

then the entries are filled in by the rule that each row (after the zeroth) begins with a 0 and every subsequent entry is the sum of the previous entry in the same row and the entry above it in the previous row.

The earliest reference we have seen for this elegant observation is Arnold [Arn91], who refers to (1.2) as the Euler–Bernoulli triangle, but it may well be of much older origin. Dumont [Dum95] refers to (1.2) as the Seidel–Entringer–Arnold triangle, referring to Seidel [Sei77].

The numbers $E_n := E_{n,n}$ appearing at the ends of the rows in (1.2) give the total number of permutations of $\{1, 2, \dots, n\}$ that alternately fall and rise, i.e. the number of “down-up permutations” of n things. The history of these numbers goes back to André [And79], [And81], [Com74], [Sch61]. They have exponential generating function (e.g.f.)

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x. \quad (1.3)$$

Conway and Guy [CG96] call (1.2) the zig-zag triangle and the E_n the zig-zag permutation numbers. The Entringer numbers have also been shown to enumerate several classes of rooted planar trees as well as other mathematical objects [Arn91], [Arn92], [Kem33], [KPP94], [Pou82].

Guy [Guy95] observed that if the entries at the beginnings of the rows in (1.2) are changed from 1, 0, 0, 0, ... to say 1, 1, 1, 1, 1, ..., or 1, 2, 4, 8, 16, ..., etc., then the numbers at the ends of the rows form interesting-looking sequences not to be found in [SP95]. Using 1, 1, 1, ... for example the triangle becomes

$$\begin{array}{cccccc}
 & & & & & 1 \\
 & & & & & 1 & 2 \\
 & & & & 4 & 3 & 1 \\
 & & 1 & 5 & 8 & 9 & \\
 24 & 23 & 18 & 10 & 1 & & \\
 1 & 25 & 48 & \dots & 66 & 76 & 77 \\
 & & & \dots & & &
 \end{array} \quad (1.4)$$

yielding the sequence

$$1, 2, 4, 9, 24, 77, 294, 1309, \dots \quad (1.5)$$

Guy asked if anything could be said about generating functions or combinatorial interpretations for these sequences. The purpose of this note is to answer this question.

2. THE BOUSTROPHEDON TRANSFORM

Given a sequence¹ $\mathbf{a} = (a_1, a_1, a_2, \dots)$ we define its boustrophedon transform to be the sequence $\mathbf{b} = (b_0, b_1, b_2, \dots)$ produced by the triangle

$$\begin{array}{ccccccc}
 & & & & a_0 = b_0 & & \\
 & & & & & & \\
 & & & & a_1 & \rightarrow & b_1 = a_0 + a_1 \\
 & & & & & & \\
 b_2 = a_1 + a_2 + b_1 & \leftarrow & a_2 + b_1 & \leftarrow & & & a_2 \\
 & & & & & & \\
 a_3 & \rightarrow & a_3 + b_2 & \rightarrow & a_2 + a_3 + b_1 + b_2 & \rightarrow & b_3 = 2a_2 + a_3 + b_1 + b_2 \\
 & & & & & & \\
 & & & & & & \dots
 \end{array} \tag{2.1}$$

when it is filled in using the rule described in Section 1. Formally, the entries $T_{n,k}$ ($n \geq k \geq 0$) in the triangle are defined by

$$\begin{aligned}
 T_{n,0} &= a_n & (n \geq 0), \\
 T_{n+1,k+1} &= T_{n+1,k} + T_{n,n-k} & (n \geq k \geq 0),
 \end{aligned} \tag{2.2}$$

and then

$$b_n = T_{n,n} \quad (n \geq 0).$$

Although many operations on sequences have been studied in the past (see [BS95] and the references therein), this transformation appears to have been overlooked.

THEOREM 1. *The boustrophedon transform \mathbf{b} of a sequence \mathbf{a} is given by*

$$b_n = \sum_{k=0}^n \binom{n}{k} a_k E_{n-k}, \quad (n \geq 0), \tag{2.3}$$

$$a_n = \sum_{k=0}^n (-1)^{n-k} \binom{n}{k} b_k E_{n-k}, \quad (n \geq 0), \tag{2.4}$$

and the e.g.f.'s of \mathbf{b} and \mathbf{a} are related by

$$\mathcal{B}(x) = (\sec x + \tan x) \mathcal{A}(x). \tag{2.5}$$

Proof. We redraw (2.1) as a directed graph Γ whose nodes are labeled by the numbers $T_{n,k}$ (see Fig. 1). Let $\pi(n, k, i)$ denote the number of paths

¹ In this paper we consider only integer-valued sequences, although the transformation can be applied to sequences over any ring.

