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Most Primitive Groups Have Messy Invariants

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Suppose G is a finite group of complex $n \times n$ matrices, and let R^G be the ring of invariants of G: i.e., those polynomials fixed by G. Many authors, from Klein to the present day, have described R^G by writing it as a direct sum $\sum_{j=1}^{\delta} \eta_j \mathbb{C}[\theta_1, ..., \theta_n]$. For example, if G is a unitary group generated by reflections, $\delta = 1$. In this note we show that in general this approach is hopeless by proving that, for any $\epsilon > 0$, the smallest possible δ is greater than $|G|^{n-1-\epsilon}$ for almost all primitive groups. Since for any group we can choose $\delta \leqslant |G|^{n-1}$, this means that most primitive groups are about as bad as they can be. The upper bound on δ follows from Dade's theorem that the θ_i can be chosen to have degrees dividing $|G|$.

Let G be a finite group of complex $n \times n$ matrices. If $f(x_1, ..., x_n)$ is a polynomial in $x_1, ..., x_n$ with complex coefficients, and $A = (a_{ij})$ is an element of G, then $Af(x_1, ..., x_n) = f(\sum a_{1j}x_j, ..., \sum a_{nj}x_j)$ is the polynomial obtained by letting A act as a linear transformation on the variables $x_1, ..., x_n$. The ring R^G of *invariants* of G consists of all polynomials f with $Af = f$ for all $A \in G$. The central problem is to find a description of R^G that is concise and easy to use. Several types of bases for R^G have been considered ([4, Ch. XVII], [32]), but we are concerned here with finding a *direct sum decomposition*

$$R^G = \sum_{j=1}^{\delta} \eta_j \mathbb{C}[\theta_1, ..., \theta_n], \tag{1}$$

where $\theta_1, ..., \theta_n$ are algebraically independent homogeneous invariants and $\eta_1 = 1$, $\eta_2, ..., \eta_\delta$ are certain other homogeneous invariants. It is known from the theory of Cohen-Macaulay rings ([15, Prop. 13]) that R^G always has a direct sum decomposition (1). Then R^G is a finitely generated free $\mathbb{C}[\theta_1, ..., \theta_n]$-module of rank δ. We are interested in the smallest value of δ that can be attained by any choice of $\theta_1, ..., \theta_n$: call this value $\delta(G)$. We shall give examples to show that $\delta(G)$ may

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be large, and then show that for most primitive groups $\delta(G)$ is about as large as it can be.

In [33, Section 3.4.12] T. A. Springer has proved a somewhat similar result for the ring of invariants of binary forms of given degree.

The θ_i and η_j in (1) form a polynomial basis for R^G. In the past century polynomial bases have been given for a number of groups ([1], [10], [13], [16], [21], [23]–[25], [27], [30]–[38]). E. Noether had shown in 1916 ([29]; see also [32], [42]) that any group has a polynomial basis containing no more than $\binom{|G|+n}{n}$ invariants, i.e. about $|G|^n/n!$ for large $|G|$. We had always thought that this was a very weak bound (cf. also [13]), but in fact it is not far off, at least for primitive groups.

We make use of the *Molien series* of G, i.e. the generating function

$$\Phi_G(\lambda) = \sum_{j=0}^{\infty} \dim_{\mathbb{C}}(R^G)_j \lambda^j,$$

in which the coefficient of λ^j is the number of linearly independent homogeneous invariants of degree j. Molien ([28]; see also [4], [27], [32]) showed that the series can be calculated from the identity

$$\Phi_G(\lambda) = \frac{1}{|G|} \sum_{A \in G} \frac{1}{\det(I - \lambda A)}. \tag{2}$$

If R^G has the decomposition (1) then the Molien series may be written as

$$\Phi_G(\lambda) = \frac{\sum_{j=1}^{\delta} \lambda^{e_j}}{(1-\lambda^{d_1})(1-\lambda^{d_2}) \cdots (1-\lambda^{d_n})}, \tag{3}$$

where

$$\deg \eta_j = e_j, \qquad \deg \theta_j = d_j.$$

THEOREM 1.

$$\delta = \frac{d_1 d_2 \cdots d_n}{|G|} \tag{4}$$

Proof. If we equate (2) and (3), multiply by $(1-\lambda)^n$, and let $\lambda \to 1$ we obtain $1/|G| = \delta/d_1 \cdots d_n$.

EXAMPLE 1. If G is a finite unitary group generated by reflections then R^G is a free ring and may be expressed in the form (1) with $\delta(G) = 1$ and $d_1 d_2 \cdots d_n = |G|$ (Shephard and Todd [30]; see also [5]).

