EXTREMAL SELF-DUAL LATTICES EXIST ONLY IN DIMENSIONS 1 TO 8, 12, 14, 15, 23, AND 24

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Abstract. It is known that if Λ is a self-dual lattice in $\mathbb{R}^n$, then

$$\min\{u \cdot u \mid u \in \Lambda, \quad u \neq 0\} \leqslant [n/8] + 1.$$

If equality holds the lattice is called *extremal*. In this paper we find all the extremal lattices: there are unique lattices in dimensions 1, 2, 3, 4, 5, 6, 7, 8, 12, 14, 15, 23, 24 and no others.

§1. *Introduction.* A lattice Λ in $\mathbb{R}^n$ has minimum squared length

$$d(\Lambda) = \min\{u \cdot u \mid u \in \Lambda, \quad u \neq 0\},$$

theta-function

$$\Theta_\Lambda(\tau) = \sum_{u \in \Lambda} q^{u \cdot u}, \qquad q = e^{\pi i \tau}, \qquad \operatorname{Im}(\tau) > 0,$$

and dual lattice

$$\Lambda^\# = \{v \in \mathbb{R}^n \mid u \cdot v \in \mathbb{Z} \quad \text{for all } u \in \Lambda\}.$$

Let

$$\theta_2(\tau) = \sum_{m=-\infty}^{\infty} q^{(m+\frac{1}{2})^2},$$

$$\theta_3(\tau) = \sum_{m=-\infty}^{\infty} q^{m^2} = \prod_{m=1}^{\infty} (1 - q^{2m})(1 + q^{2m-1})^2, \tag{1}$$

$$\theta_4(\tau) = \sum_{m=-\infty}^{\infty} (-q)^{m^2},$$

and

$$\Delta_8(\tau) = -\tfrac{1}{256}\theta_2\left(\tfrac{1}{2}(\tau - 1)\right)^8 = q \prod_{m=1}^{\infty} \{(1 - q^{2m-1})(1 - q^{4m})\}^8. \tag{2}$$

If $\Lambda = \Lambda^\#$ then Λ is called self-dual. For a self-dual lattice in $\mathbb{R}^n$,

$$\Theta_\Lambda(\tau) = \sum_{r=0}^{\mu} a_r \theta_3(\tau)^{n-8r} \Delta_8(\tau)^r, \tag{3}$$

for uniquely determined integers $a_0, \ldots, a_\mu$, where $n = 8\mu + \nu$, $0 \leqslant \nu \leqslant 7$. For a proof of (3) and an explanation of any undefined terms see, for example, **[14]**. If $a_0, \ldots, a_\mu$ are chosen so as to make the right-hand side of (3) equal to

$$1 + A^*_{\mu+1} q^{\mu+1} + A^*_{\mu+2} q^{\mu+2} + \ldots \tag{4}$$

(containing no power of q between 0 and $\mu + 1$), the result is called an *extremal theta-function*, and a lattice having an extremal theta-function, if there is one, is called an *extremal* lattice. (This is not to be confused with an *extreme* form.) It can be shown (*cf.* **[10]**, **[13]**) that

$$A^*_{\mu+1} > 0 \quad \text{for all } n,$$

which implies

$$d(\Lambda) \leqslant \mu + 1 = [n/8] + 1. \tag{5}$$

A self-dual lattice is extremal, if, and only if, equality holds in (5), by definition. Furthermore (**[10]**)

$$A^*_{\mu+2} < 0 \text{ for all sufficiently large } n,$$

implying that there is a bound n_0 such that extremal lattices exist only for $n \leqslant n_0$. In this paper we determine all the extremal lattices, and prove the result stated in the Abstract. (This solves Open Problem 6 of **[14]**.) A Type II lattice is a self-dual lattice Λ which is also even, *i.e.* satisfies $u \cdot u \in 2\mathbb{Z}$ for all $u \in \Lambda$. The analogous problem of finding all extremal Type II lattices, for which $d(\Lambda) = 2[n/24] + 2$, remains unsolved.

It should be possible to find all self-dual lattices in up to 23 dimensions from Niemeier's list **[12]** of Type II lattices in $\mathbb{R}^{24}$. Such a list would of course include our extremal lattices.

