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Doubly Multiplexed Dispersive Spectrometers

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We analyze the performance of a dispersion instrument in which light is multiplexed both in the entrance and exit slit positions. This double multiplexing scheme allows one to recover both Fellgett's advantage and the high throughput advantage normally attributed only to interferometric spectrometers. The spectrometer's performance is evaluated for a number of binary cyclic coding schemes. Optical limitations on doubly multiplexed instruments are discussed, and we show that such spectrometers compare favorably with Michelson interferometric spectrometers. Some first results obtained with a laboratory pilot model are presented.

I. Introduction

For about twenty years, one has known that there are two types of multiplexing schemes that can be used for obtaining spectra. The first of these was a scheme suggested by Golay[1] in which radiation at different frequencies is modulated at differing rates by mechanical means. Such a system involves the dispersion of radiation by means of a grating or prism and the modulation of the dispersed radiation by a mechanical chopper or mask. The theory of a number of simple systems of this type has been further discussed by Ibbett *et al.*,[2] Decker and Harwit,[3] and Sloane *et al.*[4] This type of spectrometer has, so far, not come into general use.

The second type of multiplex spectrometer is the interferometric spectrometer, as exemplified by the Michelson interferometer discussed in detail by Fellgett,[5] and by Vanasse and Sakai.[6] In recent years the interferometric spectrometers—particularly the Michelson spectrometer—have been very successful. One reason for this success lies in the very high energy throughput (or luminosity) of the interferometric instruments. This advantage of the interferometers over grating instruments was clearly pointed out first by Jacquinot.[7] He showed that the throughput of the Fabry-Perot interferometer was far greater than that of a conventional grating instrument (which, in turn, was greater than that of a prism spectrometer).

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In defense of grating spectrometers one could point out that an instrument of the type originally built by Golay did demonstrate high energy throughput and that perhaps Jacquinot's argument should be modified. As far as we know, this point has never been explicitly raised, presumably because one could raise the counterargument that the multiplexing process introduced by Golay could also be used to improve interferometric spectrometers, for the grating spectrometer multiplexing would thus increase luminosity, while for the interferometers—such as those of the Fabry-Perot variety which already have high luminosity—the corresponding advantage would lie in an ability to observe simultaneously all spectral wavelengths. The multiplexing process then converts a monochromatic device such as the Fabry-Perot spectrometer into a polychromatic instrument like the Michelson. This multiplexing advantage is the so-called Fellgett's advantage.

The tacit assumption made in the above argument is that multiplexing can be used to achieve only one out of two possible advantages. Either one can have Golay's advantage in increased luminosity, as is realized in instruments built by Golay, Girard,[8] and others; or else one can have Fellgett's advantage of polychromatic transmission. Either process increases the signal falling on the detector at any given time.

The point that perhaps has not been explicitly made before is that a double multiplexing scheme can endow an instrument with *both* increased luminosity *and* polychromatic transmission. This point was tacit in Mertz's Mock interferometer[9] but has received insufficient attention.

The purpose of the present paper is to show that multiplex spectrometry with grating instruments actually permits construction of high luminosity instruments that can compete with Michelson interferometric spectrometers under a number of conditions.

As with Michelson interferometers, these instruments show their greatest advantage under conditions where measurements are detector noise limited and where the source of radiation is diffuse, requiring a high luminosity spectrometer. With photon noise limited detectors, only the luminosity gain is realized; but such a gain could still have great significance, say, in uv or x-ray astronomy (see Ref. 9, p. 66).

