

BINARY SELF-DUAL CODES OF LENGTH 24

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ABSTRACT. There are 26 distinct indecomposable self-dual codes of length 24 over $GF(2)$, including unique codes of minimum weights 8 and 6, whose groups are, respectively, the Mathieu group M_{24} and the maximal subgroup of index 1771 in M_{24} . For each code we give the order of its group, the number of equivalent codes, and its weight distribution.

1. **Introduction.** An $[n, k]$ code C is a k -dimensional subspace of the vector space of all n -tuples of 0's and 1's with mod 2 addition. The dual code $C^\perp = \{u: u \cdot v = 0 \text{ for all } v \in C\}$ is an $[n, n - k]$ code. C is self-orthogonal if $C \subset C^\perp$, self-dual if $C = C^\perp$. Self-dual codes exist whenever the length n is even. The weight of a vector is the number of its non-zero components, and the minimum weight of C is the minimum weight of any nonzero codeword. The weight distribution of C is the set $\{\alpha_0, \alpha_1, \dots, \alpha_n\}$, where α_i is the number of codewords of weight i .

The group $G(C)$ of a code C is the set of all permutations of the coordinates which send C into itself set-wise. Two codes are equivalent if there is a coordinate permutation sending one into the other. The number of codes equivalent to C is $n!/\text{order of } G(C)$. The direct sum of codes C' and C'' , written $C' \oplus C''$, is $\{(u, v): u \in C', v \in C''\}$. If $C = C' \oplus C''$, where C' and C'' are nonzero, then C is decomposable. Otherwise C is indecomposable.

Pless [4] classified all self-dual codes of length ≤ 20 , Conway (unpublished) found the 9 self-dual codes of length 24 in which the weight of every codeword is a multiple of 4, and Niemeier [2] found the 24 even unimodular lattices in dimension 24, 9 of which correspond to the codes found by Conway.

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+ 2/bco10/boc01/ao²1², where the + 2 indicates two coordinates which do not meet any codeword of weight 4. *a'* denotes $a + b = 011010 \dots 10$. We omit the full details of W_{24}, X_{24}, Y_{24} .

3. Minimum weight 6 and 8. It is known [3], [1] that the [24, 12] Golay code is the unique code of minimum weight 8, and that its group is the Mathieu group M_{24} .

We determined that there is a unique self-dual code of minimum weight 6, which is generated (in Todd's [6] notation) by the set of 64 nonspecial hexads associated with any set of 6 mutually complementary tetrads in the Golay code. Its group is a maximal subgroup of index 1771 in M_{24} .

TABLE I
Indecomposable Self-Dual Codes of Length 24 (Page 1)

Code	{Generator Matrix Order of Group	Number ÷ ν	α_4	α_6	α_8	α_{10}	α_{12}
A_{24}	$\left\{ \begin{array}{l} d_{12}^2/ab/ba \\ (2^5 \cdot 6!)^2 \cdot 2 \end{array} \right.$	1,848	30	0	639	0	2756
B_{24}	$\left\{ \begin{array}{l} d_{10}e_7^2/bcc/aoc \\ 2^4 \cdot 5!168^2 \cdot 2 \end{array} \right.$	18,102 $\frac{6}{7}$	24	0	663	0	2720
C_{24}	$\left\{ \begin{array}{l} d_6^3(a)/abb/bab/bba \\ (2^3 \cdot 4!)^3 \cdot 3! \end{array} \right.$	46,200	18	0	687	0	2684
D_{24}	$\left\{ \begin{array}{l} d_6^4(a)/baao/obaa/aoba/aaob \\ (2^2 \cdot 3!)^4 4! \end{array} \right.$	246,400	12	0	711	0	2648
E_{24}	$\left\{ \begin{array}{l} d_{24}/a \\ 2^{11} \cdot 12 \end{array} \right.$	2	66	0	495	0	2972
F_{24}	$\left\{ \begin{array}{l} d_6^6(a)/boa^3o/oboa^3/aoboa^2/a^2oboa/a^3obo/oa^3ob \\ 4^6 \cdot 6!3 \end{array} \right.$	221,760	6	0	735	0	2612
G_{24}	$\left\{ \begin{array}{l} \text{Golay code} \\ 2^{10} \cdot 3^3 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \end{array} \right.$	8,013 $\frac{21}{23}$	0	0	759	0	2576
H_{24}	$\left\{ \begin{array}{l} d_8d_{16}/ab/ba \\ 2^3 \cdot 4!2^7 \cdot 8! \end{array} \right.$	1,980	34	64	239	960	1500
I_{24}	$\left\{ \begin{array}{l} d_4d_8d_{12}/b^3/a^2o/oa^2 \\ 2 \cdot 2!2^3 \cdot 4!2^5 \cdot 6! \end{array} \right.$	110,880	22	64	287	960	1428

TABLE I
Indecomposable Self-Dual Codes of Length 24 (Page 2)

