SOME DOUBLY EXPONENTIAL SEQUENCES

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1. INTRODUCTION

Let $x_0, x_1, x_2, \cdots$ be a sequence of natural numbers satisfying a nonlinear recurrence of the form $x_{n+1} = x_n^2 + g_n$, where $|g_n| < \frac{1}{4}x_n$ for $n \ge n_0$. Numerous examples of such sequences are given, arising from Boolean functions, graph theory, language theory, automata theory, and number theory. By an elementary method it is shown that the solution is x_n = nearest integer to k^{2^n}, for $n \ge n_0$, where k is a constant. That is, these are doubly exponential sequences. In some cases k is a "known" constant (such as $\frac{1}{2}(1+\sqrt{5})$), but in general the formula for k involves $x_0, x_1, x_2, \cdots$!

2. EXAMPLES OF DOUBLY EXPONENTIAL SEQUENCES

2.1 BOOLEAN FUNCTIONS

The simplest example is defined by

(1) $$x_{n+1} = x_n^2, \quad n \ge 0; \quad x_0 = 2$$

so that the sequence is 2, 4, 16, 256, 65536, 4294967296, $\cdots$ and $x_n = 2^{2^n}$. This is the number of Boolean functions of n variables ([12], p. 47) or equivalently the number of ways of coloring the vertices of an n-dimensional cube with two colors.

2.2 ENUMERATING PLANAR TREES BY HEIGHT

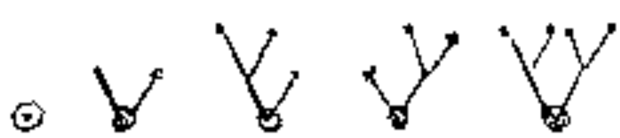
The recurrence

(2) $$x_{n+1} = x_n^2 + 1, \quad n \ge 0; \quad x_0 = 1$$

generates the sequence 1, 2, 5, 26, 677, 458330, 210066388901, $\cdots$. This arises, for example, in the enumeration of planar binary trees.

We assume the reader knows what a rooted tree ([10]) is. (The drawings below are of rooted trees.) A <u>binary rooted tree</u> is a rooted tree in which the root node has degree 2 and all other nodes have degree 1 or 3 (or else is the trivial tree consisting of the root node alone). A <u>planar binary rooted tree</u> is a particular embedding of a binary rooted tree in the plane.

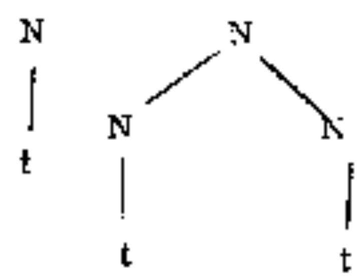
The <u>height</u> of a rooted tree is the maximum length of a path from any node to the root. For example here are the planar binary rooted trees of heights 0, 1 and 2. (Here the root is drawn at the bottom.)



Let x_n be the number of planar binary rooted trees of height at most n, so that $x_0 = 1$, $x_1 = 2$, $x_2 = 5$. Deleting the root node either leaves the empty tree or two trees of height at most $n - 1$, from which it follows that x_n satisfies (2).

Planar binary rooted trees arise in a variety of splitting processes. We give three illustrations.

a. In parsing certain context-free languages [1], [13], [18]. For example, consider a context-free grammar G with two productions $N \to NN$ and $N \to t$ where N is a nonterminal and t a terminal symbol. Derivation trees for the sentences t and tt are shown below.* Deleting the terminal symbols



and their adjacent edges converts a derivation tree into a planar binary rooted tree. Thus x_n represents the number of derivation trees for G of height at most $n + 1$.

b. Using the natural correspondence ([4], Vol. 1, p. 65) between planar binary rooted trees and the parenthesizing of a sentence, x_n is the number of ways of parenthesizing a string of symbols of any length so that the parentheses are nested to depth at most n.

c. If, in a planar binary rooted tree, we write a 0 when the path branches to the left and a 1 when the path branches to the right, the set of all paths from the root to the nodes of degree 1 forms a variable length binary code ([7]). Thus x_n is the number of variable length binary codes of maximum length at most n.

2.3 THE RECURRENCE

$$x_{n+1} = x_n^2 - 1, \quad n \geq 0; \quad x_0 = 2 \tag{3}$$

generates the sequence 2, 3, 8, 63, 3968, 15745023, 247905749270528, $\cdots$.