We will sometimes give a *generator matrix* for a lattice: this is an $n \times n$ matrix whose rows span the lattice.

§2. *Dimensions* 1–16. The following theorem is contained in Kneser **[5]**.

THEOREM 1. *The only extremal self-dual lattices in* $\mathbb{R}^n$ *for* $n \leqslant 16$ *are* $\mathbb{Z}^n$ $(1 \leqslant n \leqslant 7)$, $E_8 = D_8{}^+$, D_{12}^+, $(E_7 + E_7)^+$, A_{15}^+, *in* $\mathbb{R}^n$, $\mathbb{R}^8$, $\mathbb{R}^{12}$, $\mathbb{R}^{14}$, $\mathbb{R}^{15}$ *respectively.*

These lattices can all be constructed in a uniform way. We write

$$\Lambda = (\Lambda_1 + \Lambda_2 + \ldots + \Lambda_k)^+$$

to indicate that the n-dimensional integral lattice Λ contains the direct sum

$$\Lambda_1 \oplus \Lambda_2 \oplus \ldots \oplus \Lambda_k$$

of lattices whose dimensions add to n. In this case Λ is generated by this direct sum together with certain *glue vectors*

$$y = y_1 + y_2 + \ldots + y_k,$$

in which y_i belongs to the subspace Λ_i, and therefore to $\Lambda_i{}^\#$ (since y must have integral inner product with every vector in $0 + 0 + \ldots + \Lambda_i + 0 + \ldots + 0$). Since the same lattice will be generated if y_i is augmented by any vector of Λ_i, we may in fact regard y_i as a member of the *dual quotient* $\Lambda_i{}^\#/\Lambda_i$, a finite group.

The lattices

$\mathbb{Z}$ (one integer co-ordinate),

A_n ($n + 1$ integer co-ordinates with zero sum; $n \geqslant 1$),

D_n (n integer co-ordinates with even sum; $n \geqslant 4$),

E_6, E_7, E_8 (defined below)

are the only indecomposable integral lattices generated by vectors of squared length 1 or 2, and so are often used for the Λ_i. E_8 has eight co-ordinates $x_1, \ldots, x_8$ all in $\mathbb{Z}$, or all in $\mathbb{Z} + \frac{1}{2}$, with even sum; E_7 is that part of E_8 with $x_1 + \ldots + x_8 = 0$; and E_6 is the part with $x_1 + \ldots + x_6 = x_7 + x_8 = 0$.

In this notation, due to Kneser [6] and Niemeier [12], the extremal lattices in dimensions $\leqslant 16$ are:

$$\mathbb{Z}^n \quad (n = 1, \ldots, 7),$$

$$E_8 = D_8{}^+ = \langle D_8, (\tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2})\rangle,$$

$$D_{12}^+ = \langle D_{12}, (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\rangle,$$

$$(E_7 + E_7)^+ = \langle E_7 + E_7, ((\tfrac{1}{4})^6, (-\tfrac{3}{4})^2, (\tfrac{1}{4})^6, (-\tfrac{3}{4})^2)\rangle,$$

$$A_{15}^+ = \langle A_{15}, ((\tfrac{1}{4})^{12}, (-\tfrac{3}{4})^4)\rangle.$$

Kneser [5] has shown that these, together with the nonextremal lattice

$$D_{16}^+ = \langle D_{16}, (\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2})\rangle,$$

are the only indecomposable self-dual lattices in dimensions $\leqslant 16$.

Generator matrices for these lattices are:

$\mathbb{Z}^n$: an $n \times n$ identity matrix

$$E_8: \quad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & 0 & 0 \\ & & \cdot & \cdot & \cdot & \cdot & \cdot & \\ 1 & 0 & 0 & 0 & 0 & 0 & - & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix} \qquad (-\text{ stands for } -1)$$

$$D_{12}^+: \quad \begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & - & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ & & \cdots & & & & & \cdots & & & & \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & - & 0 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} & \frac{1}{2} \end{bmatrix}$$

