d_6 d_4^4 f_{10}$	2	8	$2^{15} 3$	7	21	4	1	0
C63	$d_6 d_4^3 f_{14}$	8	6	$2^{13} 3^2$	6	18	4	1	0
C64	$d_6 d_4^2 g_{18}$	36	2	$2^{10} 3^4$	5	12	3	1	0
C65*	$d_6 d_4 g_{16} f_6$	72	1	$2^{12} 3^3$	4	14	4	1	0
C66*	$d_6 f_{13}^3$	5616	2	$2^8 3^4 13$	3	6	2	1	0
C67	d_4^8	6	1344	$2^{23} 3^2 7$	8	2	1	0	0
C68	d_4^8	1	1152	$2^{23} 3^2$	8	3	1	0	0
C69	d_4^8	1	336	$2^{20} 3 \cdot 7$	8	2	1	0	0
C70*	$d_4^6 f_8$	4	48	$2^{18} 3$	6	7	2	0	0
C71*	$d_4^6 f_8$	1	48	$2^{16} 3$	6	9	2	0	0
C72	$d_4^4 d_4 f_{12}$	6	24	$2^{14} 3^2$	5	10	2	0	0
C73	$d_4^5 f_{12}$	1	60	$2^{12} 3 \cdot 5$	5	6	1	0	0
C74	$d_4^4 g_{16}$	8	24	$2^{18} 3$	4	7	2	0	0
C75*	$d_4^4 f_{16}$	8	8	2^{14}	4	14	2	0	0
C76	$d_4^3 g_{18} f_2$	8	6	$2^{10} 3^2$	3	13	2	0	0
C77*	$d_4^2 q_{24}^+$	6	2	$2^{15} 3^2$	2	6	1	0	0
C78*	$d_4^2 q_{24}^-$	3	2	$2^{10} 3^2$	2	8	1	0	0
C79*	$d_4 f_4^6$	16	72	$2^{11} 3^2$	2	8	1	0	0
C80*	$d_4 f_7^4$	168	8	$2^8 \cdot 3 \cdot 7$	1	8	2	0	0
C81	q_{32}	1	1	$2^5 3 \cdot 5 \cdot 31$	0	1	0	0	0
C82	r_{32}	1	1	$2^{15} 3^2 5 \cdot 7 \cdot 31$	0	1	0	0	0
C83	g_{16}^2	20160	2	$2^{15} 3^2 5 \cdot 7$	0	2	0	0	0
C84	f_4^8	256	336	$2^{12} 3 \cdot 7$	0	2	0	0	0
C85	f_2^{16}	2	11520	$2^9 3^2 5$	0	3	0	0	0

evenly. In [2] only the second of these was described and there it was called q_{24} , while q_{24}^+ was called $(g_{16} + f_8)$. (The exceptional treatment of the components of this code given on p. 52 of [2] is now eliminated.)

The glue space for either q_{24}^+ or q_{24}^- is four-dimensional and is generated by t, u, v, p, q, r with

$$t + u + v = 0 = p + q + r,$$

where p, q, r may be represented by special octads, with p orthogonal to t but not to u or v , q orthogonal to u but not to t or v , and r orthogonal to v but not to t or u (see Fig. 1).

The 16 glue components and their minimal weights are:

Representative	Minimal weight
0	0
t, u, v	4
p, q, r	8
$p + u, q + v, r + t,$ $p + v, q + t, r + u$	6
$p + t, q + u, r + v$	8 in q_{24}^+ , 4 in q_{24}^-

The groups. The order $|G|$ of the automorphism group of any of the 85 codes is given (as in [2]) by the formula

$$|G| = |G_0| |G_1| |G_2|,$$

while $|G_0|$ is the product of the $|G_0|$'s of the component codes, and $|G_1|, |G_2|$ are given in the third and fourth columns of Table A. $|G|$ itself is given