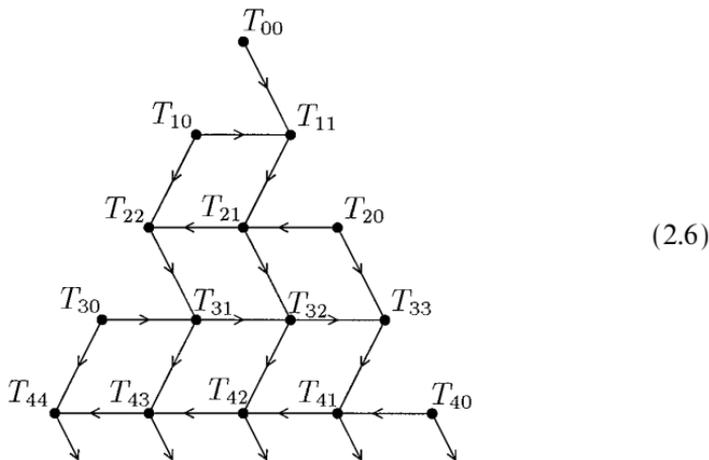


FIG. 1. Directed graph Γ underlying the boustrophedon transform.

in Γ from the node labeled $T_{i,0}$ to the node labeled $T_{n,k}$. It follows from the rule for constructing the triangle that the numbers $T_{n,k}$ are given by

$$T_{n,k} = \sum_{i=0}^n \pi(n, k, i) a_i. \tag{2.7}$$

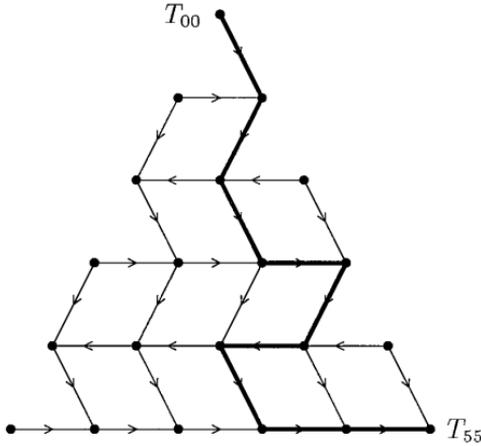
From Section 1 we know that the boustrophedon transform of the sequence $1, 0, 0, 0, \dots$ is $E_0, E_1, E_2, E_3, \dots$, and so (from (2.7))

$$E_n = \pi(n, n, 0) \quad (n \geq 0). \tag{2.8}$$

We will give a direct proof of this (although of course it is known result, cf. [Arn92]), in order to establish a bijection between paths in Γ and up-down permutations.

PROPOSITION 1. $\pi(n, n, 0)$ is equal to E_n , the number of down-up permutations of $\{1, 2, \dots, n\}$.

Proof. Let P be a path in Γ from the top node to the node labeled $T_{n,n}$. (Figure 2 shows an example for $n = 5$.) Let $T_{i,f(i)}$ be the label of the node where P arrives at level i ($1 \leq f(i) \leq i \leq n$). We construct a box diagram to represent P by the following procedure (see Fig. 3). The bottom row contains n boxes labeled $1, \dots, n$ from left to right (if n is even) or from right to left (if n is odd). The box labeled $f(n)$ is starred. We now repeatedly place a row of boxes above the empty boxes, putting a star in the $f(i)$ th box, always counting from the left if i is even or from the right if i is odd, for $i = n - 1, n - 2, \dots, 1$.

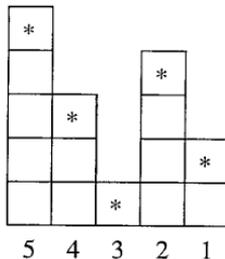
FIG. 2. A path from $T_{0,0}$ to $T_{5,5}$.

We convert the box diagram into a permutation of $\{1, \dots, n\}$ by reading the rows from the bottom up and recording the number at the foot of the column containing the star. (The permutation corresponding to the above example is $(3, 1, 4, 2, 5)$.) We omit the easy verification that this process defines a bijection between paths and down-up permutations. ■

PROPOSITION 2.

$$\pi(n, n, k) = \binom{n}{k} E_{n-k}, \quad \text{for } 0 \leq k \leq n.$$

Sketch of Proof. Consider a path from the node labeled $T_{k,0}$ to the node labeled $T_{n,n}$, such as the path from $T_{4,0}$ to $T_{9,9}$ shown in Fig. 4. The procedure used in the proof of Proposition 1 converts this into a box diagram, which for this example is shown in Fig. 5. The columns that do not contain stars identify one of the $\binom{n}{k}$ k -subsets of $\{1, \dots, n\}$, while the starred columns themselves form a box diagram (in this case it is that shown in Fig. 3) that identifies a down-up permutation of $\{1, \dots, n-k\}$. ■

FIG. 3. Box diagram corresponding to path P in Fig. 2.