$$(E_7 + E_7)^+: \quad \frac{1}{\sqrt{2}} \left|\begin{array}{ccccccc|ccccccc}
1 & 1 & 1 & 1 & 0 & 0 & 0 & & & & & & & \\
1 & 1 & 0 & 0 & 1 & 1 & 0 & \multicolumn{7}{c}{0} \\
1 & 0 & 1 & 0 & 1 & 0 & 1 & & & & & & & \\
\hline
 & & & & & & & 1 & 1 & 1 & 1 & 0 & 0 & 0 \\
\multicolumn{7}{c|}{0} & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\
 & & & & & & & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\
\hline
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\hline
2 & & & & & & & & & & & & & \\
 & 2 & & & & & & & & & & & & \\
 & & 2 & & & & & & & & & & & \\
 & & & & 2 & & & & & & & & & \\
 & & & & & & & 2 & & & & & & \\
 & & & & & & & & 2 & & & & & \\
 & & & & & & & & & 2 & & & & \\
\end{array}\right|$$

$$A_{15}^+: \quad \tfrac{1}{2} \left|\begin{array}{ccccccccccccccc}
1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 1 & 1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 \\
 & & & & \cdots & & & \cdots & & & \cdots & & & & \\
1 & 1 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 1 & 1 \\
2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2 & 2
\end{array}\right|$$

The theta-functions of these lattices are:

$$\Theta_{\mathbb{Z}^n}(\tau) = \theta_3(\tau)^n,$$

$$\Theta_{E_8}(\tau) = \tfrac{1}{2}\{\theta_2(\tau)^8 + \theta_3(\tau)^8 + \theta_4(\tau)^8\}$$

$$= 1 + 240 \sum_{r=1}^{\infty} \sigma_3(r) q^{2r},$$

where $\sigma_3(r)$ is the sum of the cubes of the divisors of r,

$$\Theta_{D_{12}^+}(\tau) = \tfrac{1}{2}\{\theta_2(\tau)^{12} + \theta_3(\tau)^{12} + \theta_4(\tau)^{12}\}$$

$$= 1 + 264 q^2 + 2048 q^3 + \ldots,$$

for $(E_7 + E_7)^+$:

$$\Theta(\tau) = \theta_3(\tau)^{14} - 28\,\theta_3(\tau)^6\,\Delta_8(\tau)$$

$$= 1 + 252 q^2 + 3136 q^3 + \ldots,$$

and

$$\Theta_{A_{15}^+}(\tau) = \theta_3(\tau)^{15} - 30\,\theta_3(\tau)^7\,\Delta_8(\tau)$$

$$= 1 + 240 q^2 + 3640 q^3 + \ldots.$$

Remarks. The 240 minimum vectors in E_8 are the 240 Cayley numbers of unit norm, or the root system of type E_8 ([**1**]). $(E_7 + E_7)^+$ may be obtained by applying Construction A of [**8**], [**14**] to the binary code generated by the first 7 rows of the above matrix. The first row of the generator matrix for A_{15}^+ is a pseudo-random sequence ([**9**]) of length 15. The other rows (except the last) are cyclic shifts of the first row.

§3. *Dimensions* 17–47.

THEOREM 2. *The only extremal self-dual lattices in* $\mathbb{R}^n$ *for* $17 \leqslant n \leqslant 47$ *are the Leech lattice* Λ_{24} *in* $\mathbb{R}^{24}$ ([**7**]), *and a certain lattice* Λ_{23} *in* $\mathbb{R}^{23}$ *which is derived from the Leech lattice.*

Proof. Suppose first that Λ is a Type II (or even) lattice in $\mathbb{R}^n$, which is also an extremal self-dual lattice. We have $d(\Lambda) \leqslant 2[n/24] + 2$ (from [**10**]), $d(\Lambda) = [n/8] + 1$, and $8 \mid n$, which imply $n \leqslant 24$. From Kneser [**5**] and Niemeier [**12**] Λ must be either E_8 or Λ_{24}.

From now on we suppose that Λ is an extremal self-dual lattice which is not even. Let Λ_0 be the even sublattice:

$$\Lambda_0 = \{u \in \Lambda \mid u \cdot u \in 2\mathbb{Z}\}.$$

Then $\det \Lambda = 1$, $\det \Lambda_0 = 2$, and

$$\Theta_{\Lambda_0}(\tau) = \tfrac{1}{2}\{\Theta_\Lambda(\tau) + \Theta_\Lambda(\tau + 1)\}.$$

To eliminate almost all of the remaining cases we make use of an argument due to Ward [**15**] (who applied it to the weight enumerators of codes). It turns out that the condition that $\Theta_{\Lambda_0^*}(\tau)$ have integer coefficients implies that the coefficients a_r in (3) must be divisible by high powers of two.