The next two examples are six-dimensional groups from Lindsey's list [22].

EXAMPLE 2. Let G be a six-dimensional representation of a nonsplitting central extension of Z_3 by the alternating group $\mathscr{A}_6$, of order $3 \cdot 360$. This group is of interest because it is generated by matrices with two eigenvalues -1 and the rest 1. Such groups are a natural generalization of the class of groups given in Example 1, ([18], [41]). We shall see that $\delta(G) \geqslant 54$, which suggests that the invariants of these groups do not have such a simple description as those of Example 1.

The Molien series may be calculated from (2), and when written in simplest terms (with numerator and denominator relatively prime) is equal to

$$\Phi^{(1)}(\lambda) = \frac{f_1(\lambda)}{\phi_1{}^6\phi_2{}^2\phi_3{}^6\phi_4\phi_5\phi_6{}^2\phi_{12}\phi_{15}} \tag{5}$$

where

$$\begin{aligned} f_1(\lambda) = 1 &- \lambda^3 + 3\lambda^6 + \lambda^9 + 4\lambda^{12} + 3\lambda^{15} + 5\lambda^{18} \\ &+ 3\lambda^{21} + 4\lambda^{24} + \lambda^{27} + 3\lambda^{30} - \lambda^{33} + \lambda^{36}, \end{aligned}$$

and $\phi_r = \phi_r(\lambda)$ is the r-th cyclotomic polynomial. The numerator of (5) (and of (6)–(10) below) is a symmetric polynomial: this is an easy consequence of the fact that all the group elements have determinant one. The Taylor series expansion of the Molien series is

$$1 + 2\lambda^3 + 7\lambda^6 + 16\lambda^9 + 38\lambda^{12} + 76\lambda^{15} + 146\lambda^{18} + \cdots$$

To obtain a lower bound on $\delta(G)$ we make use of Theorem 1. Let the Molien series be written in the form

$$\Phi^{(3)}(\lambda) = \frac{f_3(\lambda)}{(1 - \lambda^{D_1})(1 - \lambda^{D_2}) \cdots (1 - \lambda^{D_6})} \tag{6}$$

where $f_3(\lambda)$ has nonnegative coefficients and the product $D_1 D_2 \cdots D_6$ is a minimum. From Theorem 1,

$$\delta(G) \geqslant \delta_1(G) = \frac{D_1 D_2 \cdots D_6}{|G|}.$$

There are some subtleties in the determination of the minimum value of $D_1 \cdots D_6$, and it seems worthwhile to sketch a method for calculating it. By equating (5) and (6) we see that the ϕ_i's in the denominator of (5) can be partitioned into six sets, say

$$\{\phi_{t_{11}}, \ldots, \phi_{t_{1r_1}}\}$$
$$\cdots$$
$$\{\phi_{t_{61}}, \ldots, \phi_{t_{6r_6}}\}$$

such that the product of the j-th set divides $1 - \lambda^{D_j}$. (Thus the ϕ_i's in each set are distinct.) Let

$$F_j = \text{l.c.m.}\{t_{ji}, \ldots, t_{jr_j}\}.$$

Then the product of the j-th set of ϕ_i's divides $1 - \lambda^{F_j}$, F_j divides D_j, and

$$\delta_1(G) = \frac{D_1 \cdots D_6}{|G|} \geqslant \frac{F_1 \cdots F_6}{|G|}.$$

Consequently, if we multiply the numerator and denominator of $\Phi^{(1)}(\lambda)$ by

$$\rho_{12}(\lambda) = \frac{1 - \lambda^{F_1}}{\phi_{t_{11}} \cdots \phi_{t_{1r_1}}} \cdots \frac{1 - \lambda^{F_6}}{\phi_{t_{61}} \cdots \phi_{t_{6r_6}}},$$

we obtain

$$\Phi^{(2)}(\lambda) = \frac{f_2(\lambda)}{(1 - \lambda^{F_1}) \cdots (1 - \lambda^{F_6})}, \tag{7}$$

which is a form of the Molien series intermediate between $\Phi^{(1)}(\lambda)$ and $\Phi^{(3)}(\lambda)$. Then $\Phi^{(3)}(\lambda)$ is obtained by multiplying the numerator and denominator of $\Phi^{(2)}(\lambda)$ by

$$\rho_{23}(\lambda) = \frac{1 - \lambda^{D_1}}{1 - \lambda^{F_1}} \cdots \frac{1 - \lambda^{D_6}}{1 - \lambda^{F_6}}.$$