TABLE B
The Groups G_0

Component	G_0	$ G_0 $
d_{2m}	$2^{m-1} \cdot S_m$	$2^{m-1} m!$
e_7	$L_3(2)$	168
e_8	$2^3 \cdot L_3(2)$	1344
f_n	1	1
g_{16}	2^4	16
g_{18}	C_3	3
g_{20}	M_{20}	$2^6 \cdot 3 \cdot 5$
g_{21}	M_{21}	$2^6 \cdot 3^2 \cdot 5 \cdot 7$
g_{22}	M_{22}	$2^7 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11$
g_{24}	M_{24}	$2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23$
h_{24}^+	$2^6 \cdot 3 \cdot S_6$	$2^9 \cdot 3^3 \cdot 5$
q_{24}^+	$2^6 \cdot (S_3 \times 2^2)$	$2^9 \cdot 3$
q_{24}^-	$2^2 \times S_4$	$2^5 \cdot 3$

in the fifth column. Table B gives the groups G_0 (in ATLAS [1] notation) for the components.

The mass checks. We repeated various “mass checks” on the 85 codes, verifying that the number of codes containing a specified subcode (e.g., d_4 or d_6) is as predicted by the formula given on p. 28 of [2]. In particular, we rechecked that the total mass

$$\sum_{(85)} \frac{1}{|G|}$$

of the reciprocals of the numbers in the fifth column of Table A has the correct value

$$\frac{1}{32!} \prod_{i=0}^{14} (2^i + 1),$$

TABLE C
Other Alterations to Table III of [2]

Code	Components	Change
C22, C39, C53, C65, C70, C71		f_m^2 has been replaced by f_{2m} .
C10	$d_{12}^2 e_8$	$ G_2 = 2$, not 1.
C41	$d_8 d_4^4 f_8$	Change <i>boczxAB</i> to <i>boozxAB</i> .
C61	$d_6 d_4^2 d_4^3 f_3^2$	The final - was omitted.
C66	$d_6 f_{13}^2$	The last character should be - not —.
C71	$d_4^6 f_4^2$	The glue generators should be <i>ooyoxxAE</i> , <i>yooxoxBF</i> , <i>oyoxxoCD</i> , <i>oxyooxCG</i> , <i>yoxxooAG</i> , <i>xyooxoBG</i> , <i>ooxoxyFH</i> , <i>xooyoxDH</i> , <i>oxoxyoEH</i> , <i>oxxooyBD</i> , <i>xoxyooCE</i> , <i>xxooyoAF</i> , <i>ozxozx-</i> , <i>xozxoz-</i> , <i>xzoxzo-</i> .
C75	$d_4^4 f_{16}$	The printing of the glue is poorly aligned. Each glue word consists of a top line of 4 letters (chosen from <i>o, x, y, z</i>), with a 4×4 array beneath it.
C77	$d_4^2 q_{24}^+$	Redescribed above.
C78	$d_4^2 q_{24}^-$	Redescribed above.
C79	$d_4^2 f_4^6$	The parentheses around the final array indicate that its six columns are to be bodily permuted.
C80	$d_4 f_7^4$	The third and fourth glue generators should be $y(+++o+oo)(ooooooo)$ $(+oooooo)(+oooooo)$, $y(ooooooo)$ $(+++o+oo)(+oooooo)(+oooooo)$.

which is

$$\frac{391266122896364123}{532283035423762022400}$$

Weight distributions. The sixth column gives u_4 , the number of codewords of weight 4. The full weight distribution can then be obtained from Table IV of [2].

Table C summarizes the amendments to Table III of [2] other than errors in the value of n_{30} .

Additional Errata to [2]

On p. 37 of [2] the phrase “Figure (MOG)” refers to Ref. [5] (or Fig. 11.17 of [3]).

On p. 44, for the first code in Table I, change e_0 to e_8 .

On pp. 46 and 48, the heading should read $n_{30}, n_{28}, n_{26}, n_{24}, \dots$

On p. 52, the last entry in Table V should be 731, not 664.

On p. 53, the last author’s name is misspelled in Ref. [5].

2. SELF-DUAL CODES OF LENGTH LESS THAN 32

The numbers of children. We now describe how the final four columns of Table A were obtained. These give $n_{30}, n_{28}, n_{26}, n_{24}$, the number of self-dual codes (the “children”) of lengths 30, ..., 24 that arise from each of the 85 codes.