In the instrument analyzed here, an initial stage of multiplexing is introduced at the entrance of the spectrometer, and a second multiplexing stage follows at the instrument's exit plane. The signal-to-noise ratio advantage which can be achieved over an ordinary grating instrument is shown to be comparable with that of the Michelson instrument. That this should be so can be made intuitively understandable as follows: In order to describe the intensity of N spectral elements, a Michelson interferometric spectrometer will generally need to make determinations of $2N$ different mirror separations; under certain conditions this number can be reduced to N. In a similar way, we will show that there are $(2N - 1)$ determinations required for a doubly multiplexed grating instrument. The radiant power incident on the detector is roughly the same in both cases; it differs by a factor of order 2 $(1 - \gamma')/(1 - \gamma)$, where γ is the grating energy loss, and γ' is the beam splitter loss. The factor 2 enters only if the blocked radiation is wasted at both entrance and exit slit modulations. As was pointed out by Sloane *et al.*,[4] this rejection of radiation blocked by the masks is not necessary. The number $2(1 - \gamma')/(1 - \gamma)$ is likely to be of order unity and by no means amounts to the many orders of magnitude normally claimed in comparisons of grating and interferometric instruments. There appears to exist no fundamental restriction which limits grating instruments to an inferior role, provided the multiplexing techniques are fully exploited. We will attempt to show this by evaluating the relative performance of a doubly multiplexed grating instrument to the performance of singly multiplexed and single slit grating spectrometers. A comparison with the performance of the Michelson interferometer can then be made because the relative merits of simple grating instruments and interferometers have already been discussed in the literature (see Ref. 6).

While we claim that $2N - 1$ measurements suffice to reconstruct the spectrum of radiation passing through an instrument having N entrance and N exit slots positions, we will actually evaluate the performance of the system on the assumption that N^2 measurements are taken; i.e., that light is successively passed through the N exit mask positions for each of the N entrance mask positions. There is no loss of generality involved here because the various instruments will be compared on the basis of (1) an identical total observing time T, during which all spectral measurements must be taken; (2) a constant radiant energy density incident on unit area of the spectrometer entrance aperture; (3) constancy of the number N of unknown spectral elements to be determined during time T; (4) identical photodetectors; and (5) constant slot widths.

The only reason for presenting our argument on the basis of N^2 measurements, rather than $2N - 1$, lies in the greater simplicity of the mathematical treatment of the error analysis. We recognize that a need for N^2 measurements would not be convenient for large N, and, in fact, we see no compelling reason why this should be required in practical situations. It is interesting, however, that in astronomical applications, the use of N^2 measurements can give additional information about the difference in the spectral distribution of light reaching the instrument from differing portions (strips) of the source.

As a final remark, we point out that almost any grating instrument can be readily converted into a doubly multiplexed instrument simply by installing encoding masks at the instrument's entrance and exit focal planes. The instrument's performance can then be dramatically improved at very little extra cost. Curved slits present no difficulties and, in fact, may avoid some of the disadvantages mentioned by Mertz in his discussion of the Mock interferometer.

II. Operation of the Doubly Multiplexed Spectrometer

A. Encoding the Input to the Grating

The multislit spectrometer whose performance we wish to analyze makes use of an encoding mask placed at the position of the entrance slit of a conventional grating spectrometer. Instead of passing light through just one entrance slot, this mask is N slot widths wide. In any given mask position, light from the source is permitted to pass through about one-half of these N slots and is blocked from passage through the others. In general, there are M different mask positions in which different combinations of slots are permitted to pass light from the source into the spectrometer. The encoding process then consists of the successive use of each of the M mask positions to pass light through M different preselected combinations of entrance slot locations.

B. Encoding the Output

A second encoding mask is placed in the exit focal plane of the spectrometer. This mask has M' different positions, each position passing light through some of the N' exit slot locations in a preselected way. By making measurements of the intensity of radiation passing through different combinations of entrance and exit mask positions, the radiation spectrum can be recovered. In the present paper we will only discuss the situation in which $M = N = N' = M'$. We will call this number N.

III. Theoretical Analysis of a Doubly Multiplexed Spectrometer

The grating spectrometer has multiple entrance and exit slots with an entrance mask and an exit mask (see Fig. 1). Let $\epsilon = (\epsilon_{ir})$ be the $N \times N$ matrix describing the entrance mask, where $\epsilon_{ir} = 1$ or 0, according as the rth entrance slot is open or closed when the entrance mask is in position i ($1 \leq i \leq N$, $1 \leq r \leq N$). Similarly let $\chi = (\chi_{ij})$ describe the exit mask.