In the present example it is not difficult to see that only three partitions of the ϕ_i's need be considered, namely:

(I) $\{\phi_1, \phi_3\}, \{\phi_1, \phi_3\}, \{\phi_1, \phi_3\},$
$\{\phi_1, \phi_2, \phi_3, \phi_6\},$
$\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_6, \phi_{12}\},$
$\{\phi_1, \phi_3, \phi_5, \phi_{15}\},$

with

$$F_1 = F_2 = F_3 = 3, \qquad F_4 = 6, \qquad F_5 = 12, \qquad F_6 = 15.$$

(II) $\{\phi_1, \phi_3\}, \{\phi_1, \phi_3\}, \{\phi_1, \phi_3\}, \{\phi_1, \phi_3\},$
$\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_6, \phi_{12}\},$
$\{\phi_1, \phi_2, \phi_3, \phi_5, \phi_6, \phi_{15}\},$

with

$$F_1 = F_2 = F_3 = F_4 = 3, \qquad F_5 = 12, \qquad F_6 = 30.$$

(III) $\{\phi_1, \phi_3\}, \{\phi_1, \phi_3\}, \{\phi_1, \phi_3\}, \{\phi_1, \phi_3\},$
$\{\phi_1, \phi_2, \phi_3, \phi_6\},$
$\{\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6, \phi_{12}, \phi_{15}\},$

with

$$F_1 = F_2 = F_3 = F_4 = 3, \qquad F_5 = 6, \qquad F_6 = 60.$$

Suppose partition (I) occurs. Then

$$\Phi^{(2)}(\lambda) = \Phi^{(1)}(\lambda) = \frac{1 - \lambda^3 + 3\lambda^6 + \cdots + \lambda^{36}}{(1-\lambda^3)^3(1-\lambda^6)(1-\lambda^{12})(1-\lambda^{15})},$$

and $\rho_{12}(\lambda) = 1$. Since the numerator of $\Phi^{(2)}(\lambda)$ contains a negative coefficient, $\rho_{23}(\lambda) \neq 1$. If we take $\rho_{23}(\lambda) = 1 + \lambda^3$, we obtain

$$\Phi^{(3)}(\lambda) = \frac{1 + 2\lambda^6 + 4\lambda^9 + 5\lambda^{12} + \cdots + \lambda^{39}}{(1-\lambda^3)^2(1-\lambda^6)^2(1-\lambda^{12})(1-\lambda^{15})} \tag{8}$$

with nonnegative coefficients in the numerator, and $D_1 \cdots D_6 = 3^2 \cdot 6^2 \cdot 12 \cdot 15 = 58320$. Any other choice of $\rho_{23}(\lambda)$ will give a larger value of $D_1 \cdots D_6$.

Suppose instead that partition (II) occurs. Now

$$\rho_{12}(\lambda) = \phi_{10}\phi_{30}$$

and

$$\Phi^{(2)}(\lambda) = \frac{1 - 2\lambda^3 + \cdots + \lambda^{48}}{(1-\lambda^3)^4(1-\lambda^{12})(1-\lambda^{30})}.$$

Again $\rho_{23}(\lambda) \neq 1$, and so $D_1 \cdots D_6 \geqslant 58320$. Similarly for partition (III). We conclude that the minimum value of $D_1 \cdots D_6$ is 58320, and therefore

$$\delta(G) \geqslant \delta_1(G) = \frac{58320}{3 \cdot 360} = 54.$$

EXAMPLE 3. In this example we consider a member of another important class of groups, a six-dimensional representation of $SL_2(13)$ of order 2184 ([9, Section 38]). Using the method described in Example 2 (there are now 29 partitions to be considered) we find that the Molien series when written with minimum $D_1 \cdots D_6$ is

$$\begin{aligned}\Phi^{(3)}(\lambda) &= \frac{1 + \lambda^{10} + 5\lambda^{12} + 6\lambda^{14} + 12\lambda^{16} + 18\lambda^{18} + 25\lambda^{20} + \cdots + \lambda^{70}}{(1-\lambda^4)(1-\lambda^8)(1-\lambda^{12})^2(1-\lambda^{14})(1-\lambda^{26})} \\ &= 1 + \lambda^4 + 2\lambda^8 + \lambda^{10} + 9\lambda^{12} + 8\lambda^{14} + 22\lambda^{16} + 27\lambda^{18} + 54\lambda^{20} + \cdots\end{aligned} \tag{9}$$