Any self-dual code C of length 30 is obtained by taking all codewords of one of C1, ..., C85 for which some particular pair of coordinates P, Q (say) are 00 or 11, and deleting these coordinates. If C contains a weight-2 word, which it does when P and Q belong to a weight-4 word of the original code, we obtain a self-dual code with length less than 30 and $d \geq 4$ by “collapsing” C , i.e., by deleting all pairs of coordinates that support weight-2 words. All self-dual codes with length $n \leq 30$ and $d \geq 4$ can be obtained in this way. There are several cases:

If the coordinates P, Q are in an e_8 , the collapsed code is a doubly-even self-dual code of length 24, whose components are obtained by deleting the e_8 .

If P, Q are in an e_7 , the collapsed code has length 26 and its components are obtained by replacing the e_7 by an f_1 .

If P, Q form a duad (a pair of identical coordinates, cf. [2], p. 33) in a $d_m, m \geq 6$, the collapsed code has length $32 - m$ and its components are obtained by deleting the d_m .

TABLE D
Self-Dual Codes with $n \leq 22$, $d \geq 4$

n	Code	Components	[9, 11]	$ G_1 $	$ G_2 $	u_4	u_6	u_8	u_{10}	u_{12}	Generators for glue
0	C1(d_{32})	t_0	—	1	1						—
8	C2(d_{24})	e_8	A_8	1	1	14	0	1			—
12	C3(d_{20})	d_{12}	B_{12}	1	1	15	32	15	0	1	a
14	C4(d_{18})	e_7^2	D_{14}	1	2	14	49	49	14	0	dd
16	C5(d_{16})	d_{16}^2	E_{16}	1	1	28	0	198	0	28	a
	C6(d_{16})	e_8^2	$A_8 \oplus A_8$	1	2	28	0	198	0	28	—
	C7(d_{16})	d_8^2	F_{16}	1	2	12	64	102	64	12	(ab)
18	C8(d_{14})	$d_{10}e_7f_1$	I_{18}	1	1	17	51	187	187	51	$aoA, cd-$
	C9(d_{14})	d_6^3	H_{18}	1	6	9	75	171	171	75	$(abc), bbb$
20	C3(d_{12})	d_{20}	J_{20}	1	1	45	0	210	512	210	a
	C10(d_{12})	$d_{12}e_8$	$A_8 \oplus B_{12}$	1	1	29	32	226	448	226	$a-$
	C11(d_{12})	$d_{12}d_8$	K_{20}	1	1	21	48	234	416	234	(ab)
	C12(d_{12})	$d_8^2d_4$	S_{20}	1	2	13	64	242	384	242	$(ab)x, bby$
	C13(d_{12})	$e_7^2d_6$	L_{20}	1	2	17	56	238	400	238	d_0a, ddb
	C14(d_{12})	$d_6^3f_2$	R_{20}	1	6	9	72	246	368	246	$aaaa, cccB, (abc)-$
	C15(d_{12})	d_4^2	M_{20}	1	120	5	80	250	352	250	$(ooxyx)$
22	C8(d_{10})	$d_{14}e_7f_1$	N_{22}	1	1	28	49	246	700	700	aoA, bdA
	C16(d_{10})	$d_{10}f_2^2$	P_{22}	1	2	20	57	270	676	676	$(ao)^*, cc-$
	C17(d_{10})	$d_{10}d_6^2$	Q_{22}	1	2	16	61	282	664	664	aoc, oaa, bbb
	C18(d_{10})	$e_8e_7^2$	$E_8 \oplus D_{14}$	1	2	28	49	246	700	700	$-dd$
	C19(d_{10})	$d_8e_7d_6f_1$	R_{22}	1	1	16	61	282	664	664	$odbA, boaA, aob-$
	C20(d_{10})	$d_8^2d_6^2f_2$	S_{22}	1	2	12	65	294	652	652	$ba0A, ao0AB, abb-, occ-$
	C21(d_{10})	$d_8^2d_4^2f_2$	T_{22}	1	4	8	69	306	640	640	$aoxoA, ooyyAB, atyo-, bozx-, obxz-$
	C22(d_{10})	$d_4^4f_6$	U_{22}	6	24	4	73	318	628	628	$aoxzBC, ozxyAC, ooxxAE, oyyoAD, ozzoAF, xxxx-, yyyy-$
	C23(d_{10})	g_{22}	G_{22}	2	1	0	77	330	616	616	the all-ones vector

