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The Binary Self-Dual Codes of Length up to 32: A Revised Enumeration

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This paper presents a revised enumeration of the binary self-dual codes of length up to 32 given by Conway and Pless in 1978–1980. The list of 85 doubly-even self-dual codes of length 32 is essentially correct, but several of their descriptions need amending. The principal change is that there are 731 (not 664) inequivalent self-dual codes of length 30. Furthermore, there are three (not two) [28, 14, 6] and 13 (not eight) [30, 15, 6] self-dual codes. Some additional information is provided about the self-dual codes of length less than 32.

1. THE 85 DOUBLY-EVEN CODES OF LENGTH 32

In the course of preparing Ref. [4] we discovered certain errors in [10], and this led us to recheck the list of 85 doubly-even self-dual codes of length 32 given in [2]. The actual enumeration of these 85 codes in [2] was subject to many computer checks and was correct, but unfortunately several errors and obscurities have crept into their descriptions as printed in [2]. There are also serious errors in the numbers of children of length 30

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(found by hand) for many of these codes. We therefore give (in Table A) an amended version of Table III of [2], omitting the glue vectors. The remainder of this section contains comments on this table and further errata to [2].

Names. The 85 codes are given in the same order as in Table III of [2]. We label them C1, ..., C85 (in the first column of Table A). A star indicates that the code is mentioned in Table C below.

Components. The second column gives the components. Although the component codes $d_n, e_n, \ldots$ are described in [2], some additional remarks are appropriate.

The code g_{24-m} $(m = 0, 2, 3, 4, 6, 8)$ is obtained by taking the words of the extended binary Golay code g_{24} (see [3, 7]) that vanish on m digits (and then deleting those digits). For the [16, 5, 8] first-order Reed–Muller code g_{16} (wrongly called a second-order code in [2]) the eight digits must be a special octad, while for g_{18} they must be an umbral hexad (see [3 or 5] for terminology). For $0 \leqslant m \leqslant 6$, g_{24-m} is a $[24-m,\ 12-m,\ 8]$ code.

The [24, 11, 8] half-Golay code h_{24} consists of the Golay codewords that intersect a given tetrad evenly.

Under the action of $\mathrm{Aut}(g_{24})$ there are two distinct ways to select tetrads $t = \{c, d, e, f\}$, $u = \{a, b, e, f\}$, $v = \{a, b, c, d\}$ so that $t + u + v = 0$, depending on whether $\{a, b, \ldots, f\}$ is a special hexad or an umbral hexad (see Fig. 1). Correspondingly there are two [24, 10, 8] quarter Golay codes q_{24}^{+}, q_{24}^{-}, consisting of the codewords of g_{24} that intersect all of t, u, v

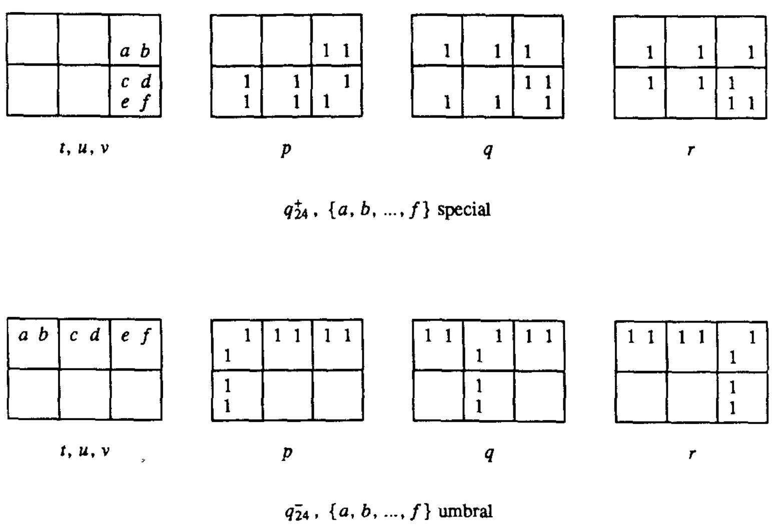

FIG. 1. The codes q_{24}^{+}, q_{24}^{-}.