A. The Basic Equation

When the entrance mask is in position i and the exit mask is in position j, the detector measures

$$\eta_{i,j} = \sum_{r=1}^{N} \sum_{s=1}^{N} \epsilon_{i,r}\psi_{r,s}\chi_{j,s} + \nu_{i,j}, \tag{1}$$

where $\psi_{r,s}$ is the measurement that would be made by a noiseless detector placed at the sth exit slot when signal enters only through the rth entrance slot, and $\nu_{i,j}$ is the detector noise for measurement (i,j). We assume that the optical source is spatially homogeneous, and that the spectrometer optics are arranged so that the spectrum produced by the rth entrance slot acting alone is a shift by r places of the spectrum produced by the first entrance slot acting alone. Thus there are $2N - 1$ unknown spectral components:

$$\psi_{-(N-1)}, \ldots, \psi_{-1}, \psi_0, \psi_1, \ldots, \psi_{N-1}$$

given by $\psi_{r,s} \equiv \psi_{r-s}$. Then Eq. (1) may be rewritten in matrix form as

$$\eta = \epsilon\bar{\psi}\chi^T + \nu, \tag{2}$$

where $\eta = (\eta_{i,j})$, $\nu = (\nu_{i,j})$, and

$$\psi = \begin{pmatrix} \psi_0 & \psi_{-1} & \cdots & \psi_{-N+1} \\ \psi_1 & \psi_0 & \cdots & \psi_{-N} \\ \psi_{N-1} & \psi_{N-2} & \cdots & \psi_0 \end{pmatrix}.$$

B. The Detector Noise

$\nu_{i,j}$ in the (i,j)th measurement is assumed to have the following characteristics: (1) the expected value of $\nu_{i,j}$ is 0, i.e., $E(\nu_{i,j}) = 0$; (2) the noise obtained in different measurements is uncorrelated; i.e.,

$$E(\nu_{i,j}\nu_{k,l}) = \sigma^2\delta_{i,k}\delta_{j,l},$$

where σ is the root mean square noise level. For purposes of comparison we observe that if M measurements are made in time T, then

$$\sigma^2 = KM/T, \tag{3}$$

where K is a constant depending upon the photodetector.

C. The Estimation Problem

We have the possibility of making up to N^2 measurements $\eta_{i,j}$ ($i,j = 1, \ldots, N$) to estimate $2N - 1$ unknown ψ_t's. Given these measurements, we must then decide how to estimate the unknown ψ_t's. As justified by Sloane *et al.*[4] for a singly multiplexed spectrometer, an unbiased estimator for ψ_t will be used which is a linear function of the measurements,

$$\hat{\psi}_t = \sum_{i,j} \alpha_{t,i,j}\eta_{i,j},$$

where the coefficients $\alpha_{t,i,j}$ are to be determined. Unbiasedness means that $E\hat{\psi}_t = \psi_t$. When all N^2 measurements are used, this implies that the α's must satisfy

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{t,i,j} \sum_{r=u+1}^{N} \epsilon_{i,r}\chi_{j,r-u} = \delta_{t,u} \text{ for } u \geq 0, \tag{4a}$$

$$\sum_{i=1}^{N}\sum_{j=1}^{N} \alpha_{t,i,j} \sum_{r=1}^{u+N} \epsilon_{i,r}\chi_{j,r-u} = \delta_{t,u} \text{ for } u < 0. \tag{4b}$$

As a measure of the accuracy of the estimate we adopt the mean square error criterion: minimize

$$\sum_{t=-N+1}^{N-1} \sigma_t^2 \triangleq \sum_{t=-N+1}^{N-1} E(\hat{\psi}_t - \psi_t)^2 = \sigma^2 \sum_{t,i,j} (\alpha_{t,i,j})^2. \tag{5}$$

The estimation problem is then to choose the masks, ϵ, χ, the number of measurements to be made, and the α's, so as to satisfy the unbiasedness condition (4) and to minimize Eq. (5). Unfortunately, we do not know how to perform this minimization; we can only evaluate a given scheme once it is presented. In the next paragraph we evaluate σ_t^2 for a scheme where $\epsilon = \chi$, where N^2 measurements are made, and where the α's are given by Eq. (6).