TABLE E
Self-Dual Codes with Length 24 and $d \geq 4$

Code	Components	[11]	d	Code	Components	[11]	d
C2(e_8)	d_{24}	E_{24}	4	C32(d_8)	$d_8 e_7^2 f_2$	J_{24}	4
C6(e_8)	$d_{16} e_8$	—	4	C33(d_8)	$d_8 d_6^2 f_4$	R_{24}	4
C7(d_8)	$d_{16} d_8$	H_{24}	4	C34(d_8)	$d_8 d_4^4$	T_{24}	4
C10(e_8)	d_{12}^2	A_{24}	4	C35(d_8)	$e_7 d_6^2 d_4 f_1$	P_{24}	4
C11(d_8)	d_{12}^2	—	4	C26(e_8)	d_6^4	D_{24}	4
C12(d_8)	$d_{12} d_8 d_4$	I_{24}	4	C36(d_8)	d_6^4	Q_{24}	4
C18(e_8)	$d_{10} e_7^2$	B_{24}	4	C37(d_8)	$d_6^2 d_6 d_4 f_2$	S_{24}	4
C19(d_8)	$d_{10} e_7 d_6 f_1$	K_{24}	4	C38(d_8)	$d_6^2 d_4^2 f_4$	U_{24}	4
C20(d_8)	$d_{10} d_6^2 f_2$	N_{24}	4	C39(d_8)	$d_6 d_4^3 f_6$	W_{24}	4
C24(e_8)	e_8^3	—	4	C27(e_8)	d_4^6	F_{24}	4
C25(d_8)	$e_8 d_8^2$	—	4	C40(d_8)	d_4^6	V_{24}	4
C25(e_8)	d_8^3	C_{24}	4	C41(d_8)	$d_4^4 f_8$	X_{24}	4
C29(d_8)	d_8^3	L_{24}	4	C42(d_8)	$d_4^2 g_{16}$	Y_{24}	4
C30(d_8)	d_8^3	M_{24}	4	C43(d_8)	h_{24}	Z_{24}	6
C31(d_8)	$d_8^2 d_4^2$	O_{24}	4	C28(e_8)	g_{24}	G_{24}	8

If P, Q are in a $d_m, m \geq 6$, but do not form a duad, or if P, Q belong to a d_4 component, the collapsed code has length 28.

In all other cases C does not collapse and we have a self-dual code with length 30 and $d \geq 4$.

As an example we consider the code C61. There are 19 orbits of $\text{Aut}(C61)$ on unordered pairs of distinct coordinates, so there are $n_{30} = 19$ length 30 children, 6 of which collapse to shorter lengths, as follows. One length 26 child is obtained from a duad in the d_6 (thus $n_{26} = 1$).

From two coordinates in the d_6 not forming a duad we obtain the first child of length 28, while a second child of length 28 is obtained from any

TABLE F
Weight Distributions of Self-Dual Codes with
 $n = 26, 28, 30$ and $d \geq 6$

	u_6	u_8	u_{10}	u_{12}	u_{14}
A_{26}	52	390	1313	2340	2340
A_{28}	26	442	1560	3653	5020
B_{28}, C_{28}	42	378	1624	3717	4680
A_{30}, B_{30}, C_{30}	19	393	1848	5192	8931
D_{30}	27	369	1848	5256	8883
E_{30}, \dots, M_{30}	35	345	1848	5320	8835