TABLE A

Doubly-Even Self-Dual Codes of Length 32

Code	Components	$\lvert G_1\rvert$	$\lvert G_2\rvert$	$\lvert G\rvert$	u_4	n_{30}	n_{28}	n_{26}	n_{24}
C1	d_{32}	1	1	$2^{30}3^6 5^3 7^2 11.13$	120	2	2	1	1
C2	$d_{24}e_8$	1	1	$2^{27}3^6 5^2 7^2 11$	80	4	3	2	2
C3	$d_{20}d_{12}$	1	1	$2^{26}3^6 5^3 7$	60	5	4	2	2
C4	$d_{18}e_7^2$	1	2	$2^{22}3^6 5.7^3$	50	5	3	2	1
C5	d_{16}^2	1	2	$2^{29}3^4 5^2 7^2$	56	3	2	1	1
C6	$d_{16}e_8^2$	1	2	$2^{27}3^4 5.7^3$	56	5	3	2	2
C7	$d_{16}d_8^2$	1	2	$2^{27}3^4 5.7$	40	6	4	2	2
C8	$d_{14}d_{10}e_7f_1$	1	1	$2^{20}3^4 5^2 7^2$	38	11	5	3	2
C9	$d_{14}d_6^3$	1	6	$2^{20}3^6 5.7$	30	6	4	2	1
C10*	$d_{12}^2 e_8$	1	2	$2^{25}3^5 5^2 7$	44	5	3	2	2
C11	$d_{12}^2 d_8$	1	2	$2^{25}3^5 5^2$	36	6	4	2	2
C12	$d_{12}d_8^2 d_4$	1	2	$2^{24}3^4 5$	28	11	7	2	2
C13	$d_{12}e_7^2 d_6$	1	2	$2^{19}3^5 5.7^2$	32	9	5	3	1
C14	$d_{12}d_6^3 f_2$	1	6	$2^{19}3^6 5$	24	9	4	2	1
C15	$d_{12}d_4^5$	1	120	$2^{22}3^3 5^2$	20	5	3	1	1
C16	$d_{10}^3 f_2$	1	6	$2^{22}3^4 5^3$	30	5	2	1	1
C17	$d_{10}^2 d_6^2$	1	4	$2^{22}3^4 5^2$	26	7	4	2	1
C18	$d_{10}e_8 e_7^2$	1	2	$2^{20}3^4 5.7^3$	38	8	4	3	2
C19	$d_{10}d_8 e_7 d_6 f_1$	1	1	$2^{19}3^4 5.7$	26	17	7	4	2
C20	$d_{10}d_8 d_6^2 f_2$	1	2	$2^{20}3^4 5$	22	15	6	3	2
C21	$d_{10}d_6^2 d_4^2 f_2$	1	4	$2^{19}3^3 5$	18	16	6	2	1
C22*	$d_{10}d_4^4 f_6$	6	24	$2^{19}3^3 5$	14	9	3	1	1
C23	$d_{10}\,g_{22}$	2	1	$2^{15}3^3 5^2 7.11$	10	4	2	1	1
C24	e_8^4	1	24	$2^{27}3^5 7^4$	56	2	1	1	1
C25	$e_8 d_8^3$	1	6	$2^{25}3^5 7$	32	5	3	2	2
C26	$e_8 d_6^4$	1	24	$2^{21}3^6 7$	26	5	3	2	1
C27	$e_8 d_4^6$	3	720	$2^{22}3^4 5.7$	20	4	2	1	1
C28	$e_8\,g_{24}$	1	1	$2^{16}3^4 5.7^2 11.23$	14	3	1	1	1
C29	d_8^4	1	24	$2^{27}3^5$	24	3	2	1	1
C30	d_8^4	1	8	$2^{27}3^4$	24	4	2	1	1
C31	$d_8^3 d_4^2$	1	6	$2^{23}3^4$	20	6	3	1	1
C32	$d_8^2 e_7^2 f_2$	1	4	$2^{20}3^4 7^2$	26	10	3	2	1
C33	$d_8^2 d_6^2 f_4$	1	4	$2^{20}3^4$	18	14	4	2	1
C34	$d_8^2 d_4^4$	1	16	$2^{24}3^2$	16	8	4	1	1
C35	$d_8 e_7 d_6^2 d_4 f_1$	1	2	$2^{18}3^4 7$	20	18	7	3	1
C36	$d_8 d_6^4$	1	8	$2^{21}3^5$	18	7	4	2	1
C37	$d_8 d_6^2 d_6 d_4 f_2$	1	2	$2^{18}3^4$	16	22	9	3	1
C38	$d_8 d_6^2 d_4^2 f_4$	1	4	$2^{18}3^3$	14	20	7	2	1
C39*	$d_8 d_6 d_4^3 f_6$	2	6	$2^{17}3^3$	12	17	6	2	1
C40	$d_8 d_4^6$	1	48	$2^{22}3^2$	12	7	4	1	1
C41*	$d_8 d_4^4 f_8$	2	24	$2^{18}3^2$	10	12	4	1	1
C42	$d_8 d_4^2\,g_{16}$	36	2	$2^{17}3^3$	8	9	3	1	1
C43	$d_8 h_{24}$	1	1	$2^{16}3^4 5$	6	5	2	1	1
C44	$e_7^4 d_4$	1	24	$2^{17}3^5 7^4$	29	4	2	1	0