2.4 THE RECURRENCE

$$y_{n+1} = y_n^2 - y_n + 1, \quad n \geq 1; \quad y_1 = 2 \tag{4}$$

generates the sequence 2, 3, 7, 43, 1807, 3263443, 10650056950807, $\cdots$. This sequence occurs (a) in Lucas' test for the primality of Mersenne numbers ([11], p. 233) and (b) in approximating numbers by sums of reciprocals. Any positive real number $y < 1$ admits a unique expansion of the form

$$y = \frac{1}{y_1} + \frac{1}{y_2} + \frac{1}{y_3} + \cdots ,$$

*In language theory, it is customary to draw trees with the root at the top.

where the y_i are integers so chosen that after i terms, when the sum s_i has been obtained, y_{i+1} is the least integer such that $s_i + 1/y_{i+1}$ does not exceed y ([16]). It follows that $y_{i+1} = y_i^2 - y_i + \epsilon_i$, $\epsilon_i \geq 1$. The most slowly converging such series is

$$\frac{1}{2} + \frac{1}{3} + \frac{1}{7} + \frac{1}{43} + \cdots$$

when $\epsilon_i = 1$ for $i \geq 1$; this converges to 1, and the denominators satisfy (4). Recurrence (4) is a special case of the next example.

2.5 GOLOMB'S NONLINEAR RECURRENCES

For $r = 1, 2, \cdots$, Golomb [9] has defined a sequence $[y_n^{(r)}]$ by

$$y_{n+1}^{(r)} = y_0^{(r)} y_1^{(r)} \cdots y_n^{(r)} + r, \quad n \geq 0; \quad y_0^{(r)} = 1 \ . \tag{5}$$

Equivalent definitions are

$$y_0^{(r)} = 1, \quad y_1^{(r)} = r + 1$$

$$y_{n+1}^{(r)} = \left(y_n^{(r)}\right)^2 - r y_n^{(r)} + r, \quad n \geq 1 \tag{6}$$

and

$$y_0^{(r)} = 1, \quad y_1^{(r)} = r + 1$$

$$y_{n+1}^{(r)} = \left(y_n^{(r)} - \rho\right)^2 + (2\rho - \rho^2), \quad n \geq 1 \ , \tag{7}$$

where $\rho = \frac{r}{2}$.

From (6) $[y_n^{(1)}]$ is the sequence of example 2.4. The Fermat numbers are $y_n^{(2)}$. The sequences $[y_n^{(2)}] - [y_n^{(5)}]$ begin:

1, 3, 5, 17, 257, 65537, 4294967297, $\cdots$

1, 4, 7, 31, 871, 756031, 571580604871, $\cdots$

1, 5, 9, 49, 2209, 4870849, 23725150497409, $\cdots$

1, 6, 11, 71, 4691, 21982031, 483209576974811, $\cdots$.

(Note that the value of $y_6^{(3)}$ given in [9] is incorrect.)

The substitution $x_n = y_n^{(r)} - \rho$, $n \geq 1$, converts (7) to

$$x_{n+1} = x_n^2 + \rho(1 - \rho), \quad n \geq 0; \quad x_0 = (1 + \rho^2)^{\frac{1}{2}} \tag{8}$$

2.6 THE RECURRENCE

$$y_0 = 1, \quad y_1 = 2 \ ,$$

$$y_{n+1} = 2y_n(y_n - 1), \quad n \geq 1 \tag{9}$$

generates the sequence 1, 2, 4, 24, 1104, 2435424, 11862575248704, $\cdots$, which also arises in approximating numbers by sums of reciprocals [16]. The substitution $x_n = 2y_n - 1$, $n \geq 1$, converts (9) to

$$x_0 = \sqrt{5},$$

(10) $$x_{n+1} = x_n^2 - 2, \qquad n \geq 0 .$$

Sequences generated by (10) with different initial values are also used in primality testing. With the initial value $x_0 = 3$ we obtain the sequence 3, 7, 47, 2207, 4870847, 23725150497407, $\cdots$ ([17], p. 280), and with $x_0 = 4$ the sequence 4, 14, 194, 37634, 1416317954, $\cdots$ ([19]).

2.7 THE RECURRENCE

$$y_0 = 1, \qquad y_1 = 2$$

(11) $$y_{n+1} = y_n^2 - y_{n-1}^2, \qquad n \geq 1$$

generates the sequence 1, 2, 3, 5, 16, 231, 53105, 2820087664, $\cdots$. In [3] it was given as a puzzle to guess the recurrence satisfied by this sequence.

The substitution $x_n = y_n - \frac{1}{2}$, $n \geq 0$, converts (11) to

$$x_0 = \tfrac{1}{2}, \qquad x_1 = 1\tfrac{1}{2}, \qquad x_2 = 2\tfrac{1}{2}$$

(12) $$x_{n+1} = x_n^2 - x_{n-2}^2 - x_{n-2} - 1, \qquad n \geq 2 .$$

3. SOLVING THE RECURRENCES

Recurrences (1)–(3), (8), (10) and (12) all have the form

(13) $$x_{n+1} = x_n^2 + g_n, \qquad n \geq 0$$

with boundary conditions, and are such that

(i) $x_n > 0$

(ii) $|g_n| < \frac{1}{4} x_n$ and $1 \leq x_n$ for $n \geq n_0$ and

(iii) g_n satisfies condition (16) below.

