

## Note

# New Bounds on the Number of Unit Spheres That Can Touch a Unit Sphere in $n$ Dimensions

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New upper bounds are given for the maximum number,  $\tau_n$ , of nonoverlapping unit spheres that can touch a unit sphere in  $n$ -dimensional Euclidean space, for  $n \leq 24$ . In particular it is shown that  $\tau_3 = 240$  and  $\tau_{24} = 196560$ .

The problem of finding the maximum number,  $\tau_3$ , of billiard balls that can touch another billiard ball has a long and fascinating history (see [2]); the answer is known to be 12. But up to now no corresponding numbers  $\tau_n$  have been determined for higher dimensions.

We shall use the following theorem.

**THEOREM.** *Assume  $n \geq 3$ . If  $f(t)$  is a real polynomial which satisfies*

(C1)  $f(t) \leq 0$  for  $-1 \leq t \leq \frac{1}{2}$ , and

(C2) *the coefficients in the expansion of  $f(t)$  in terms of Jacobi polynomials [1, Chap. 22]*

$$f(t) = \sum_{i=0}^k f_i P_i^{\alpha, \alpha}(t),$$

where  $\alpha = (n - 3)/2$ , satisfy  $f_0 > 0, f_1 \geq 0, \dots, f_k \geq 0$ , then  $\tau_n$  is bounded by

$$\tau_n \leq \frac{f(1)}{f_0}.$$

This theorem may be found (implicitly or explicitly) in [3, 4, 6], but for completeness we sketch a simplified proof. A *spherical code*  $C$  is any finite

subset of the unit sphere in  $n$  dimensions. For  $-1 \leq t \leq 1$  let

$$A_t = \delta_t \cdot (1/|C|) \cdot (\text{number of ordered pairs } c, c' \in C \text{ such that } \langle c, c' \rangle = t),$$

where  $\delta_t$  is a Dirac delta-function,  $|C|$  is the cardinality of  $C$ , and  $\langle \cdot, \cdot \rangle$  is the usual inner product. Then  $\int_{-1}^1 A_t dt = |C|$ . For all  $k \geq 0$  we have

$$\int_{-1}^1 A_t P_k^{\alpha, \alpha}(t) dt = \frac{1}{|C|} \sum_{c, c' \in C} P_k^{\alpha, \alpha}(\langle c, c' \rangle) \geq 0,$$

since the kernel  $P_k^{\alpha, \alpha}(\langle x, y \rangle)$  is positive definite.

If there is an arrangement of  $\tau$  unit spheres  $S_1, \dots, S_\tau$  touching another unit sphere  $S_0$ , the points of contact of  $S_0$  with  $S_1, \dots, S_\tau$  form a spherical code  $C$  with  $A_t = 0$  for  $\frac{1}{2} < t < 1$ . It follows that an upper bound to  $\tau_n$  is given by the optimal solution to the following linear programming problem: choose the  $A_t$  ( $-1 \leq t \leq \frac{1}{2}$ ) so as to maximize  $\int_{-1}^{1/2} A_t dt$  subject to the constraints

$$A_t \geq 0 \quad \text{for } -1 \leq t \leq 1/2,$$

and

$$\int_{-1}^{1/2} A_t P_k^{\alpha, \alpha}(t) dt \geq -P_k^{\alpha, \alpha}(1), \quad \text{for } k = 0, 1, \dots$$

The theorem now follows by passing to the dual problem, and using the fact that any feasible solution to the dual problem is an upper bound to the optimal solution of the original problem.

For  $n = 8$  we apply the theorem with

$$\begin{aligned} f(t) &= \frac{320}{3} (t+1) \left(t + \frac{1}{2}\right)^2 t^2 \left(t - \frac{1}{2}\right) \\ &= P_0 + \frac{16}{7} P_1 + \frac{200}{63} P_2 + \frac{832}{231} P_3 \\ &\quad + \frac{1216}{429} P_4 + \frac{5120}{3003} P_5 + \frac{2560}{4641} P_6, \end{aligned} \tag{1}$$

where  $P_i$  stands for  $P_i^{2.5, 2.5}(t)$ , and obtain  $\tau_8 \leq 240$ . Similarly for  $n = 24$  we take

$$\begin{aligned}
 f(t) &= \frac{1490944}{15} (t + 1) \left(t + \frac{1}{2}\right)^2 \left(t + \frac{1}{4}\right)^2 t^2 \left(t - \frac{1}{4}\right)^2 \left(t - \frac{1}{2}\right) \\
 &= P_0 + \frac{48}{23} P_1 + \frac{1144}{425} P_2 + \frac{12992}{3825} P_3 + \frac{73888}{22185} P_4 \\
 &\quad + \frac{2169856}{687735} P_5 + \frac{59062016}{25365285} P_6 + \frac{4472832}{2753575} P_7 \\
 &\quad + \frac{23855104}{28956015} P_8 + \frac{7340032}{20376455} P_9 + \frac{7340032}{80848515} P_{10}, \tag{2}
 \end{aligned}$$

where  $P_i$  stands for  $P_i^{10.5,10.5}(t)$ , and obtain  $\tau_{24} \leq 196560$ . Since each sphere in the  $E_8$  lattice packing in 8 dimensions touches 240 others, and each sphere in the Leech lattice packing in 24 dimensions touches 196560 others [5], we have determined  $\tau_8$  and  $\tau_{24}$ .