TABLE A—*Continued*

Code	Components	$\|G_1\|$	$\|G_2\|$	$\|G\|$	u_4	n_{30}	n_{28}	n_{26}	n_{24}
C45	$e_7^2 d_6^3$	1	6	$2^{16}3^6 7^2$	23	6	3	2	0
C46	$e_7 d_6^3 d_4 f_3$	1	6	$2^{15}3^5 7$	17	13	4	2	0
C47	$e_7 d_6 d_4^4 f_3$	1	24	$2^{17}3^3 7$	14	12	4	2	0
C48	$e_7 d_4^4 f_9$	18	24	$2^{15}3^4 7$	11	7	2	1	0
C49	$e_7 d_4 g_{21}$	6	1	$2^{12}3^4 5.7^2$	8	6	2	1	0
C50	$d_6^5 f_2$	1	10	$2^{16}3^5 5$	15	6	2	1	0
C51	$d_6^4 d_4^2$	1	48	$2^{20}3^5$	14	6	3	1	0
C52	$d_6^4 d_4 f_2^2$	1	8	$2^{17}3^4$	13	13	4	1	0
C53*	$d_6^4 f_8$	2	24	$2^{16}3^5$	12	8	2	1	0
C54	$d_6^3 d_4^3 f_2$	1	6	$2^{16}3^4$	12	12	5	1	0
C55	$d_6^3 d_4^2 f_6$	1	6	$2^{14}3^4$	11	15	3	1	0
C56	$d_6^2 d_4^4 f_2^2$	1	8	$2^{17}3^2$	10	16	4	1	0
C57	$d_6^2 d_4^4 f_4$	2	16	$2^{19}3^2$	10	13	4	1	0
C58	$d_6^2 d_4^3 f_2 f_6$	1	12	$2^{14}3^3$	9	18	4	1	0
C59	$d_6^2 d_4^2 f_{12}$	12	4	$2^{14}3^3$	8	14	3	1	0
C60	$d_6^2 g_{20}$	4	2	$2^{15}3^3 5$	6	6	2	1	0
C61*	$d_6 d_4^2 d_4^3 f_3^2$	1	12	$2^{15}3^2$	8	19	6	1	0
C62	$d_6 d_4^4 f_{10}$	2	8	$2^{15}3$	7	21	4	1	0
C63	$d_6 d_4^3 f_{14}$	8	6	$2^{13}3^2$	6	18	4	1	0
C64	$d_6 d_4^2 g_{18}$	36	2	$2^{10}3^4$	5	12	3	1	0
C65*	$d_6 d_4 g_{16} f_6$	72	1	$2^{12}3^3$	4	14	4	1	0
C66*	$d_6 f_{13}^2$	5616	2	$2^8 3^4 13$	3	6	2	1	0
C67	d_4^8	6	1344	$2^{23}3^2 7$	8	2	1	0	0
C68	d_4^8	1	1152	$2^{23}3^2$	8	3	1	0	0
C69	d_4^8	1	336	$2^{20}3.7$	8	2	1	0	0
C70*	$d_4^6 f_8$	4	48	$2^{18}3$	6	7	2	0	0
C71*	$d_4^6 f_8$	1	48	$2^{16}3$	6	9	2	0	0
C72	$d_4^4 d_4 f_{12}$	6	24	$2^{14}3^2$	5	10	2	0	0
C73	$d_4^5 f_{12}$	1	60	$2^{12}3.5$	5	6	1	0	0
C74	$d_4^4 g_{16}$	8	24	$2^{18}3$	4	7	2	0	0
C75*	$d_4^4 f_{16}$	8	8	2^{14}	4	14	2	0	0
C76	$d_4^3 g_{18} f_2$	8	6	$2^{10}3^2$	3	13	2	0	0
C77*	$d_4^2 q_{24}^+$	6	2	$2^{15}3^2$	2	6	1	0	0
C78*	$d_4^2 q_{24}^-$	3	2	$2^{10}3^2$	2	8	1	0	0
C79*	$d_4 f_4^6$	16	72	$2^{11}3^2$	2	8	1	0	0
C80*	$d_4 f_7^4$	168	8	$2^8 3.7$	1	8	2	0	0
C81	q_{32}	1	1	$2^5 3.5.31$	0	1	0	0	0
C82	r_{32}	1	1	$2^{15}3^2 5.7.31$	0	1	0	0	0
C83	g_{16}^2	20160	2	$2^{15}3^2 5.7$	0	2	0	0	0
C84	f_4^8	256	336	$2^{12}3.7$	0	2	0	0	0
C85	f_2^{16}	2	11520	$2^9 3^2 5$	0	3	0	0	0

evenly. In [2] only the second of these was described and there it was called q_{24}, while q_{24}^{+} was called $(g_{16}+f_8)$. (The exceptional treatment of the components of this code given on p. 52 of [2] is now eliminated.)