D. Using N^2 Measurements

N^2 measurements $\eta_{i,j}$ are made; i.e., one measurement is made for every combination of entrance and exit mask positions. Then

$$\Omega = \begin{pmatrix} \omega_{11} & \omega_{12} & \cdots & \omega_{1N} \\ \omega_{21} & \omega_{22} & \cdots & \omega_{2N} \\ \omega_{N1} & \omega_{N2} & \cdots & \omega_{NN} \end{pmatrix} \equiv \epsilon^{-1}\eta(\chi^T)^{-1}$$

would equal ψ if there were no noise. In the presence of noise each element $\omega_{r,r-t}$ of the tth diagonal of Ω—counting the main diagonal as the 0th, etc.—is an unbiased estimate of ψ_t; and since each $\omega_{r,\,r-t}$ has the same mean (ψ_t) and can be shown, at least in the case of the S matrix discussed below, to have the same variance (for fixed t), we use as our estimate of ψ_t an equally weighted, linear combination of the $\omega_{r,\,r-t}$:

$$\psi_t = \begin{cases} \dfrac{1}{N-t} \displaystyle\sum_{r=t+1}^{N} \omega_{r,r-t} \text{ if } t \geq 0 & \text{(6a)} \\[2ex] \dfrac{1}{N-t} \displaystyle\sum_{r=1}^{N+t} \omega_{r,r-t} \text{ if } t < 0. & \text{(6b)} \end{cases}$$

We now treat the special case where ϵ and χ are both taken to be the matrix S defined as follows: Let H be a normalized $(N + 1) \times (N + 1)$ Hadamard matrix of $+1$'s and -1's with the first row and column consisting of $+1$'s. S is obtained by deleting the first row and column of H and replacing $+1$'s by 0's and -1's by 1's. Then $S^{-1} = [2/(N + 1)](2S^T - J)$, where I is a unit matrix and J is an $N \times N$ matrix

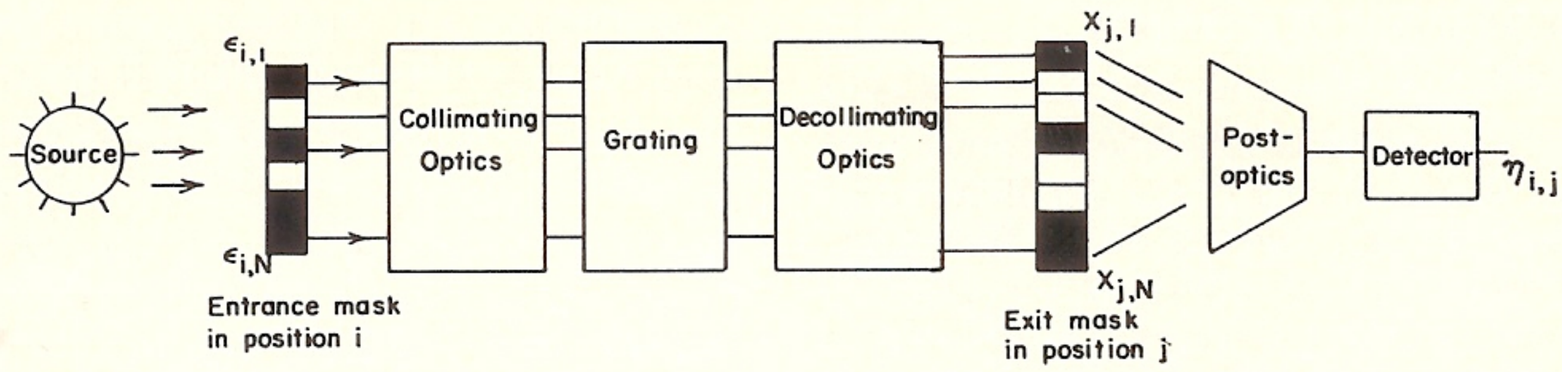

Fig. 1. Schematic diagram of doubly multiplexed spectrometer.

