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THE OPTIMAL ISODUAL LATTICE QUANTIZER IN THREE DIMENSIONS

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Abstract. The mean-centered cuboidal (or m.c.c.) lattice is known to be the optimal packing and covering among all isodual three-dimensional lattices. In this note we show that it is also the best quantizer. It thus joins the isodual lattices $\mathbb{Z}$, A_2 and (presumably) D_4, E_8 and the Leech lattice in being simultaneously optimal with respect to all three criteria.

1. Introduction

An isodual lattice [10] is one that is geometrically similar to its dual. Let Λ be an n-dimensional lattice and let V denote the Voronoi cell containing the origin. We may assume Λ is scaled so that V has unit volume. Three important parameters of Λ are the packing radius (the in-radius of V), the covering radius (the circum-radius of V) and its quantization error, which is the normalized second moment

$$G := \frac{1}{n}\int_V x \cdot x \, dx \tag{1}$$

(see [8], also [11], [12], [13]).

It is known that the face-centered cubic (or f.c.c.) lattice A_3 has the largest packing radius of any three-dimensional lattice (Gauss), while its dual, the body-centered cubic (or b.c.c.) lattice A_3^* has both the smallest covering radius [1] and the smallest quantization error [2]. In [10] it was shown that among all *isodual* three-dimensional lattices, the mean-centered cuboidal (or m.c.c.) lattice has the largest packing radius and the smallest covering radius.

The m.c.c. lattice, denoted here by M_3, has Gram matrix

$$\frac{1}{2}\begin{bmatrix} 1+\sqrt{2} & -1 & -1 \\ -1 & 1+\sqrt{2} & 1-\sqrt{2} \\ -1 & 1-\sqrt{2} & 1+\sqrt{2} \end{bmatrix},$$

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determinant 1, packing radius $\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{\sqrt{2}}}$, center density $\delta = 0.1657\ldots$ (which is between the values for the f.c.c. and b.c.c. lattices), covering radius $3^{0.5}2^{-1.25}$ (again between the values for the f.c.c. and b.c.c. lattices), kissing number 8 and automorphism group of order 16. The m.c.c. lattice received a brief mention in [3] and was studied in more detail in [10]. It also arises from the period matrix of the hyperelliptic Riemann surface $w^2 = z^8 - 1$ [4].

The purpose of this note is to prove:

Theorem 1.1. *The m.c.c. lattice M_3 is the optimal quantizer among all isodual three-dimensional lattices.*

In higher dimensions, less is known. The lattice D_4 is optimal among all four-dimensional lattice packings, and has a lower quantization error than any other four-dimensional lattice presently known (see [8] for references). It is *not* the best four-dimensional lattice covering (A_4^* is better), but it is isodual and may be optimal among isodual lattices with respect to all three criteria. Similar remarks apply to the isodual eight-dimensional lattice E_8 (again A_8^* is a better covering but is not isodual). In 24 dimensions the isodual Leech lattice is known to be the best lattice packing [6], and may well also be the optimal covering and quantizer.

2. Proof of Theorem 1.1

We will specify three-dimensional lattices by giving Gram matrices and conorms (or Selling parameters) — cf. [7], [9], [10]. It was shown in [10] that, up to equivalence, indecomposable isodual three-dimensional lattices of determinant 1 have Gram matrices of the form

$$\frac{1}{2-\alpha\beta}\begin{bmatrix} \frac{2\alpha}{\beta} & -\alpha\beta & -\alpha(2-\beta) \\ -\alpha\beta & \frac{2\beta}{\alpha} & -\frac{2\beta(1-\alpha)}{\alpha} \\ -\alpha(2-\beta) & \frac{-2\beta(1-\alpha)}{\alpha} & \frac{\alpha^2\beta+2\alpha+2\beta-4\alpha\beta}{\alpha} \end{bmatrix}, \tag{2}$$

where α, β are any real numbers satisfying $0 < \alpha < 1$, $0 < \beta < 1$; and decomposable lattices have Gram matrices

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & \alpha & -h \\ 0 & -h & \beta \end{bmatrix}, \tag{3}$$

where α, β, h are any real numbers satisfying $0 \le 2h \le \alpha \le \beta$, $\alpha\beta - h^2 = 1$.

In the indecomposable case the nonzero conorms are:

$$\begin{aligned} p_{01} &= \frac{\alpha(2-\beta)}{\gamma},\ p_{02} = \frac{\alpha\beta}{\gamma},\ p_{03} = \frac{2\alpha(1-\beta)}{\beta\gamma}, \\ p_{12} &= \frac{2\beta(1-\alpha)}{\alpha\gamma},\ p_{13} = \frac{2(1-\alpha)(1-\beta)}{\gamma},\ p_{23} = \frac{\beta(2-\alpha)}{\gamma}, \end{aligned} \tag{4}$$

where $\gamma = 2 - \alpha\beta$; in the decomposable case they are:

$$p_{01} = 1,\ p_{02} = \alpha - h,\ p_{03} = \beta - h,\ p_{12} = 0,\ p_{13} = 0,\ p_{23} = h\,. \tag{5}$$

The m.c.c. lattice corresponds to the case $\alpha = \beta = 2 - \sqrt{2}$ of Eqs. (2) and (4).