The glue space for either q_{24}^{+} or q_{24}^{-} is four-dimensional and is generated by t, u, v, p, q, r with

$$t+u+v=0=p+q+r,$$

where p, q, r may be represented by special octads, with p orthogonal to t but not to u or v, q orthogonal to u but not to t or v, and r orthogonal to v but not to t or u (see Fig. 1).

The 16 glue components and their minimal weights are:

Representative	Minimal weight
0	0
t, u, v	4
p, q, r	8
$p+u, q+v, r+t,$ $p+v, q+t, r+u$	6
$p+t, q+u, r+v$	8 in q_{24}^{+}, 4 in q_{24}^{-}

The groups. The order $|G|$ of the automorphism group of any of the 85 codes is given (as in [2]) by the formula

$$|G|=|G_0|\ |G_1|\ |G_2|,$$

while $|G_0|$ is the product of the $|G_0|$'s of the component codes, and $|G_1|$, $|G_2|$ are given in the third and fourth columns of Table A. $|G|$ itself is given

TABLE B

The Groups G_0

Component	G_0	$\|G_0\|$
d_{2m}	$2^{m-1}.S_m$	$2^{m-1}m!$
e_7	$L_3(2)$	168
e_8	$2^3.L_3(2)$	1344
f_n	1	1
g_{16}	2^4	16
g_{18}	C_3	3
g_{20}	M_{20}	$2^6 3.5$
g_{21}	M_{21}	$2^6 3^2 5.7$
g_{22}	M_{22}	$2^7 3^2 5.7.11$
g_{24}	M_{24}	$2^{10} 3^3 5.7.11.23$
h_{24}	$2^6{:}3S_6$	$2^9 3^3 5$
q_{24}^{+}	$2^6.(S_3\times 2^2)$	$2^9 3$
q_{24}^{-}	$2^2\times S_4$	$2^5 3$

in the fifth column. Table B gives the groups G_0 (in ATLAS [1] notation) for the components.

The mass checks. We repeated various "mass checks" on the 85 codes, verifying that the number of codes containing a specified subcode (e.g., d_4 or d_6) is as predicted by the formula given on p. 28 of [2]. In particular, we rechecked that the total mass

$$\sum_{(85)} \frac{1}{|G|}$$

of the reciprocals of the numbers in the fifth column of Table A has the correct value

$$\frac{1}{32!} \prod_{i=0}^{14} (2^i + 1),$$

TABLE C

Other Alterations to Table III of [2]

Code	Components	Change
C22, C39, C53, C65, C70, C71		f_m^2 has been replaced by f_{2m}.
C10	$d_{12}^2 e_8$	$\|G_2\| = 2$, not 1.
C41	$d_8 d_4^4 f_8$	Change *boczxAB* to *boozxAB*.
C61	$d_6 d_4^2 d_4^3 f_3^2$	The final - was omitted.
C66	$d_6 f_{13}^2$	The last character should be - not —.
C71	$d_4^6 f_4^2$	The glue generators should be *ooyoxxAE, yooxoxBF, oyoxxoCD, oxyooxCG, yoxxooAG, xyooxoBG, ooxoxyFH, xooyoxDH, oxoxyoEH, oxxooyBD, xoxyooCE, xxooyoAF, ozxozx-, xozxoz-, xzoxzo-*.
C75	$d_4^4 f_{16}$	The printing of the glue is poorly aligned. Each glue word consists of a top line of 4 letters (chosen from o, x, y, z), with a 4×4 array beneath it.
C77	$d_4^2 q_{24}^+$	Redescribed above.
C78	$d_4^2 q_{24}^-$	Redescribed above.
C79	$d_4^2 f_4^6$	The parentheses around the final array indicate that its six columns are to be bodily permuted.
C80	$d_4 f_7^4$	The third and fourth glue generators should be $y(+++o+oo)(ooooooo)$ $(+oooooo)(+oooooo)$, $y(ooooooo)$ $(+++o+oo)(+oooooo)(+oooooo)$.

which is

$$\frac{391266122896364123}{5322830354237620224 00}.$$

Weight distributions. The sixth column gives u_4, the number of codewords of weight 4. The full weight distribution can then be obtained from Table IV of [2].

Table C summarizes the amendments to Table III of [2] other than errors in the value of n_{30}.

Additional Errata to [2]

On p. 37 of [2] the phrase "Figure (MOG)" refers to Ref. [5] (or Fig. 11.17 of [3]).

On p. 44, for the first code in Table I, change e_0 to e_8.