Let

$$y_n = \log x_n, \qquad \alpha_n = \log\left(1 + \frac{g_n}{x_n^2}\right) .$$

Then by taking logarithms of (13) we obtain

(14) $$y_{n+1} = 2y_n + \alpha_n, \qquad n \geq 0 .$$

For any sequence $\{\alpha_n\}$, the solution of (14) is (see for example [15], p. 26)

$$y_n = 2^n\left(y_0 + \frac{\alpha_0}{2} + \frac{\alpha_1}{2^2} + \cdots + \frac{\alpha_{n-1}}{2^n}\right)$$

$$= Y_n - r_n \quad ,$$

where

$$Y_n = 2^n y_0 + \sum_{i=0}^{\infty} 2^{n-1-i}\alpha_i$$

(15)

$$r_n = \sum_{i=n}^{\infty} 2^{n-1-i}\alpha_i \quad .$$

Assuming that the g_n are such that

(16) $$|\alpha_n| \geq |\alpha_{n+1}| \quad \text{for} \quad n \geq n_0 \ ,$$

it follows from (15) that $|r_n| \leq |\alpha_n|$. Then

(17) $$x_n = e^{y_n} = e^{Y_n - r_n} = X_n e^{-r_n} \quad ,$$

where

(18) $$X_n = e^{Y_n} = k^{2^n} \quad ,$$

(19) $$k = x_0 \exp\left(\sum_{i=0}^{\infty} 2^{-i-1}\alpha_i\right) \quad .$$

Also

$$X_n = x_n e^{r_n} \leq x_n e^{|\alpha_n|}$$

$$\leq x_n\left(1 + \frac{2|g_n|}{x_n^2}\right) \quad \text{for} \quad n \geq n_0 \ ,$$

using (ii), and the fact that $(1 - u)^{-1} \leq 1 + 2u$ for $0 \leq u \leq \frac{1}{2}$,

$$= x_n + \frac{2|g_n|}{x_n}$$

and

$$X_n \geq x_n e^{-|\alpha_n|} \geq x_n\left(1 - \frac{|g_n|}{x_n^2}\right) = x_n - \frac{|g_n|}{x_n} \quad .$$

From assumption (ii), this means that

$$|x_n - X_n| < \tfrac{1}{4} \quad \text{for } n \geq n_0 .$$

If x_n is an integer, as in recurrences (1)-(3), (8) for r even, and (10), then the solution to the recurrence (13) is

(20) $$x_n = \text{nearest integer to } k^{2^n}, \qquad \text{for } n \geq n_0$$

while if x_n is half an odd integer, as in (8) for r odd and (12), the solution is

(21) $$x_n = (\text{nearest integer to } k^{2^n} + \tfrac{1}{2}) - \tfrac{1}{2}, \qquad \text{for } n \geq n_0 ,$$

where k is given by (19).

Note that if g_n is always positive, then $\alpha_n > 0$, $r_n > 0$, $X_n > x_n$, and (20) may be replaced by

(22) $$x_n = [k^{2^n}] \qquad \text{for } n \geq n_0 ,$$

where $[a]$ denotes the integer part of a. Similarly if g_n is always negative then $X_n < x_n$ and

(23) $$x_n = \lceil k^{2^n} \rceil \qquad \text{for } n \geq n_0 ,$$

where $\lceil a \rceil$ denotes the smallest integer $\geq a$.

In some cases (see below) k turns out to be a "known" constant (such as $\frac{1}{2}(1 + \sqrt{5})$). But in general Eqs. (20)-(23) are not legitimate solutions to the recurrence (13), since the only way we have to calculate k involves knowing the terms of the sequence. Nevertheless, they accurately describe the asymptotic behavior of the sequence.

We now apply this result to the preceding examples. For all except 2.7 the proofs of properties (ii) and (iii) are by an easy induction, and are omitted.

Example 2.1.

Here $g_n = 0$, $k = 2$ and (20) correctly gives the solution $x_n = 2^{2^n}$.

Example 2.2.

Condition (ii) holds for $n_0 = 2$, and (iii) requires $x_n \leq x_{n+1}$, which is immediate. From (20) $x_n = [k^{2^n}]$ for $n \geq 1$, where

$$k = \exp\left(\frac{1}{2}\log 2 + \frac{1}{4}\log\frac{5}{4} + \frac{1}{8}\log\frac{26}{25} + \frac{1}{16}\log\frac{677}{676} + \cdots\right)$$
$$= 1.502837 \cdots$$

The comparison of k^{2^n} with x_n is as follows:

n	0	1	2	3	4	5
x_n	1	2	5	26	677	458330
k^{2^n}	1.50284	2.25852	5.10091	26.01924	677.00074	458330.00000

Example 2.3 is similar, and $x_n = [k^{2^n}]$ where $k = 1.678459 \cdots$.