	C66		C78
	11110000000000000000000000000000		11110000000000000000000000000000
	00111100000000000000000000000000		00001111000000000000000000000000
	01010111010000010001000000000000		0000000101111010110011001010000
	01010101101000001000100000000000		0000000100111101011001100101000
	01010100110100000100010000000000		0000000100011110101100110010100
	01010100011010000010001000000000		0000000100001111010110011001010
	01010100011010000000010000000000		000000011000011110101100110010
	01010100011010000000010000000000		00000000101001110101100101001
	01010100010001101000000001000000		0000000111001010000011110101100
	01010100010001101000000001000000		00000000011010011110101100101010
	01010100001000110100000000100000		0000000111101100010110010001100
	01010100001000110100000000100000		00000001111111111111111111111111
	01010110000010001100000000000100		01010110100000110010000011100001
	01010101000001000110000000000010		0110001101001010110010000001001
	01010110100000100010000000000001		0110010100000000000111100000000
	01010111111111111100000000000000		00110110000000000000011110000000
A ₂₆	ααββγγ	B ₃₀	α α
A ₂₈	αβαβ		
	C77		C79
	11110000000000000000000000000000		11110000000000000000000000000000
	00001111000000000000000000000000		00001111000000000000000000000000
	00000001111111000000000000000000		00110011001100000000011000000000
	00000000000000011111110000000000		00110101000000110000010100000000
	00000000000000000000000001111111		00110110000000000011011000000000
	00000000101010101010101000000000		010100110101000000000000110000
	00000000011001100110011000000000		0101010100000101000000001010000
	00000000000111100001111000000000		0101011000000000010100000110000
	00000101000001010000010100000101		0110001101100000000000000000011
	00000011000000110000001100000011		01100101000001100000000000000101
	01010000010100000101000001010000		0000000011110001000100010001000
	00110000001100000011000000110000		00000000100001111000100010001000
	0000000011110000100010001000100		00000000100010000111100010001000
	0000000011110000010001000100010		00000000100010001000011110001000
	0000000011110000001000100010001		00000000100010001000100001111000
	000000000000000000000111100001111		00000000100010001000100010000111
A ₃₀	α α	C ₃₀	α α

FIG. 2. Generator matrices for C66, C77, ..., C85 and their children of length 26, 28, 30 and $d = 6$.

two coordinates in the first type of d_4 , and three further children of length 28 arise from the three different ways of choosing a pair of coordinates from the other type of d_4 . Thus $n_{28} = n_{26} + 5 = 6$. Finally, there are 13 children with length 30 and $d \geq 4$, so that $n_{30} = n_{28} + 13 = 19$.

In [2, 10] the actions of the automorphism groups of the 85 codes on pairs of coordinates were found (unfortunately often incorrectly) by hand. In the present version most of this work has been redone by computer (using in particular the graph-automorphism program Nauty [8]). The numbers of children of lengths $n \leq 28$ given in [2] are correct. There are numerous errors in n_{30} , however (now corrected in Table A), and the total number of self-dual codes of length 30 is 731, not 664 as stated in Table V of [2].

	C80		C82
	11110000000000000000000000000000		10011100100000100000011000000000
	01011000000100000010010110000000		10001110010000010000001100000000
	01010100000010000011001010000000		10000111001000001000000110000000
	01010010000001000011100100000000		10000011100100000100000011000000
	01010001000000100001110010000000		10000001110010000010000001100000
	01010000100000010010111000000000		10000000111001000001000000110000
	01010000010000001001011100000000		10000000011100100000100000011000
	01011101000000000100000010000000		10000000001110010000010000001100
	01010111010000000001000000100000		10000000000111001000001000000011
	010100111010000000000100000010000		11000000000001110010000010000001
	010110011100000000000100000010000		11100000000000111001000001000000
	010101001110000000000010000001000		10110000000000011100100000100000
	010110100110000000000001000000010		10011000000000001110010000010000
	010110100100000000000001000000010		10001100000000000111001000001000
	00111111110000000111111100000000		10000110000000000111001000001000
B_{28}	$\alpha\alpha\beta\beta$	F_{30}	$\alpha\alpha$
C_{28}	$\alpha\beta\alpha\beta$		
D_{30}	$\alpha \quad \alpha$		
	C81		C83
	10110110111100010101110000100100		11101000111010000000000000000000
	10011011011110001010111000010010		10110100101101000000000000000000
	10001101101111000101011100001001		10011010001101000000000000000000
	11000110110111100010101110000100		00000000111010001110100011101000
	10100011011011110001010111000010		000000001001101001011010010110100
	10010001101101111000101011100001		000000001001101001011010010110100
	11001000110110111100010101110000		00000000100011011000110110001101
	10100100011011011110001010111000		11011000110110001101100000000000
	10010010001101101111000101011100		10101100101011001010111000000000
	10000100100011011011110001010111		10010110100101101001011000000000
	11000010010001101101111000101011		10001011100010111000010110000000
	11100001001000110110111100010101		00000000000000001101100011011000
	11110000100100011011011110001010		000000000000000001010110010101100
	10111000010010001101101111000101		000000000000000001001011010010110
	11011100001001000110110111100010		000000000000000001000101110001011
E_{30}	$\alpha\alpha$	G_{30}	$\alpha\alpha$
		H_{30}	$\alpha \quad \alpha$