From [2] we know that the normalized second moment (1) for a decomposable or indecomposable three-dimensional lattice Λ is given by

$$G = \frac{DS_1 + 2S_2 + K}{36D^{4/3}}, \tag{6}$$

where $D = \det \Lambda$,

$$\begin{aligned}
S_1 &= p_{01} + p_{02} + p_{03} + p_{12} + p_{13} + p_{23}\,, \\
S_2 &= p_{01}p_{02}p_{13}p_{23} + p_{01}p_{03}p_{12}p_{23} + p_{02}p_{03}p_{12}p_{13}\,, \\
K &= p_{01}p_{02}p_{03}(p_{12} + p_{13} + p_{23}) + p_{01}p_{12}p_{13}(p_{02} + p_{03} + p_{23}) \\
&\quad + p_{02}p_{12}p_{23}(p_{01} + p_{03} + p_{13}) + p_{03}p_{13}p_{23}(p_{01} + p_{02} + p_{12})\,.
\end{aligned}$$

For our lattices, $D = 1$.

We first consider the indecomposable case. From (4), (6) we find that

$$G = \frac{f(\alpha,\beta)}{36\alpha\beta(2-\alpha\beta)^4} \tag{7}$$

where

$$\begin{aligned}
f(\alpha,\beta) &= 3\alpha^5\beta^5 - 8\alpha^4\beta^4(\alpha+\beta) + 4\alpha^3\beta^3(\alpha^2 + 10\alpha\beta + \beta^2) \\
&- 48\alpha^3\beta^3(\alpha+\beta) + 8\alpha^2\beta^2(3\alpha^2 + 4\alpha\beta + 3\beta^2) + 32\alpha^2\beta^2(\alpha+\beta) \\
&- 8\alpha\beta(5\alpha^2 + 6\alpha\beta + 5\beta^2) + 16(\alpha^2 + \alpha\beta + \beta^2)\,.
\end{aligned}$$

It turns out that there is exactly one choice for (α, β) in the range $0 < \alpha < 1$, $0 < \beta < 1$ for which both partial derivatives $\frac{\partial G}{\partial \alpha}$ and $\frac{\partial G}{\partial \beta}$ vanish, namely $\alpha = \beta = 2 - \sqrt{2}$, corresponding to the m.c.c. lattice. At all other points in the interior of this region, one of the two partial derivatives does not vanish and so the point cannot be a local minimum. To show this we used the computer algebra system Maple [5] to compute the numerators of $\frac{\partial G}{\partial \alpha}$ and $\frac{\partial G}{\partial \beta}$, giving a pair of simultaneous equations in α and β, symmetrical in α and β. By eliminating β, we obtain a single equation for α:

$$\begin{aligned}
&\alpha(\alpha-1)(\alpha-2)(\alpha^2+2\alpha-2)(\alpha^2-4\alpha+2)(\alpha^4-4\alpha^3+6\alpha^2-4)\ \times \\
&\times\ (57\alpha^6 - 220\alpha^5 - 102\alpha^4 + 1448\alpha^3 - 1860\alpha^2 + 832\alpha - 152) = 0\,.
\end{aligned}$$

The only roots in the range $0 < \alpha < 1$ are $2 - \sqrt{2}$, $\sqrt{3} - 1$ and the root $0.9894\ldots$ of the sixth-degree factor. By symmetry, β must also take one of these three values. At only one of these nine points do both partial derivatives vanish, namely $\alpha = \beta = 2 - \sqrt{2}$. At that point the matrix of second partial derivatives is

$$\begin{bmatrix} 0.1448\ldots & -0.1127\ldots \\ -0.1127\ldots & 0.1448\ldots \end{bmatrix},$$

which is positive definite, showing that m.c.c. is a local minimum. The resulting value of G is

$$\frac{17 + 4\sqrt{2}}{288} = 0.0786696\ldots,$$

between the values for the b.c.c. and f.c.c. lattices, which are respectively

$$\frac{19}{384}2^{2/3} = 0.0785432\ldots \text{ and } \frac{2^{1/3}}{16} = 0.0787450\ldots.$$

On the boundary of the region the lattices are either degenerate (if α or β is 0) or decomposable (if α or β is 1). In the latter case we assume $\alpha \leq \beta$ and find that there is a unique point where $\frac{\partial G}{\partial \alpha}$ and $\frac{\partial G}{\partial \beta}$ vanish, when $\alpha = \sqrt{3} - 1$, $\beta = 1$. This is the lattice $\mathbb{Z} \oplus 3^{-1/4} A_2$, for which $G = \frac{5\sqrt{3}}{162} + \frac{1}{36} = 0.0812361\ldots$. It is not a local minimum.

It remains to consider the decomposable case. From (5), (6), we find that

$$G = \frac{1}{36}\left\{\alpha\beta(\alpha+\beta) + 2(\alpha\beta-1)^{3/2} + 2\alpha + 2\beta + 1\right\}. \tag{8}$$

Again we solve $\frac{\partial G}{\partial \alpha} = \frac{\partial G}{\partial \beta} = 0$, and find that the only possibilities in the range $0 \le \alpha \le \beta$ are $\alpha = \beta = 1$, $G = \frac{1}{12}$; $\alpha = 1$, $\beta = 2$, $G = \frac{1}{12}$; and $\alpha = \beta = \sqrt{3} - 1$ (the decomposable lattice mentioned above). None of these are local minima. This completes the proof.

The proof also shows that the m.c.c. lattice is the only isodual lattice where G has a local minimum.

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