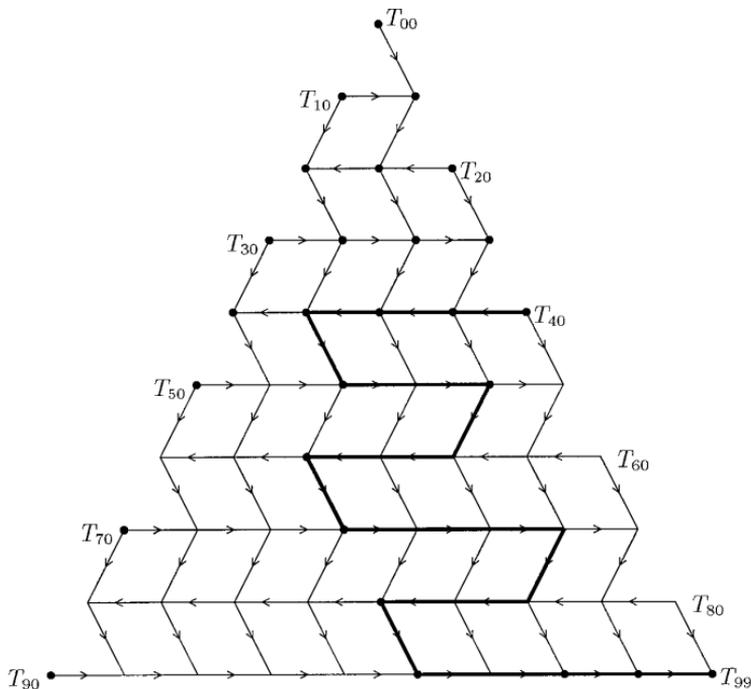


FIG. 4. A path from $T_{4,0}$ to $T_{9,9}$.

From Proposition 2 and (2.7) we obtain

$$b_n = T_{n,n} = \sum_{k=0}^n \binom{n}{k} E_{n-k} a_k,$$

which establishes (2.3). Eqs. (2.5) and (2.4) now follow immediately. This completes the proof of the theorem. ■

Remark. With only a little more effort we can determine all the “boustrophedon numbers” $\pi(n, k, i)$. Note that $\pi(n, 0, i) = 0$ for $n \geq 1$, $0 \leq i \leq n - 1$, and $\pi(n, 0, n) = 1$.

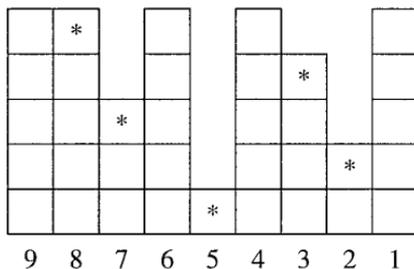


FIG. 5. Box diagram for path shown in Fig. 4.

PROPOSITION 3. For $n \geq 1$, $0 \leq k \leq n-1$,

$$\pi(n, k, 0) = E_{n,k} = \sum_{r=0}^{\lfloor (k-1)/2 \rfloor} (-1)^r \binom{k}{2r+1} E_{n-2r-1}.$$

Proof. $\pi(n, k, 0) = E_{n,k}$ follows from (2.7) and the definition of $E_{n,k}$ (see (1.2)), and the formula for $E_{n,k}$ is given in [Ent66]. ■

Remark. If the path is extended to reach the node labeled $T_{n+1, n+1}$, the corresponding box diagram has the same format as those arising in Proposition 1, except that the star in the last row is constrained to appear in the box labeled $k+1$.

PROPOSITION 4. For $n \geq 2$, $0 < k < n$, $0 < i \leq n$,

$$\pi(n, k, i) = \sum_{s=0}^{\min\{k, n-i\}} \binom{k}{s} \binom{n-k}{n-i-s} \pi(n-i, s, 0). \quad (2.9)$$

Sketch of Proof. Consider a path P from $T_{i,0}$ to $T_{n,k}$, and complete it to a path Q from $T_{i,0}$ to $T_{n+1, n+1}$ by extending P by a downward sloping edge and a series of horizontal edges, as illustrated in Fig. 6. We form the