FIGURE 2 — continued

Tables of Self-Dual Codes of Length $n \leq 24$

The self-dual codes of length $n \leq 20$ were first enumerated in [9], and those of lengths 22 and 24 in [11], although there they are not described in the terminology later used in [2]. For completeness we therefore give the components, values of $|G_1|$, $|G_2|$, weight distributions $\{u_i\}$, and glue generators for the codes with $n \leq 22$ in Table D. The column headed "Code" gives the parent code of length 32 (with the component to be deleted in parentheses). The column headed "[9, 11]" gives the names used in these papers. ([9, 11] also give generator matrices for these codes.)

Table E lists the codes of length 24, although to save space we just

C84

```

1110100000000001110100011101000
1011010000000001011010010110100
1001101000000001001101010011010
1000110100000001000110110001101
0000000111010001110100010110100
0000000101101001011010010011010
0000000100110101001101010001101
0000000100011011000110111000110
1101100011011000110110000000000
1010110010101100101011000000000
1001011010010110100101100000000
1000101110001011100010110000000
11011000101100010000000011011000
1010110011011000000000010101100
1001011010101100000000010010110
1000101110010110000000010001011

```

I_{30}	$\alpha\alpha$	
J_{30}	α	α

C85

```

1000000000000001111100010001000
0100000000000001111010001000100
0010000000000001111001000100010
0001000000000001111000100010001
0000100000000001000111110001000
00000100000000000100111101000100
0000001000000000010111100100010
0000000100000000001111100010001
0000000100000001000100011111000
0000000010000000100010011110100
0000000001000000010001011110010
0000000000100000001000111110001
0000000000010001000100010001111
0000000000001000100010001001111
00000000000000100010001000101111
0000000000000010001000100011111

```

K_{30}	$\alpha\alpha$	
L_{30}	α	α
M_{30}	α	α

FIGURE 2 — continued

describe each code by giving its parent (and in parentheses the component to be deleted), its components, and the minimal distance d . If the deleted component (in parentheses) is an e_8 , the code is doubly-even; otherwise (when the deleted component is a d_8) it is singly-even.

Codes with $d \geq 6$. Similar tables could easily be constructed to list the self-dual codes of lengths 26, 28, and 30, but they would occupy too much space. Instead we just describe the 17 codes with $d \geq 6$ (and correct some errors in [10]).

There is one code (A_{26}) of length 26, arising from C66; three codes of length 28, one (A_{28}) from C66 and two (B_{28}, C_{28}) from C80; and 13 codes

(A_{30}, \dots, M_{30}) of length 30, one from each of C77, ..., C82, two from each of C83, C84, and three from C85.

Figure 2 contains generator matrices for C66, C77, ..., C85. Each row of Greek letters below one of these matrices specifies a self-dual child with $d \geq 6$. The child is obtained by restricting to the subcode consisting of the words that are equal on pairs of coordinates described by the same Greek letter, and then deleting these coordinates. (For C83 and C84 we have used the generator matrices described in [6, 12].) The weight distributions are given in Table F (compare [4]).

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