Consequently

$$\delta(G) \geqslant \delta_1(G) = 768.$$

EXAMPLE 4. Let G be the 24-dimensional representation of the Conway group $\cdot 0$, of order 8315553613086720000 ([6]). The Molien series is

$$\Phi_{\cdot 0}(\lambda) = p(\lambda)/q(\lambda), \tag{10}$$

where

$$p(\lambda) = 1 + \lambda^{12} + \lambda^{26} + 2\lambda^{28} + \lambda^{30} + \lambda^{32} + 3\lambda^{34} + \cdots$$
$$+ 401078\lambda^{100} + \cdots + \lambda^{916}$$

and

$$q(\lambda) = (1 - \lambda^2)(1 - \lambda^{16})(1 - \lambda^{18})(1 - \lambda^{20})(1 - \lambda^{22})(1 - \lambda^{24})^3(1 - \lambda^{30})^2$$
$$\cdot (1 - \lambda^{32})^2(1 - \lambda^{36})^2(1 - \lambda^{40})(1 - \lambda^{42})(1 - \lambda^{46})(1 - \lambda^{52})(1 - \lambda^{56})$$
$$\cdot (1 - \lambda^{60})(1 - \lambda^{66})(1 - \lambda^{70})(1 - \lambda^{78})(1 - \lambda^{84}), \text{ of degree } 940.$$

Thus

$$\Phi_{\cdot 0}(\lambda) = \frac{1}{1 - \lambda^2} (1 + \lambda^{12} + \lambda^{16} + \lambda^{18} + \lambda^{20} + \lambda^{22} + 3\lambda^{24}$$
$$+ \lambda^{26} + 3\lambda^{28} + 4\lambda^{30} + 5\lambda^{32} + 5\lambda^{34} + 10\lambda^{36} + 8\lambda^{38}$$
$$+ 14\lambda^{40} + 17\lambda^{42} + 22\lambda^{44} + 27\lambda^{46} + 44\lambda^{48} + 45\lambda^{50} + \cdots)$$

We have not attempted to minimize $D_1 \cdots D_{24}$ in this example. However, if (10) is the minimum—and our experience suggests it should be close to the minimum then $\delta(G) \geqslant \delta_1(G) = D_1 \cdots D_{24}/|G| = 205679393714995200$.

Example 5. The direct product $\mathscr{S}_3 \times Z_m$ of the symmetric group of order 6 with a cyclic group of order m, where m is prime to 6, has a two-dimensional representation ([16]) for which $\delta(G)$ is m.

These examples illustrate the fact that in general $\delta(G)$ is large. We now give our bounds on $\delta(G)$. The upper bound is a consequence of the following (unpublished) theorem of Dade, which we include with his permission.

Let $R = \mathbb{C}[x_1, ..., x_n]$. A sequence $\theta_1, ..., \theta_m$ in R is called an *R-sequence* ([20, p. 84], [43, II, p. 394]) if the image of θ_{k+1} is neither a unit nor a zero-divisor in $R/(R\theta_1 + \cdots + R\theta_k)$, for $0 \leqslant k < m$.

Theorem 2 (Dade [7]). *With G as above, there exists an R-sequence $\theta_1, ..., \theta_n$ such that each θ_i is a homogeneous invariant in R^G of degree dividing $|G|$.*

Remark. Once such an R-sequence has been found it is straightforward to show that there are further invariants $\eta_1, ..., \eta_\delta$ such that (1) holds (see for example [35]).

Proof. Let R_1 consist of the homogeneous linear polynomials in R. Choose any non-zero $f_1 \in R_1$, and let $f_{11} = f_1, f_{12}, ..., f_{1a_1}$ be the distinct images of f_1 under G. Set

$$\theta_1 = \prod_{j=1}^{a_1} f_{1j}$$

Suppose $\theta_1, \ldots, \theta_i$ ($1 \leqslant i < n$) have been found. Choose $f_{i+1} \in R_1$ such that

$$f_{i+1} \text{ is not in the prime ideal } (f_{1c_1}, f_{2c_2}, \ldots, f_{ic_i}) \tag{11}$$

for all $1 \leqslant c_1 \leqslant a_1, \ldots, 1 \leqslant c_i \leqslant a_i$. (Since there are only finitely many of these ideals, and each intersects R_1 in a vector space of dimension $i < n$, such an f_{i+1} can always be found.) Let $f_{i+1,1} = f_{i+1}, f_{i+1,2}, \ldots, f_{i+1,a_{i+1}}$ be the distinct images of f_{i+1} under G, and set