of +1's. (See Sloane *et al.*[4] for more details.) Using the properties of the S matrix from Sloane *et al.*,[4] one finds in this case that

$$\sigma_t^2 = \frac{16}{(N+1)^2}\left(\frac{N^2-1}{N-|t|}+1\right)\sigma^2 \sim \frac{16\,\sigma^2}{(N-|t|)N^2} \text{ for } N \text{ large,} \tag{7}$$

where $t = -(N-1), \ldots, (N-1)$.

E. $2N - 1$ Measurements Are Enough

It is interesting to observe that if ϵ and χ have rank N, it is possible to obtain unbiased estimates of the $2N - 1$ unknown ψ_t's by measuring only $2N - 1$ $\eta_{i,j}$'s. However, there seems to be no simple rule for finding which $\eta_{i,j}$'s to use, nor for finding the α's to use in Eq. (3). We have analyzed the situation for $N = 7$, and Fig. 2 shows spectra obtained with a laboratory pilot model using both methods of reduction.

IV. Comparisons with Other Grating Spectrometers

To make a fair comparison, we assume (1) a fixed total measuring time T; (2) constant incident energy density; (3) an equal number N of unknown spectral elements to be estimated; (4) identical photodetectors; and (5) constant slot widths. For the doubly multiplexed system of Sec. III there are initially $2N - 1$ unknowns $\psi_{-(N-1)}, \ldots, \psi_0, \ldots, \psi_{N-1}$, but for this comparison we will suppose we only wish to estimate the N central elements,

$$\psi_{-(N-1)/2}, \ldots, \psi_0, \ldots, \psi_{(N-1)/2}$$

(taking N odd for convenience). Of course, we can still obtain some estimate of the ends of the spectrum, but the errors there will not appear in the comparisons.

As a measure of performance we take, as before, the total mean square error in all the unknowns:

$$\sigma_{\text{total}}^2 = \sum_{t=-(N-1)/2}^{(N-1)/2} \sigma_t^2.$$

Table I compares three different grating spectrometers. The first column is for a single entrance and exit slot. N measurements are made in time T, with a mean square error σ^2 in each. The second column is for a singly multiplexed instrument with an exit mask $= S$, and is taken from Sloane *et al.*[4] The last column is for the doubly multiplexed system of Sec. III, using Eq. (7) for σ_t^2, and has, by Eq. (3), been multiplied by a factor of N to allow for having to make N^2 measurements in time T.

A more accurate expression for the error in the double mask case is provided by

$$\sigma_{\text{total}}^2 = \sigma^2[(22.18/N) - (40.72/N^2) + 0(1/N^3)]. \tag{8}$$

Remarks on the "wide aperture advantage." By using more entrance slots than are necessary, a doubly multiplexed spectrometer may be used to obtain a total mean square error for N unknowns that is much less than $(22.2/N)\sigma^2$ of Table I. Suppose that N spectral elements are to be estimated. A doubly multiplexed spectrometer is used with W entrance and exit slots, where W is greater than N, and W^2 measurements are made. There are now $2W - 1$ unknowns, but Eq. (6) is used to estimate only the N elements that we are interested in. The total mean square error is now, from Eqs. (3) and (7),

$$\sigma_{\text{total}}^2 \sum_{t=-(N-1)/2}^{(N-1)/2} \frac{16\sigma_0^2}{(W-|t|)W^2},$$

where $\sigma_0^2 = W^2\sigma^2/N$. Therefore $\sigma_{\text{total}}^2 \sim 16\sigma^2/W$ for $W \gg N$, Eq. (9) which is much less than $(22.2/N)\sigma^2$.

