# A LATTICE WITHOUT A BASIS OF MINIMAL VECTORS

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*Abstract.* It is shown that in all dimensions $n \geqslant 11$ there exists a lattice which is generated by its minimal vectors but in which no set of $n$ minimal vectors forms a basis.

It was shown by Ryskov [5] (see also Csóka [2] and Martinet [4, Theorem 5.4]) that in dimensions $n \leqslant 7$ every perfect lattice contains a basis of minimal vectors. At a conference in Luminy in 1992, Louis Michel asked if any lattice was known with the property that it is generated by its minimal vectors, but no subset of the minimal vectors forms a basis. In this note we exhibit an 11-dimensional lattice with this property. From this it is trivial to construct such lattices in all higher dimensions. We note that (in spite of suggestions to the contrary in [3] and [6]), it appears that the related question of whether every lattice has a basis of Voronoi (or facet) vectors is still unsolved.

We follow the notation of [1]. The norm of a vector $x$ is its squared length $x \, . \, x$.

THEOREM 1. *The* 11*-dimensional lattice* $\Lambda$ *with Gram matrix*

$$\begin{bmatrix} 60 & 5 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 60 & 5 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 60 & 5 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 60 & 5 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 5 & 60 & 5 & -12 & -12 & -12 & -12 & -7 \\ 5 & 5 & 5 & 5 & 5 & 60 & -12 & -12 & -12 & -12 & -7 \\ -12 & -12 & -12 & -12 & -12 & -12 & 60 & -1 & -1 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & 60 & -1 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & 60 & -1 & -13 \\ -12 & -12 & -12 & -12 & -12 & -12 & -1 & -1 & -1 & 60 & -13 \\ -7 & -7 & -7 & -7 & -7 & -7 & -13 & -13 & -13 & -13 & 96 \end{bmatrix} \tag{1}$$

*has minimal norm* 60, *is generated by its* 24 *minimal vectors, but no set of* 11 *minimal vectors forms a basis.*

*Proof.* Let the generating vectors corresponding to (1) be denoted by $u_1, \ldots, u_6, v_1, \ldots, v_4, w$, respectively, and let

$$x = -2(u_1 + \ldots + u_6) - 3(v_1 + \ldots + v_4).$$

Then $\Lambda$ also contains $u_7 = 3w - x$ and $v_5 = x - 2w$. It is straightforward to check that $u_1, \ldots, u_7, v_1, \ldots, v_5$ have Gram matrix

$$\begin{array}{c} u_i \\ \\ v_i \end{array} \left[\begin{array}{ccc|ccc} 60 & & 5 & & & \\ & \cdots & & & -12 & \\ 5 & & 60 & & & \\ \hline & & & 60 & & -1 \\ & -12 & & & \cdots & \\ & & & -1 & & 60 \end{array}\right],$$

and satisfy

$$2(u_1 + \ldots + u_7) + 3(v_1 + \ldots + v_5) = 0. \tag{2}$$

Thus $\Lambda$ contains at least $2 \times 12$ vectors of norm 60. We will show below that there are no nonzero vectors in $\Lambda$ of norm less than 60, and no other vectors of norm 60.

Since $w = u_7 + v_5$, the lattice is generated by the 12 vectors $u_1, \ldots u_7, v_1, \ldots, v_5$ (subject to (2)). However, no subset of 11 of these vectors generates $\Lambda$. In fact, omitting one of the $u_i$'s generates a sublattice of index 2, and omitting one of the $v_i$'s generates a sublattice of index 3.

To show that there are no other nonzero vectors of norm $\leqslant 60$ we proceed as follows. We choose 13-dimensional coordinates, setting

$$u_1 = \left(\frac{3\alpha}{4}, \frac{3\beta}{4}, \frac{-\beta}{4}, \frac{-\beta}{4}, \frac{-\beta}{4}, \frac{-\beta}{4}, \frac{-\beta}{4}, \frac{-\beta}{4}; 0, 0, 0, 0, 0\right),$$

$$v_1 = \left(\frac{-7\alpha}{10}, \frac{\beta}{10}, \frac{\beta}{10}, \frac{\beta}{10}, \frac{\beta}{10}, \frac{\beta}{10}, \frac{\beta}{10}, \frac{\beta}{10}; \frac{4\gamma}{5}, \frac{-\gamma}{5}, \frac{-\gamma}{5}, \frac{-\gamma}{5}, \frac{-\gamma}{5}\right),$$

where $\alpha = \sqrt{15}$, $\beta = \sqrt{55}$ and $\gamma = \sqrt{61}$. Then $u_2, \ldots u_7$ are obtained from $u_1$ by cyclic shifts of coordinates 2 through 8, and $v_2, \ldots, v_5$ from $v_1$ by cyclic shifts of the last five coordinates. Note that, in these coordinates,

$$v_1 + \ldots + v_5 = -\left(\frac{7\alpha}{2}, \left(\frac{-\beta}{2}\right)^7; 0^5\right).$$

Any linear combination of $v_1, \ldots, v_5$ that vanishes on the last five coordinates is a multiple of this vector.

Write a typical lattice vector as

$$y = (\alpha y_1, \beta y_2, \ldots, \beta y_8; \gamma y_9, \ldots, \gamma y_{13}).$$

Then $(y_9, \ldots, y_{13})$ belongs to the lattice $A_4^*$ (in the notation of [1]), and can have norm at most 60/61 if $N(y) \leqslant 60$. The only sufficiently short vectors of $A_4^*$ are 0 and the vectors $\pm((4/5)^1(-1/5)^4)$ of norm 4/5, so we must have $y = z$ or $y = z \pm v_i$ $(1 \leqslant i \leqslant 5)$, where $z$ is an integral linear combination of $u_1, \ldots, u_7, v_1 + \ldots + v_5$. Therefore $z = (\alpha e_1, \beta e_2, \ldots, \beta e_8)$, where $(e_1, \ldots, e_8)$ is in the lattice $E_7^*$. The coordinates $e_i$ are rational numbers with denominators 1, 2 or 4. If $z \neq 0$, examination of vectors in $E_7^*$ shows that the numbers

$|e_1|, \ldots, |e_8|$ are bounded below by some rearrangement of one of

$$\text{(a)} \quad \left(\frac{1}{2}\right)^8,$$

$$\text{(b)} \quad 1^2 0^6,$$

$$\text{(c)} \quad \left(\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^6,$$

$$\text{(d)} \quad \left(\frac{7}{4}\right)^1 \left(\frac{1}{4}\right)^7.$$

These conditions are enough to ensure that if $y=z$ then

$$15y_1^2 + 55(y_2^2 + \ldots + y_8^2) \geqslant 60,$$

with equality only for the vectors $\pm u_i$.

If $y = z \pm v_i$ then we need

$$15y_1^2 + 55(y_2^2 + \ldots + y_8^2) \leqslant 60 - \tfrac{4}{5}61 = 11{\cdot}2. \qquad (*)$$

But in case (a) all of $|y_2|, \ldots, |y_8|$ are $\geqslant 0{\cdot}4$, and in cases (b), (c), (d) either one of $|y_2|, \ldots, |y_8|$ is $\geqslant 0{\cdot}65$ or $|y_1| \geqslant 1{\cdot}65$, and in each case $(*)$ is violated. Therefore $z = 0$ and $y = \pm v_i$.

*Remark.* Our lattice has determinant $2 \times 55^6 \times 61^4$ and automorphism group of order $7! \times 5! \times 2$.

## *References*


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