$$\theta_{i+1} = \prod_{j=1}^{a_{i+1}} f_{i+1,j}.$$

The construction implies that $f_{i+1,j}$ is not in the ideal (11) for all $1 \leqslant c_1 \leqslant a_1, \ldots, 1 \leqslant c_i \leqslant a_i$. Repeat until $\theta_1, \ldots, \theta_n$ have been obtained. Clearly the θ_j are homogeneous invariants of degrees dividing $|G|$. That $\theta_1, \ldots, \theta_n$ is an R-sequence follows from N. Bourbaki, Groupes et Algebras de Lie, Chap. 5, Exercise 5, Hermann, Paris, 1968, p. 137.

COROLLARY

$$\delta(G) \leqslant |G|^{n-1}.$$

Proof. Immediate from Theorems 1 and 2.

THEOREM 3. *Let n be fixed, and let $\mathscr{C}$ be any infinite family of finite, inequivalent, $n \times n$ complex groups that satisfies*

(A1) *every element of $\mathscr{C}$ is irreducible, and*

(A2) *there are finitely many abstract groups $H_1, \ldots, H_s$ such that if $G \in \mathscr{C}$ then $G/Z(G) \simeq H_k$ for some k, where $Z(G)$ is the center of G.*

Then for any fixed $\epsilon > 0$,

$$\frac{\delta(G)}{|G|^{n-1-\epsilon}} \to \infty \qquad as \quad |G| \to \infty \tag{13}$$

and so $\delta(G) > |G|^{n-1-\epsilon}$ for almost all $G \in \mathscr{C}$.

Proof. For $G \in \mathscr{C}$ let $Z(G)$ have order m. Then by [11, Theorem 1.4], $Z(G) = \{\omega^j I : 0 \leqslant j < m\}$, where $\omega = e^{2\pi i/m}$. Suppose $G/Z(G) \simeq H_k$. Then $|G| = m\,|H_k| \leqslant mh$, where $h = \max\{|H_1|, \ldots, |H_s|\}$. The degree of any homogeneous invariant of G must be divisible by m, so in (4) each $d_j \geqslant m$. Then

$$\frac{\delta(G)}{|G|^{n-1-\epsilon}} = \frac{d_1 \cdots d_n}{|G|^{n-\epsilon}} \geqslant \frac{m^n}{(mh)^{n-\epsilon}} = \frac{m^\epsilon}{h^{n-\epsilon}}.$$

As $|G| \to \infty$ we must have $|Z(G)| = m \to \infty$ (since there are only finitely many H_k's), which proves (8) and the theorem.

Let G be an irreducible finite group of $n \times n$ matrices acting on a complex n-dimensional vector space V. Then G is *imprimitive* if V is a direct sum of vector spaces:

$$V = V_1 \oplus \cdots \oplus V_m \tag{14}$$

where $m \geqslant 2$, each $V_i \neq 0$, and for all $A \in G$ and all i, $A(V_i) = V_j$ for some $j = j(A, i)$. Otherwise G is *primitive*. G is *quasiprimitive* if for any normal subgroup N of G, V is a direct sum (14) where the V_i are invariant under N and afford equivalent representations of N. By Clifford's theorem ([8], [9], [14]) a primitive group is quasiprimitive. In the classification of linear groups it is customary to restrict attention to primitive or quasiprimitive groups ([1], [2], [12], [16], [18], [22], [40]) since imprimitive groups are induced from a representation of smaller degree of a proper subgroup (cf. [9, Th. 14.1 (4)]).

THEOREM 4. *The conclusion of Theorem* 3 *applies to the family of inequivalent* $n \times n$ *primitive groups or quasiprimitive groups.*

Proof. Suppose G is quasiprimitive, and let H be any abelian normal subgroup of G. By hypothesis V is the direct sum of one-dimensional subspaces that are invariant under H, and the elements of H can be taken to be scalar matrices. Hence $H \subset Z(G)$. Therefore by Jordan's theorem ([1], [9]), $|G/Z(G)|$ is bounded, and so (A2) holds, proving the theorem.

Remark. It seems likely that a similar result will hold for the class of all $n \times n$ groups, or all $n \times n$ irreducible groups. The conclusion will not be as strong, however, since for any $n \geqslant 2$ there is an infinite family of irreducible imprimitive unitary groups generated by reflections with $\delta(G) = 1$ (the groups $G(m, p, n)$ in [30]).

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