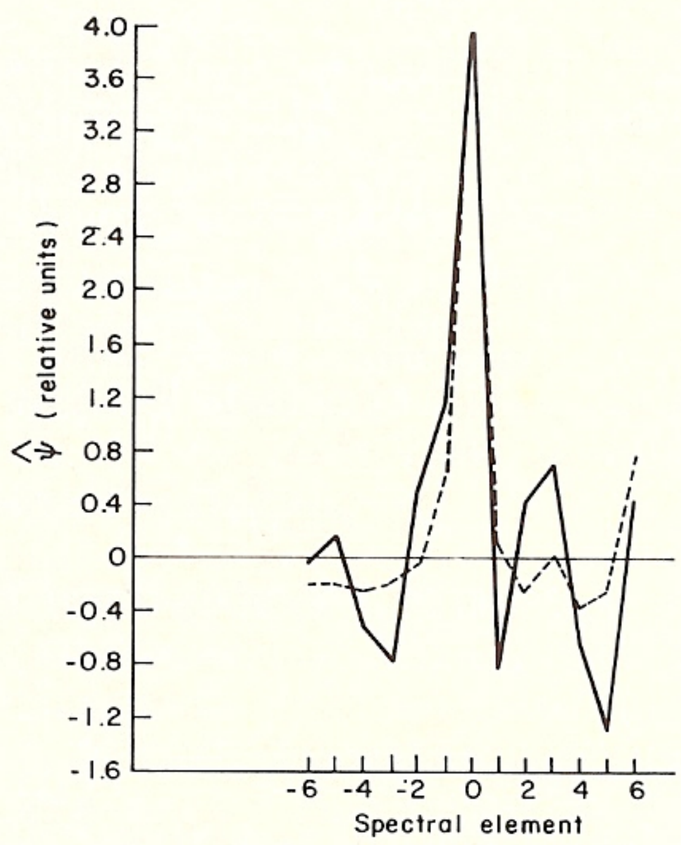

Fig. 2. Results from laboratory pilot model showing mercury green line. The broken line is for $N^2 = 49$ measurements and the solid line for $2N - 1 = 13$ measurements. All measurements have the same integration time.

Table I. Comparison of Total Mean Square Error for Three Grating Spectrometers in Estimating N Unknowns

N	No mask	Single mask	Double masks
3	$3\ \sigma^{2a}$	$2.25\ \sigma^2$	$2.56\ \sigma^2$
7	$7\ \sigma^2$	$3.06\ \sigma^2$	$2.00\ \sigma^2$
11	$11\ \sigma^2$	$3.36\ \sigma^2$	$1.51\ \sigma^2$
19	$19\ \sigma^2$	$3.61\ \sigma^2$	$0.98\ \sigma^2$
N	$N\ \sigma^2$	$\{2 - [2/(N + 1)]\}^2\ \sigma^2$	$\sim(22.2/N)\ \sigma^2$ for N large[b]

[a] σ^2 is the mean square noise in a single measurement made in time T/N; $\sigma^2 = (\text{constant}\cdot N/T)$

[b] This number becomes $16\ \sigma^2/W$ when W slots are used to detect N spectral elements and $W \gg N$.

This is the *wide aperture advantage*.

In practice this means that W should always be made as large as possible, the limitations on W arise from inhomogeneities in the image at the entrance and the deterioration of resolving power with increasing total aperture width. This is an engineering limitation discussed in the Appendix.

V. Comparison of Doubly Multiplexed and Michelson Spectrometers

We first compare the Michelson spectrometer to a single slit grating instrument. As is well known, one can separate the Michelson's advantages into two parts. First, there is the multiplex advantage, which for N spectral elements is equivalent to a reduction of σ^2 by a factor $2/N$. Second, there is the throughput advantage which gives a greater effective signal for a constant detector noise. As viewed here, this increases the figure of merit for the Michelson by another factor of $(A\Omega/a\omega)$, where a and ω are the grating aperture and acceptance solid angle, and A and Ω play the same role for the interferometer.

ω and Ω normally are limited by the resolving power that has to be attained, while A and a are limited by practical size limitations on beam splitters and gratings available to the experimenter.