From the Jacobi identity (see *e.g.* [**14**; Th. 2])

$$\Theta_\Lambda(\tau) = \left(\frac{i}{\tau}\right)^{n/2} \Theta_\Lambda\left(-\frac{1}{\tau}\right),$$

$$\Theta_{\Lambda_0^*}(\tau) = \Theta_\Lambda(\tau) + \left(\frac{i}{\tau}\right)^{n/2} \Theta_\Lambda\left(-\frac{1}{\tau} + 1\right). \tag{6}$$

Hence

$$\left(\frac{i}{\tau}\right)^{n/2} \Theta_\Lambda\left(-\frac{1}{\tau} + 1\right)$$

has integer coefficients in its q-expansion. Suppose the extremal theta-function (4) is

$$\Theta_\Lambda(\tau) = \sum_{r=0}^{\mu} a_r^* \, \theta_3(\tau)^{n-8r} \Delta_8(\tau)^r. \tag{7}$$

Then

$$\left(\frac{i}{\tau}\right)^{n/2} \Theta_\Lambda\left(-\frac{1}{\tau} + 1\right) = \left(\frac{i}{\tau}\right)^{n/2} \sum_{r=0}^{\mu} (-1)^r 2^{-8r} a_r^* \, \theta_4\left(-\frac{1}{\tau}\right)^{n-8r} \theta_2\left(-\frac{1}{2\tau}\right)^{8r}$$

$$= \sum_{r=0}^{\mu} (-1)^r 2^{-4r} a_r^* \, \theta_2(\tau)^{n-8r} \theta_4(2\tau)^{8r}, \tag{8}$$

which implies

$$2^{12r-n} \mid a_r^* \quad \text{for} \quad n < 12r. \tag{9}$$

The a_r^* are easily computed (by equating (4) and (7)) and condition (9) eliminates all n in the range $17 \leqslant n \leqslant 47$ except 22, 23 and 24. For example, when $n = 17$, $a_0^* = 1$, $a_1^* = -34$, $a_2^* = -204$, and $2^7 \nmid a_2^*$.

Finally, $n = 22$ is eliminated by calculating $\Theta_{\Lambda_0^\#}(\tau)$ explicitly. It is $1 - 11q^{3/2} + \dots$, and the negative coefficient shows that Λ does not exist.

When $n = 24$ the extremal theta-function is equal to the theta-function of the Leech lattice:

$$\begin{aligned}\Theta_\Lambda(\tau) &= \Theta_{\Lambda_{24}}(\tau) \\ &= \Theta_{E_8}(\tau)^3 - \tfrac{45}{16}\{\theta_2(\tau)\,\theta_3(\tau)\,\theta_4(\tau)\}^8 \\ &= 1 + 196560q^4 + 16773120q^6 + \dots .\end{aligned}$$

The uniqueness of the Leech lattice ([**3**], [**12**]) shows that $\Lambda = \Lambda_{24}$. The automorphism group of this lattice has been studied in [**2**], [**4**].

In dimension 23 an extremal lattice Λ must have theta-function

$$\begin{aligned}&\theta_3(\tau)^{23} - 46\,\theta_3(\tau)^{15}\,\Delta_8(\tau) \\ &= 1 + 4600q^3 + 93150q^4 + \dots .\end{aligned}$$

The even sublattice Λ_0 (with theta-function $1 + 93150q^4 + \dots$) has dual lattice $\Lambda_0^\#$ with theta-function (from (6), (8))

$$\begin{aligned}&1 + 4600q^3 + 93150q^4 + \dots \\ &\quad + 2^{23}q^{23/4}(1 + 23q^2 + 253q^4 + \dots) \\ &\quad + 23.2^{12}q^{15/4}(1 + 15q^2 + 105q^4 + \dots)(1 - 16q^2 + 112q^4 + \dots) \\ &= 1 + 4600q^3 + 94208q^{3\frac{3}{4}} + 93150q^4 + \dots .\end{aligned}$$

From this we can see that the four cosets of Λ_0 in $\Lambda_0^\#$ contain nonzero vectors of

squared length	$4 + 2n$	$3\frac{3}{4} + 2n$	$3 + 2n$	$3\frac{3}{4} + 2n$	
coset representative	$y_0 = 0$	y_1	y_2	y_3	(say),

for various integers $n \geqslant 0$. The union of the cosets Λ_0 and $\Lambda_0 + y_2$ is Λ.