On pp. 46 and 48, the heading should read $n_{30}, n_{28}, n_{26}, n_{24}, \ldots$.

On p. 52, the last entry in Table V should be 731, not 664.

On p. 53, the last author's name is misspelled in Ref. [5].

2. Self-Dual Codes of Length Less than 32

The numbers of children. We now describe how the final four columns of Table A were obtained. These give $n_{30}, n_{28}, n_{26}, n_{24}$, the number of self-dual codes (the "children") of lengths 30, ..., 24 that arise from each of the 85 codes.

Any self-dual code C of length 30 is obtained by taking all codewords of one of C1, ..., C85 for which some particular pair of coordinates P, Q (say) are 00 or 11, and deleting these coordinates. If C contains a weight-2 word, which it does when P and Q belong to a weight-4 word of the original code, we obtain a self-dual code with length less than 30 and $d \geqslant 4$ by "collapsing" C, i.e., by deleting all pairs of coordinates that support weight-2 words. All self-dual codes with length $n \leqslant 30$ and $d \geqslant 4$ can be obtained in this way. There are several cases:

If the coordinates P, Q are in an e_8, the collapsed code is a doubly-even self-dual code of length 24, whose components are obtained by deleting the e_8.

If P, Q are in an e_7, the collapsed code has length 26 and its components are obtained by replacing the e_7 by an f_1.

If P, Q form a duad (a pair of identical coordinates, cf. [2], p. 33) in a d_m, $m \geqslant 6$, the collapsed code has length $32-m$ and its components are obtained by deleting the d_m.

TABLE D

Self-Dual Codes with $n \leqslant 22$, $d \geqslant 4$

n	Code	Components	[9, 11]	$\lvert G_1 \rvert$	$\lvert G_2 \rvert$	u_4	u_6	u_8	u_{10}	u_{12}	Generators for glue
0	C1(d_{32})	i_0	—	1	1						—
8	C2(d_{24})	e_8	A_8	1	1	14	0	1			—
12	C3(d_{20})	d_{12}	B_{12}	1	1	15	32	15	0	1	*a*
14	C4(d_{18})	e_7^2	D_{14}	1	2	14	49	49	14	0	*dd*
16	C5(d_{16})	d_{16}	E_{16}	1	1	28	0	198	0	28	*a*
	C6(d_{16})	e_8^2	$A_8 \oplus A_8$	1	2	28	0	198	0	28	—
	C7(d_{16})	d_8^2	F_{16}	1	2	12	64	102	64	12	*(ab)*
18	C8(d_{14})	$d_{10}e_7f_1$	I_{18}	1	1	17	51	187	187	51	*aoA, cd-*
	C9(d_{14})	d_6^3	H_{18}	1	6	9	75	171	171	75	*(abc), bbb*
20	C3(d_{12})	d_{20}	J_{20}	1	1	45	0	210	512	210	*a*
	C10(d_{12})	$d_{12}e_8$	$A_8 \oplus B_{12}$	1	1	29	32	226	448	226	*a-*
	C11(d_{12})	$d_{12}d_8$	K_{20}	1	1	21	48	234	416	234	*(ab)*
	C12(d_{12})	$d_8^2d_4$	S_{20}	1	2	13	64	242	384	242	*(ab)x, bby*
	C13(d_{12})	$e_7^2d_6$	L_{20}	1	2	17	56	238	400	238	*doa, ddb*
	C14(d_{12})	$d_6^3f_2$	R_{20}	1	6	9	72	246	368	246	*aaaA, cccB, (abc)-*
	C15(d_{12})	d_4^5	M_{20}	1	120	5	80	250	352	250	*(ooxyx)*
22	C8(d_{10})	$d_{14}e_7f_1$	N_{22}	1	1	28	49	246	700	700	*aoA, bdA*
	C16(d_{10})	$d_{10}^2f_2$	P_{22}	1	2	20	57	270	676	676	*(ao)*, cc-*
	C17(d_{10})	$d_{10}d_6^2$	Q_{22}	1	2	16	61	282	664	664	*aoc, oaa, bbb*
	C18(d_{10})	$e_8e_7^2$	$E_8 \oplus D_{14}$	1	2	28	49	246	700	700	*-dd*
	C19(d_{10})	$d_8e_7d_6f_1$	R_{22}	1	1	16	61	282	664	664	*odbA, boaA, aob-*
	C20(d_{10})	$d_8d_6^2f_2$	S_{22}	1	2	12	65	294	652	652	*baoA, aooAB, abb-, occ-*
	C21(d_{10})	$d_6^2d_4^2f_2$	T_{22}	1	4	8	69	306	640	640	*aoxoA, ooyyAB, aayo-, bozx-, obxz-*
	C22(d_{10})	$d_4^4f_6$	U_{22}	6	24	4	73	318	628	628	*oxyzBC, ozxyAC, ooxxAE, oyoyAD, ozzoAF, xxxx-, yyyy-*
	C23(d_{10})	g_{22}	G_{22}	2	1	0	77	330	616	616	the all-ones vector