As shown in the Appendix, the single slit grating instrument has a throughput

$$\mathcal{L}_G = a\omega = (L/d)(Gm\lambda/bR)^2,$$

where L and d are the slit height and width; G, M, and b the grating width, order, and spacing; λ the wavelength; and R the resolving power. For the interferometer the corresponding value is

$$\mathcal{L}_I = A\Omega = A(2\pi/R).$$

Thus the interferometer is particularly valuable at high resolving power. Placing some practical numbers into these equations, e.g., $A \sim 30$ cm^2, gives $\mathcal{L}_I \sim 200/R$ cm.^{2} For similar resolving power $\mathcal{L}_I \approx 1/15$ and $\mathcal{L}_G \approx 1/480$. Quite generally, we may also choose $m\lambda/b \sim 1$, $L/d \sim 200$, and $G^2 \sim 200$, so that $\mathcal{L}_G \sim (200)^2/R^2$ and $\mathcal{L}_I/\mathcal{L}_G \sim R/200$. Thus for very low resolving power $R \sim 200$, the interferometer's main advantage lies not in its high throughput, but only in its ability to multiplex—an advantage which the Michelson interferometer has, but not, for example, the Fabry-Perot.

We turn next to the multiplexed instrument. As shown in the Appendix, the grating resolving power for wide slits is given by $R \approx 8/\delta^2$, where δ is the angle subtended by the width of the slits at the grating. The slit height can subtend a much greater angle, typically of order 1/12 rad; hence the throughput of the multiplexed grating instrument is

$$\mathcal{L}_{MS} \approx a\omega \approx a \cos\alpha (8/R)^{\frac{1}{2}} 1/12,$$

where α is the angle of incidence on the grating. If R is to be limited by the slit width, and is to have a value of, say, 3×10^3, then $\delta = 5 \times 10^{-2}$. A considerable increase in the acceptance angle, therefore, results from the multiplexing technique. Equation (A1) shows that

$$d/f \approx (m\lambda/b)(1/R \cos\alpha) \approx 5 \times 10^{-4}.$$

The number of slits $N = \delta/(d/f) = 100$, and the throughput is

$$\mathcal{L}_{MS} \approx (200)\cdot\tfrac{1}{2}\cdot 8/(3 \times 10^3)^{\frac{1}{2}}\ \tfrac{1}{12} \approx 0.4 \text{ cm}^2 \text{ sr}.$$

These figures should be compared only in order of magnitude. The resolving power is not defined in precisely the same way in the different instruments, hence an exact comparison is inappropriate. Moreover, the numerical values inserted into the equations are somewhat a matter of taste.

Nevertheless, our comparison shows that even when realistic grating and mask transmission losses are taken into account, a multiplexed grating instrument should still compare favorably with an interferometer in total energy throughput.

In summary, one knows that the Michelson interferometer's multiplex advantage gives an improvement in signal-to-noise ratio by a factor of order N. We find in Table I a similar improvement in going from a single slit spectrometer to a single mask system. The Michelson also has an energy throughput advantage over a single slit spectrometer; the present section has shown that the additional gain of a factor of order N in the double mask situation of Table I gives a throughput advantage which also corresponds to that of the Michelson. A detailed comparison of the relative merits of interferometers and doubly multiplexed dispersive systems would involve reflection and transmission losses, sizes of available grating or beam splitters, and other instrumental questions.

For these reasons we feel that multiplexed grating instruments should be seriously considered, particularly for work in spectral regions where transmission optics present problems.

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Appendix. Resolving Power of Grating Instruments