For other values of  $n$  below 24 we were unable to find such simple and effective polynomials. The best polynomial we have found for  $n = 4$ , for example, is  $f(t) = P_0 + a_1 P_1 + a_2 P_2 + \dots + a_9 P_9$ , where  $a_1 = 2.412237$ ,  $a_2 = 3.261973$ ,  $a_3 = 3.217960$ ,  $a_4 = 2.040011$ ,  $a_5 = 0.853848$ ,  $a_6 = a_7 = a_8 = 0$ ,  $a_9 = 0.128520$  (shown to 6 decimal places, although we actually used 17 places), and  $P_i$  stands for  $P_i^{0.5,0.5}(t)$ . This implies  $\tau_4 \leq 25.5585$ . This polynomial was found by the following method. First replace (C1) by a finite set of inequalities at the points  $t_j = -1 + 0.0015j$  ( $0 \leq j \leq 1000$ ). Second, choose a value of  $k$ , and use linear programming to find  $f_1^*, \dots, f_k^*$  so as to minimize

$$\sum_{i=1}^k f_i^* P_i^{\alpha,\alpha}(1)$$

subject to the constraints

$$f_i^* \geq 0 \quad (1 \leq i \leq k), \quad \sum_{i=1}^k f_i^* P_i^{\alpha,\alpha}(t_j) \leq -1 \quad (0 \leq j \leq 1000).$$

Let  $f^*(t)$  denote the polynomial  $1 + \sum_{i=1}^k f_i^* P_i^{\alpha,\alpha}(t)$ . Of course this need not satisfy (C1) for *all* points  $t$  on the interval  $[-1, \frac{1}{2}]$ . Let  $\epsilon$  be chosen to be greater than the maximum value of  $f^*(t)$  on  $[-1, \frac{1}{2}]$  ( $\epsilon$  may be calculated by finding the zeros of the derivative of  $f^*(t)$ ). Then  $f(t) = f^*(t) - \epsilon$  satisfies (C1) and (C2), and so

$$\tau_n \leq \frac{f^*(1) - \epsilon}{1 - \epsilon}.$$

All the upper bounds shown in Table I, except for  $n = 17$ , were obtained in this way. The degree  $k$  was allowed to be as large as 30, but in all the cases considered the degree of the best polynomial (given in the third column of the

table) did not exceed 14. For  $n = 8$  and  $n = 24$  the form of the polynomials obtained in this way led us to (1) and (2), but for the other values of  $n$  no such simple expression suggested itself.

For  $n = 17$  we made use of the additional inequalities

$$\int_{-1}^{-3^{1/2}/2} A_t dt \leq 1 \quad \text{and} \quad \int_{-1}^{-(2/3)^{1/2}} A_t dt \leq 2$$

TABLE I

Range of Possible Values of  $\tau_n$ , the Maximum Number of Unit Spheres That Can Touch a Unit Sphere in  $n$  Dimensions

$n$	$\tau_n$	deg <sup>a</sup>
1	2	
2	6	
3	12	
4	24-25	9
5	40-46	10
6	72-82	10
7	126-140	10
8	240	6
9	306-380	11
10	500-595	11
11	582-915	11
12	840-1416	11
13	1130-2233	12
14	1582-3492	12
15	2564-5431	12
16	4320-8313	13
17	5346-12215	13
18	7398-17877	13
19	10668-25901	13
20	17400-37974	13
21	27720-56852	13
22	49896-86537	14
23	93150-128096	14
24	196560	10

<sup>a</sup> The degree of the polynomial used to obtain the upper bound.

to obtain  $\tau_{17} \leq 12215$ . Other inequalities of this type could probably be used to obtain further improvements of these results. Unfortunately for  $n = 3$  our methods only give  $\tau_3 \leq 13$ .

These upper bounds are a considerable improvement over the old bounds [2, 5, 7]. For example, the bounds given in [5] (which are based on a still unproved conjecture of Coxeter [2]) are 26, 48, 85, 146, and 244 for  $n = 4, 5, 6, 7,$  and  $8,$  respectively. The lower bounds in the table are taken from [5, 8, 9].

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