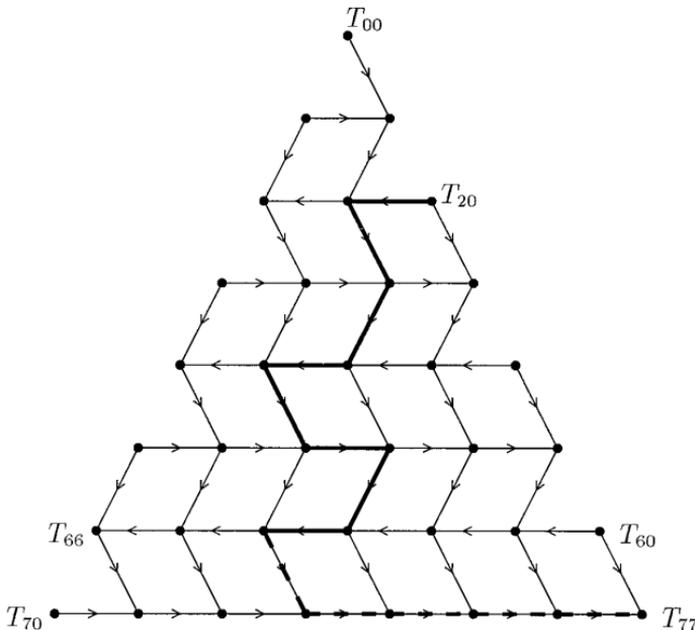


FIG. 6. Path from $T_{2,0}$ to $T_{6,4}$ (solid line) and its continuation to $T_{7,7}$ (broken line).

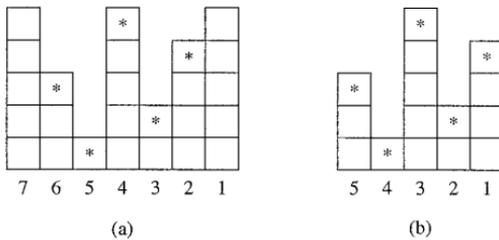


FIG 7. (a) Box diagram corresponding to path shown in Fig. 6. (b) Reduced box diagram, representing path of type $\pi(4, 3, 0)$.

box diagram for Q , as in Proposition 2 (see Fig. 7a). After deleting all the unstarred columns we obtain the box diagram for a path of type $\pi(n - i, s, 0)$, for some s (Fig. 7b).

In the box diagram for Q itself, the star in the last row divides the remaining stars into two sets of sizes s (to the right) and $n - i - s$ (to the left), and the binomial coefficients in (2.9) count the ways in which the corresponding columns can be selected. ■

Propositions 1–4 together express all the boustrophedon numbers in terms of the E_n 's, and via (2.7) give an explicit formula for every entry in the triangle (2.1).

3. COMBINATORIAL INTERPRETATIONS AND EXAMPLES

Equation (2.3) yields many possible combinatorial interpretations for the numbers b_n . For example, if a_n is the number of arrangements of n labeled objects so that they have some property Q , then b_n is the number of ways of dividing n objects into two groups so that the first group has property Q and the second forms a down-up sequence. Since E_n is also the number of ordered binary trees on n nodes (cf. [Pou82], [KPP94]), other interpretations for the b_n can be given in terms of graphs.

EXAMPLE 1. We can see now that (1.5) has e.g.f. $e^x(\sec x + \tan x)$, and that the n th term of this sequence gives the number of ways we can form a down-up sequence of some length $l \geq 0$ from $\{1, \dots, n\}$. E.g. for $n = 3$ there are 9 possibilities: $\phi, 1, 2, 3, 21, 31, 32, 213, 312$.

EXAMPLE 2. The boustrophedon transform of the Bell numbers (cf. [SP95], Fig. M4981) produces the sequence 1, 2, 5, 16, 60, 258, ..., whose n th term gives the number of ways to take blocks labeled 1, ..., n and to partition some of them into heaps and to arrange the rest so they form a down-up sequence.

and

$$L_{n+1, k+1} = L_{n+1, k} + L_{n, n-k}, \quad R_{n+1, k+1} = R_{n+1, k} + R_{n, n-k}.$$

We were happy to find that Theorem 1 leads to an equally simple description of this transformation. The proof is left to the reader.

THEOREM 2. *The e.g.f.'s of \mathbf{a} , \mathbf{m} and \mathbf{b} are related by*

$$\mathcal{M}(x) = \frac{1}{\cos x - \sin x} \mathcal{A}(x),$$

$$\mathcal{B}(x) = \frac{\cos x + \sin x}{\cos x - \sin x} \mathcal{A}(x).$$

ACKNOWLEDGMENTS

We thank Richard Guy for suggesting the problem, and J. H. Conway, I. M. Pak, and S. Plouffe for helpful discussions. A referee suggested that since the boustrophedon and double-ox transforms are associated with the root systems A_n and C_n , respectively (cf. [Arn92]), it would be worthwhile studying analogues of these transforms for other root systems.

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