(1) For a conventional single slit grating instrument, the dispersion equation is

$$(b/m)\cos\alpha = d\lambda/d\alpha \approx \lambda f/Rd, \tag{A1}$$

where b is the ruling spacing, m is the order of the spectrum, α is the angle of incidence, λ is the wavelength, f is the focal length, and the resolving power is $R = \lambda/\Delta\lambda$. $\Delta\lambda$ is the spread in wavelength of radiation passing through the exit slit of width d. If the system speed is given in terms of an angle Ⓗ, the grating length G is given by

$$G = \text{Ⓗ}f/\cos\alpha = bdRd\text{Ⓗ}/m\lambda. \tag{A2}$$

If the height of the entrance slit is L, the instrument's throughput is

$$\mathcal{L} = Ld\text{Ⓗ}^2 = (L/d)(Gm\lambda/bR)^2. \tag{A3}$$

(2) For a multislit instrument the luminosity is determined by somewhat different considerations. Here it is important to image entrance slot κ on exit slot l, while simultaneously imaging entrance slot $\kappa + n$ on exit slot $l + n$ at the same wavelength λ. Moreover, for the simplest case of a cyclic encoding pattern in which each mask need only be translated one slot width at a time, we require that all the slots in each mask be equally wide.

With these restrictions we then wish to find the angular slit width $\delta = D/f$ consistent with resolving power R, where D is the mask width. The number of permissible slots of width d is $N = D/d$. Clearly, resolving power obtained in this way must still be subject to slot width limitations as shown in Eq. (A1).

An analysis for distortion of an extended image by a grating is given by Laws.[10] Here, by a different analysis, we show its effect on resolving power.

Let the angle of incidence on the grating be α for radiation entering the first slot of the entrance mask and let $\alpha + \delta$ be the angle of incidence for radiation entering the Nth entrance slot. The corresponding angles of diffraction will be β and $\beta + \delta''$. The grating equation will read

$$\sin\alpha - \sin\beta = m\lambda/b \tag{A4a}$$

and

$$\sin(\alpha + \delta) - \sin(\beta + \delta'') = m\lambda/b, \tag{A4b}$$

where λ is the central wavelength for which each entrance slot is to be imaged on its corresponding exit slot. To second order, these equations are simultaneously satisfied when

$$\delta'' = (1/\cos\beta)\{\delta\cos\alpha + (\delta^2/2)[(\sin\beta/\cos^2\beta)\cos^2\alpha - \sin\alpha]\}.$$

If δ'' is set equal to δ, a Littrow mode of operation can be useful. We have found this to be true in the laboratory. In the case $\delta = \delta''$, these equations are satisfied when the left-hand sides of Eqs. (A4a) and (A4b) are identical. In expanded form, this criterion is

$$(\sin\alpha - \sin\beta)(\cos\delta - 1) + (\cos\alpha - \cos\beta)\sin\delta = 0, \tag{A5}$$

which, after expansion, gives

$$(\cos\alpha - \cos\beta) = (m\lambda/b)[(\sin\delta/2) + (\sin^3\delta/8) + \ldots] \quad \text{for } \delta \ll 1. \tag{A6}$$

Given that the imaging criterion is satisfied at the extreme mask positions, one asks about the deviation in wavelength, $\Delta\lambda$, at some intermediate angular positions δ', where

$$\sin(\alpha + \delta') - \sin(\beta + \delta') = m(\lambda + \Delta\lambda)/b. \tag{A4c}$$

By an expansion similar to Eqs. (A5) and (A6) one can obtain

$$\frac{\Delta\lambda}{\lambda} = -\left(\frac{\sin^2\delta'}{2} + \frac{\sin^4\delta'}{8}\right) + \ldots + \frac{\sin\delta'\sin\delta}{2} + \frac{\sin\delta'\sin^3\delta}{8} + \ldots. \tag{A7a}$$

One notes that $\Delta\lambda = 0$ for $\delta' = 0$ or δ, as required. For small angles δ we can neglect higher terms and approximate Eq. (A7a) by

$$\Delta\lambda/\lambda = \delta'(\delta - \delta')/2. \tag{A7b}$$

This has a maximum at $\delta' = \delta/2$. The limiting (minimum) resolving power corresponds to this maximum so that

$$R = \lambda/\Delta\lambda = 8/\delta^2. \tag{A8}$$

Since the resolving power depends on δ^2, it is clear that a very rapid deterioration at wide slit openings is to be expected.

A laboratory test, using the sodium D line separation, shows that the relation (A8) seems reasonably well satisfied, at least for values of $R \sim 2000$.

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