TABLE E

Self-Dual Codes with Length 24 and $d \geqslant 4$

Code	Components	[11]	d	Code	Components	[11]	d
C2(e_8)	d_{24}	E_{24}	4	C32(d_8)	$d_8 e_7^2 f_2$	J_{24}	4
C6(e_8)	$d_{16} e_8$	—	4	C33(d_8)	$d_8 d_6^2 f_4$	R_{24}	4
C7(d_8)	$d_{16} d_8$	H_{24}	4	C34(d_8)	$d_8 d_4^4$	T_{24}	4
C10(e_8)	d_{12}^2	A_{24}	4	C35(d_8)	$e_7 d_6^2 d_4 f_1$	P_{24}	4
C11(d_8)	d_{12}^2	—	4	C26(e_8)	d_6^4	D_{24}	4
C12(d_8)	$d_{12} d_8 d_4$	I_{24}	4	C36(d_8)	d_6^4	Q_{24}	4
C18(e_8)	$d_{10} e_7^2$	B_{24}	4	C37(d_8)	$d_6^2 d_6 d_4 f_2$	S_{24}	4
C19(d_8)	$d_{10} e_7 d_6 f_1$	K_{24}	4	C38(d_8)	$d_6^2 d_4^2 f_4$	U_{24}	4
C20(d_8)	$d_{10} d_6^2 f_2$	N_{24}	4	C39(d_8)	$d_6 d_4^3 f_6$	W_{24}	4
C24(e_8)	e_8^3	—	4	C27(e_8)	d_4^6	F_{24}	4
C25(d_8)	$e_8 d_8^2$	—	4	C40(d_8)	d_4^6	V_{24}	4
C25(e_8)	d_8^3	C_{24}	4	C41(d_8)	$d_4^4 f_8$	X_{24}	4
C29(d_8)	d_8^3	L_{24}	4	C42(d_8)	$d_4^2 g_{16}$	Y_{24}	4
C30(d_8)	d_8^3	M_{24}	4	C43(d_8)	h_{24}	Z_{24}	6
C31(d_8)	$d_8^2 d_4^2$	O_{24}	4	C28(e_8)	g_{24}	G_{24}	8

If P, Q are in a d_m, $m \geqslant 6$, but do not form a duad, or if P, Q belong to a d_4 component, the collapsed code has length 28.

In all other cases C does not collapse and we have a self-dual code with length 30 and $d \geqslant 4$.

As an example we consider the code C61. There are 19 orbits of Aut(C61) on unordered pairs of distinct coordinates, so there are $n_{30} = 19$ length 30 children, 6 of which collapse to shorter lengths, as follows. One length 26 child is obtained from a duad in the d_6 (thus $n_{26} = 1$).

From two coordinates in the d_6 not forming a duad we obtain the first child of length 28, while a second child of length 28 is obtained from any

TABLE F

Weight Distributions of Self-Dual Codes with $n = 26$, 28, 30 and $d \geqslant 6$

	u_6	u_8	u_{10}	u_{12}	u_{14}
A_{26}	52	390	1313	2340	2340
A_{28}	26	442	1560	3653	5020
B_{28}, C_{28}	42	378	1624	3717	4680
A_{30}, B_{30}, C_{30}	19	393	1848	5192	8931
D_{30}	27	369	1848	5256	8883
E_{30}, ..., M_{30}	35	345	1848	5320	8835

```
C66
11110000000000000000000000000000
00111100000000000000000000000000
01010111010000010001000000000000
01010101101000001000100000000000
01010100110100000100010000000000
01010100011010000010001000000000
01010110001101000000000100000000
01010101000110100000000010000000
01010100100011010000000001000000
01010100010001101000000000100000
01010100001000110100000000010000
01010100000100011010000000001000
01010110000010001100000000000100
01010101000001000110000000000010
01010110100000100010000000000001
01011011111111111110000000000000
```

A_{26} ααββγγ

A_{28} αβαβ

```
C78
11110000000000000000000000000000
00001111000000000000000000000000
00000000101111010110011001010000
00000000100111101011001100101000
00000000100011110101100110010100
00000000100001111010110011001010
00000000110000011110101100110010
00000000010100111010110010101001
00000000111001010000011110101100
00000000001101001110101100101010
00000000111101100010110010001100
00000000111111111111111111111111
01010110100000110010000011100001
01100011010010101100100000001001
01100101000000000001111000000000
00110110000000000000011110000000
```