Example 2.5.

It is found that (ii) is valid for $n_0 = 1$ if $r = 1$ and for $n_0 = 3$ if $r \geq 3$. The solution of (5) for $r = 1$ (and of example 2.4) is

$$y_n^{(1)} = [k^{2^n} + \tfrac{1}{2}], \quad n \geq 0,$$

and for $r \geq 3$ is

$$y_n^{(r)} = [k^{2^n} + \tfrac{r}{2}], \quad n \geq 3,$$

where k is given by (19). The first few values of k are as follows.

r	1	3	4	5
k	1.264085	1.526526	1.618034	1.696094

For $r = 4$, the value of k is seen to be very close to the "golden ratio"

$$\varphi = \tfrac{1}{2}(1 + \sqrt{5}) = 1.6180339887 \cdots .$$

In fact we may take $k = \varphi$ for

$$y_1^{(4)} = 5,$$
$$y_{n+1}^{(4)} = (y_n^{(4)} - 2)^2, \quad n \geq 1$$

is solved exactly by

$$y_n^{(4)} = \varphi^{2^n} + \varphi^{-2^n} + 2, \quad n \geq 1,$$

and so

$$y_n^{(4)} = [\varphi^{2^n} + 2], \quad n \geq 1.$$

(This was pointed out to us by D. E. Knuth.) So far, none of the other values of k have been identified. Golomb [9] has studied the solution of (5) by a different method.

Example 2.6.

The solution to (9) is

$$y_n = [\tfrac{1}{2}(1 + k^{2^n})] \quad \text{for } n \geq 1,$$

where $k = 1.618034 \cdots$, and again, as pointed out by D. E. Knuth, we may take

$$k = \varphi = \tfrac{1}{2}(1 + \sqrt{5}),$$

since

$$x_n = \varphi^{2^n} + \varphi^{-2^n}, \quad n \geq 0$$

solves (10) exactly. A similar exact solution can be given for (10) for any initial value x_0.

Example 2.7.

This is the only example for which (ii) and (iii) are not immediate. Bounds on x_n and y_n are first established by induction:

$$2^{2^{n-2.1}} \le x_n \le y_n \le 2^{2^{n-2}} \quad \text{for } n \ge 4 ,$$

then

$$g_n = -(x_{n-2} + \tfrac{1}{2})^2 - \frac{3}{4} = -y_{n-2}^2 - \frac{3}{4}$$

and

$$2^{2^{n-3.1}} \le g_n \le 2^{2^{n-3}} \quad \text{for} \quad n \ge 7 .$$

It is now easy to show that (ii) and (iii) hold for $n \ge n_0 = 5$. The solution is

$$y_n = [k^{2^n} + \tfrac{1}{2}] , \quad n \ge 1 ,$$

where $k = 1.185305 \cdots$.

EXERCISES

The technique may sometimes be applied to recurrences not having the form of (13). We invite the reader to tackle the following.

(1) $$y_{n+1} = y_n^2 - 3y_n, \quad n \ge 0; \quad y_0 = 3 ,$$

which generates the sequence 3, 18, 5778, 192900153618, $\cdots$ used in a rapid method of extracting a square root ([5]).

(2) $$y_0 = 1, \quad y_1 = 3$$

$$y_{n+1} = y_n y_{n-1} + 1, \quad n \ge 1 ,$$

which generates the sequence 1, 3, 4, 13, 53, 690, 36571, 25233991, 922832284862, $\cdots$ ([2]).

(3) $$y_0 = 1$$

$$y_{n+1} = y_0 + y_0y_1 + \cdots + y_0y_1 \cdots y_n, \quad n \ge 0$$

which generates the sequence 1, 1, 2, 4, 12, 108, 10476, 108625644, 11798392680793836, $\cdots$.

(4) $$y_0 = 1$$

$$y_{n+1} = y_n^2 + y_n + 1, \quad n \ge 0$$

which generates the sequence 1, 3, 13, 183, 33673, 1133904603, $\cdots$, the coefficients of the least rapidly converging continued cotangent ([14]).

(5) $$y_0 = 1$$
$$y_{n+1} = (y_n + 1)^2, \quad n \ge 0$$

which generates the sequence 1, 4, 25, 676, 456329, 210066388900, ⋯ ([6]).

(6)
$$y_0 = y_1 = 1$$
$$y_{n+1} = y_n^2 + 2y_n(y_0 + y_1 + \cdots + y_{n-1}), \quad n \ge 1.$$

which generates the sequence 1, 1, 3, 21, 651, 457653, 210065930571, ⋯, arising in the enumeration of shapes ([6]).

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