Code	Generator Matrix Order of Group	Number $\div \nu$	α_4	α_6	α_8	α_{10}	α_{12}
J_{24}	$\left\{ \begin{array}{l} d_8 e_7^2 + 2/bco10/boc01/ao^2 1^2 \text{ (see (1))} \\ 2^3 \cdot 4!168^2 \cdot 2 \end{array} \right.$	181,028 $\frac{4}{7}$	20	64	295	960	1416
K_{24}	$\left\{ \begin{array}{l} d_6 d_{10} e_7 + 1/b^2 c1/oa01/abo1 \\ 2^2 \cdot 3!2^4 \cdot 5!168 \end{array} \right.$	253,440	20	64	295	960	1416
L_{24}	$\left\{ \begin{array}{l} d_8^3(b)/b^3/a^2 o/oa^2 \\ (2^3 \cdot 4!)^3 \cdot 3! \end{array} \right.$	46,200	18	64	303	960	1404
M_{24}	$\left\{ \begin{array}{l} d_8^3(c)/a^3/ba'o/boa' \\ (2^3 \cdot 4!)^3 \cdot 2 \end{array} \right.$	138,600	18	64	303	960	1404
N_{24}	$\left\{ \begin{array}{l} d_4^2 d_{10} + 2/b^3 11/oa^2 11/abo01/bao10 \\ (2^2 \cdot 3!)^2 2^4 \cdot 5!2 \end{array} \right.$	887,040	16	64	311	960	1392
O_{24}	$\left\{ \begin{array}{l} d_4^2 d_8^2 /ab^2 o/boao/oboa/baob \\ (2 \cdot 2!)^2 (2^3 \cdot 4!)^2 \cdot 2 \end{array} \right.$	1,663,200	14	64	319	960	1380
P_{24}	$\left\{ \begin{array}{l} d_4 d_6^2 e_7 + 1/ob^2 c1/ab^2 o0/oa'a'o0/boao1 \\ 2 \cdot 2!(2^2 \cdot 3!)^2 168 \cdot 2 \end{array} \right.$	2,534,400	14	64	319	960	1380
Q_{24}	$\left\{ \begin{array}{l} d_6^4(b)aoao/boa^2/oa0a'/oba'a \\ (2^2 \cdot 3!)^4 \cdot 8 \end{array} \right.$	739,200	12	64	327	960	1368
R_{24}	$\left\{ \begin{array}{l} d_4^2 d_8 + 4/b^2 o1^4/bob1^2 o^2/o^2 a01^2 o/ao^2 01^3/oa01^3 0 \\ (2^2 \cdot 3!)^2 2^3 \cdot 4! \cdot 2 \end{array} \right.$	8,870,400	12	64	327	960	1368
S_{24}	$\left\{ \begin{array}{l} d_4 d_6^3 + 2/abo^2 1^2/oaob10/aob^2 0^2/boao01/bo^2 a10 \\ 2 \cdot 2!(2^2 \cdot 3!)^3 \cdot 2 \end{array} \right.$	17,740,800	10	64	335	960	1356

4. General enumeration theorems. The following theorems, and others, were used to check Table I.

THEOREM 2. Let $\alpha_C(x) = \sum_{i=0}^n \alpha_i x^i$ be the weight enumerator of C . Then

$$\sum \alpha_C(x) = \prod_{j=1}^{n/2-2} (2^j + 1) \cdot \left[2^{n/2-1}(1 + x^n) + \sum_{2|i} \binom{n}{i} x^i \right],$$

where the sum extends over all self-dual codes C of even length n .

THEOREM 3. *If n is even, the number of self-dual codes with length n and minimum weight ≥ 4 is*

$$\sum_{i=0}^{n/2} \frac{(-1)^i n!}{2^i i! (n-2i)!} \prod_{j=1}^{n/2-i-1} (2^j + 1).$$

TABLE I
Indecomposable Self Dual Codes of Length 24 (Page 3)

Code	{Generator Matrix Order of Group	Number $\div \nu$	α_4	α_6	α_8	α_{10}	α_{12}
T_{24}	$\left\{ \begin{array}{l} d_4^4 d_8 / babab / ba^2 oa / oab^2 a' / aoba^2 / b^2 oaa' \\ 4^4 \cdot 2^3 \cdot 4! \cdot 8 \end{array} \right.$	4,989,600	10	64	335	960	1356
U_{24}	$\left\{ \begin{array}{l} d_4^2 d_6^2 + 4 / ob^2 o1^2 0^2 / oa^2 o0^3 1 / obob0^2 1^2 / oaoa010^2 / b^2 o^2 1^4 / a^2 o^2 1010 \\ 4^2 (2^2 \cdot 3!)^2 \cdot 4 \end{array} \right.$	53,222,400	8	64	343	960	1344
V_{24}	$\left\{ \begin{array}{l} d_4^6 (b) / babo^3 / obabo^2 / o^2 babo / o^3 bab / bo^3 ba / abo^3 b \\ 4^6 \cdot 6 \cdot 8 \end{array} \right.$	9,979,200	6	64	351	960	1332
W_{24}	$\left\{ \begin{array}{l} d_4^3 d_6 + 6 / \dots \\ 4^3 \cdot 2^2 \cdot 3! \cdot 3! \cdot 2 \end{array} \right.$	106,444,800	6	64	351	960	1332
X_{24}	$\left\{ \begin{array}{l} d_4^4 + 8 / \dots \\ 4^4 \cdot 4! \cdot 2 \end{array} \right.$	159,667,200	4	64	359	960	1320
Y_{24}	$\left\{ \begin{array}{l} d_4^2 + 16 \cdot o / \dots \\ 2^{11} \cdot 3^2 \end{array} \right.$	106,444,800	2	64	367	960	1308
Z_{24}	$\left\{ \begin{array}{l} \text{see } \S 3 \\ 2^{10} \cdot 3^3 \cdot 5 \end{array} \right.$	14,192,640	0	64	375	960	1296
Total:		556,041,557	$\frac{86}{1127} \cdot \nu$				

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