B_{30} α   α

```
C77
11110000000000000000000000000000
00001111000000000000000000000000
00000000111111110000000000000000
00000000000000001111111100000000
00000000000000000000000011111111
00000000010101010101010100000000
00000000001100110011001100000000
00000000000011110000111100000000
00000101000001010000010100000101
00000011000000110000001100000011
01010000010100000101000001010000
00110000001100000011000000110000
00000000011110000100010001000100
00000000011110000010001000100010
00000000011110000001000100010001
00000000000000000000111100001111
```

A_{30} α   α

```
C79
11110000000000000000000000000000
00001111000000000000000000000000
00110011001100000000001100000000
00110101000000110000010100000000
00110110000000000011011000000000
01010011010100000000000000110000
01010101000001010000000001010000
01010110000000000101000001100000
01100011011000000000000000000011
01100101000001100000000000000101
00000000011110001000100010001000
00000000100001111000100010001000
00000000100010000111100010001000
00000000100010001000011110001000
00000000100010001000100001111000
00000000100010001000100010000111
```

C_{30} α   α

FIG. 2. Generator matrices for C66, C77, ..., C85 and their children of length 26, 28, 30 and $d = 6$.

two coordinates in the first type of d_4, and three further children of length 28 arise from the three different ways of choosing a pair of coordinates from the other type of d_4. Thus $n_{28} = n_{26} + 5 = 6$. Finally, there are 13 children with length 30 and $d \geqslant 4$, so that $n_{30} = n_{28} + 13 = 19$.

In [2, 10] the actions of the automorphism groups of the 85 codes on pairs of coordinates were found (unfortunately often incorrectly) by hand. In the present version most of this work has been redone by computer (using in particular the graph-automorphism program Nauty [8]). The numbers of children of lengths $n \leqslant 28$ given in [2] are correct. There are numerous errors in n_{30}, however (now corrected in Table A), and the total number of self-dual codes of length 30 is 731, not 664 as stated in Table V of [2].

```
                 C80
     11110000000000000000000000000000
     01011000000100000010010110000000
     01010100000010000011001010000000
     01010010000001000011100100000000
     01010001000000100001110010000000
     01010000100000010010111000000000
     01010000010000001001011100000000
     01010000001000000100101110000000
     01011110100000000010000001000000
     01010111010000000001000000100000
     01010011101000000000100000010000
     01011001110000000000010000001000
     01010100111000000000001000000100
     01011010011000000000000100000010
     01011101001000000000000010000001
     00111111111000000011111110000000
B28  ααββ
C28  αβαβ
D30  α   α
```

```
                 C82
     10011100100000100000011000000000
     10001110010000010000001100000000
     10000111001000001000000110000000
     10000011100100000100000011000000
     10000001110010000010000001100000
     10000000111001000001000000110000
     10000000011100100000100000011000
     10000000001110010000010000001100
     10000000000111001000001000000110
     10000000000011100100000100000011
     11000000000001110010000010000001
     11100000000000111001000001000000
     10110000000000011100100000100000
     10011000000000001110010000010000
     10001100000000000111001000001000
     10000110000000000011100100000100
F30  αα
```

```
                 C81
     10110110111100010101110000100100
     10011011011110001010111000010010
     10001101101111000101011100001001
     11000110110111100010101110000100
     10100011011011110001010111000010
     10010001101101111000101011100001
     11001000110110111100010101110000
     10100100011011011110001010111000
     10010010001101101111000101011100
     10001001000110110111100010101110
     10000100100011011011110001010111
     11000010010001101101111000101011
     11100001001000110110111100010101
     11110000100100011011011110001010
     10111000010010001101101111000101
     11011100001001000110110111100010
E30  αα
```

```
                 C83
     11101000111010000000000000000000
     10110100101101000000000000000000
     10011010100110100000000000000000
     10001101100011010000000000000000
     00000000111010001110100011101000
     00000000101101001011010010110100
     00000000100110101001101010011010
     00000000100011011000110110001101
     11011000110110001101100000000000
     10101100101011001010110000000000
     10010110100101101001011000000000
     10001011100010111000101100000000
     00000000000000001101100011011000
     00000000000000001010110010101100
     00000000000000001001011010010110
     00000000000000001000101110001011
G30  αα
H30  α               α
```

FIGURE 2 — *continued*

Tables of Self-Dual Codes of Length $n \leqslant 24$

The self-dual codes of length $n \leqslant 20$ were first enumerated in [9], and those of lengths 22 and 24 in [11], although there they are not described in the terminology later used in [2]. For completeness we therefore give the components, values of $|G_1|$, $|G_2|$, weight distributions $\{u_i\}$, and glue generators for the codes with $n \leqslant 22$ in Table D. The column headed "Code" gives the parent code of length 32 (with the component to be deleted in parentheses). The column headed "[9, 11]" gives the names used in these papers. ([9, 11] also give generator matrices for these codes.)