Now the lattice M of even integers also has four cosets in its dual $M^\#$, whose nonzero vectors have

squared length	$4 + 2n$	$\frac{1}{4} + 2n$	$1 + 2n$	$\frac{1}{4} + 2n$
coset representative	$z_0 = 0$	$z_1 = \frac{1}{2}$	$z_2 = 1$	$z_3 = -\frac{1}{2}$.

It follows that there is a 24-dimensional lattice

$$(\Lambda_0 + M)^+$$

obtained by extending $\Lambda_0 + M$ by the glue vectors $y_i + z_i$ ($i = 0, 1, 2, 3$), which is extremal and so must be the Leech lattice. Λ is therefore uniquely characterized as the projection onto the 23-space orthogonal to v of the vectors of Λ_{24} which have even inner product with a minimal vector v in Λ_{24}. This completes the proof of Theorem 2.

§4. *Dimensions* $n \geqslant 48$.

THEOREM 3. *No extremal lattice exists in* $\mathbb{R}^n$ *for* $n \geqslant 48$.

Proof. With the same notation as before we show that

$$\Theta_{\Lambda_0^*}(\tau) = \Theta_\Lambda(\tau) + \sum_{r=0}^{\mu} (-16)^{-r} a_r^* \theta_2(\tau)^{n-8r} \theta_4(2\tau)^{8r}$$

contains a negative coefficient for $n \geqslant 48$, which proves the theorem. The q-expansion of the right-hand side is

$$\begin{aligned} &1 + (-1)^\mu 2^{\nu-4\mu} a_\mu^* q^{\nu/4} \\ &+ (-1)^{\mu+1} 2^{\nu-4\mu} q^{\nu/4+2} \{(16\mu - \nu) a_\mu^* + 4096 a_{\mu-1}^*\} \\ &+ \dots . \end{aligned}$$

We will show that $a_\mu^* < 0$ and $a_{\mu-1}^* < 0$ for $\mu \geqslant 6$, from which the negative coefficient is apparent. Applying the Bürmann–Lagrange theorem [**16**; p. 128] to (4) and (7) we obtain, exactly as in [**10**; Eq. (6)],

$$a_s^* = -\frac{n}{s!} \frac{d^{s-1}}{dq^{s-1}} \left\{ \frac{d\theta_3}{dq} \,.\, \theta_3^{\,8s-n-1} \,.\, h^s \right\}_{q=0}, \tag{10}$$

for $0 \leqslant s \leqslant \mu$, where from (2)

$$h = \prod_{m=1}^{\infty} \{(1 - q^{2m-1})(1 - q^{4m})\}^{-8}.$$

Now $d\theta_3/dq$ has non-negative coefficients in its q-expansion. When $s = \mu$ or $\mu - 1$ we have

$$-1 \geqslant 8s - n - 1 \geqslant -16.$$

Let $1 \leqslant k \leqslant 16$. Then from (1)

$$\theta_3^{-k} = \prod_{m=1}^{\infty} (1 - q^{2m})^{-k} \prod_{m=1}^{\infty} (1 + q^{2m-1})^{-2k}.$$

The first product has non-negative coefficients. On the other hand,

$$h^s \prod_{m=1}^{\infty} (1 + q^{2m-1})^{-2k} = \prod_{m=1}^{\infty} (1 - q^{2m-1})^{-(8s-2k)} \times \prod_{m=1}^{\infty} (1 - q^{4m-2})^{-2k} \prod_{m=1}^{\infty} (1 - q^{4m})^{-8s}.$$

If $n \geqslant 48$, $\mu \geqslant 6$, $s \geqslant 5$, then every product on the right-hand side has non-negative coefficients and the first has strictly positive coefficients. Therefore the expression in braces in (10) has strictly positive coefficients, showing that $a_\mu < 0$ and $a_{\mu-1} < 0$. This completes the proof of Theorem 3.

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