Table E lists the codes of length 24, although to save space we just

C84

```
      11101000000000001110100011101000
      10110100000000001011010010110100
      10011010000000001001101010011010
      10001101000000001000110110001101
      00000000111010001110100010110100
      00000000101101001011010010011010
      00000000100110101001101010001101
      00000000100011011000110111000110
      11011000110110001101100000000000
      10101100101011001010110000000000
      10010110100101101001011000000000
      10001011100010111000101100000000
      11011000101100010000000011011000
      10101100110110000000000010101100
      10010110101011000000000010010110
      10001011100101100000000010001011
I30   αα
J30   α       α
```

C85

```
      10000000000000001111100010001000
      01000000000000001111010001000100
      00100000000000001111001000100010
      00010000000000001111000100010001
      00001000000000001000111110001000
      00000100000000000100111101000100
      00000010000000000010111100100010
      00000001000000000001111100010001
      00000000100000001000100011111000
      00000000010000000100010011110100
      00000000001000000010001011110010
      00000000000100000001000111110001
      00000000000010001000100010001111
      00000000000001000100010001001111
      00000000000000100010001000101111
      00000000000000010001000100011111
K30   αα
L30   α   α
M30   α               α
```

FIGURE 2 — *continued*

describe each code by giving its parent (and in parentheses the component to be deleted), its components, and the minimal distance d. If the deleted component (in parentheses) is an e_8, the code is doubly-even; otherwise (when the deleted component is a d_8) it is singly-even.

Codes with $d \geqslant 6$. Similar tables could easily be constructed to list the self-dual codes of lengths 26, 28, and 30, but they would occupy too much space. Instead we just describe the 17 codes with $d \geqslant 6$ (and correct some errors in [10]).

There is one code (A_{26}) of length 26, arising from C66; three codes of length 28, one (A_{28}) from C66 and two (B_{28}, C_{28}) from C80; and 13 codes

(A_{30}, ..., M_{30}) of length 30, one from each of C77, ..., C82, two from each of C83, C84, and three from C85.

Figure 2 contains generator matrices for C66, C77, ..., C85. Each row of Greek letters below one of these matrices specifies a self-dual child with $d \geqslant 6$. The child is obtained by restricting to the subcode consisting of the words that are equal on pairs of coordinates described by the same Greek letter, and then deleting these coordinates. (For C83 and C84 we have used the generator matrices described in [6, 12].) The weight distributions are given in Table F (compare [4]).

References

1. J. H. Conway, R. T. Curtis, S. P. Norton, R. A. Parker, and R. A. Wilson, "ATLAS of Finite Groups," Oxford Univ. Press, London, 1985.
2. J. H. Conway and V. Pless, On the enumeration of self-dual codes, *J. Combin. Theory Ser. A* **28** (1980), 26–53.
3. J. H. Conway and N. J. A. Sloane, "Sphere Packings, Lattices and Groups," Springer-Verlag, New York, 1988.
4. J. H. Conway and N. J. A. Sloane, A new upper bound on the minimal distance of self-dual codes, *IEEE Trans. Inform. Theory* **36** (1990), 1319–1333.
5. R. T. Curtis, A new combinatorial approach to M_{24}, *Math. Proc. Cambridge Philos. Soc.* **79** (1976), 25–42.
6. H. V. Koch, "On Self-Dual, Doubly-Even Codes of Length 32," Report P-Math-32/84, Institut für Mathematik, Akademie der Wissenschaften der DDR, Berlin, 1984.
7. F. J. MacWilliams and N. J. A. Sloane, "The Theory of Error-Correcting Codes," North-Holland, Amsterdam, 1977.
8. B. D. McKay, "NAUTY User's Guide (Version 1.2)," Dept. Computer Science Technical Report TR-CS-87-03, Australian National University, Canberra, Australia, 1987.
9. V. Pless, A classification of self-orthogonal codes over *GF*(2), *Discrete Math.* **3** (1972), 209–246.
10. V. Pless, The children of the (32, 16) doubly even codes, *IEEE Trans. Inform. Theory* **24** (1978), 738–746.
11. V. Pless and N. J. A. Sloane, On the classification and enumeration of self-dual codes, *J. Combin. Theory Ser. A* **18** (1975), 313–335.
12. V. Y. Yorgov, On the extremal doubly-even codes of length 32, "Proceedings, Fourth Joint Swedish–Soviet International Workshop on Information Theory, Gotland, Sweden, 1989," pp. 275–279.