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UNIQUENESS OF CERTAIN SPHERICAL CODES

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1. Introduction. In this paper we show that there is essentially only one way of arranging 240 (resp. 196560) nonoverlapping unit spheres in $\mathbf{R}^8$ (resp. $\mathbf{R}^{24}$) so that they all touch another unit sphere, and only one way of arranging 56 (resp. 4600) spheres in $\mathbf{R}^8$ (resp. $\mathbf{R}^{24}$) so that they all touch two further, touching spheres. The following tight spherical t-designs are unique: the 5-design in Ω_7, the 7-designs in Ω_8 and Ω_{23}, and the 11-design in Ω_{24}. It was shown in [**20**] that the maximum number of nonoverlapping unit spheres in $\mathbf{R}^8$ (resp. $\mathbf{R}^{24}$) that can touch another unit sphere is 240 (resp. 196560). Arrangements of spheres meeting these bounds can be obtained from the E_8 and Leech lattices, respectively. The present paper shows that these are the only arrangements meeting these bounds. In [**2**], [**3**], it was shown that there are no tight spherical t-designs for $t \geqq 8$ except for the tight 11-design in Ω_{24}. The present paper shows that this and three other tight t-designs are also unique. There is already a considerable body of literature concerning the uniqueness of these lattices and their associated codes and groups ([**5**], [**6**], [**8**], [**11**], [**13**], [**17**]–[**19**], [**21**], [**22**], [**27**], [**28**]). However the results given here are believed to be new.

Our notation is that Ω_n denotes the unit sphere in $\mathbf{R}^n$ and (,) is the usual inner product. An *(n, M, s) spherical code* is a subset C of Ω_n of size M such that $(\mathbf{u}, \mathbf{v}) \leqq s$ for all $\mathbf{u}, \mathbf{v} \in C$, $\mathbf{u} \neq \mathbf{v}$.

Examples of spherical codes may be obtained from sphere packings ([**15**], [**25**]) via the following theorem, whose elementary proof is omitted.

THEOREM 1. *In a packing of unit spheres in $\mathbf{R}^n$ let $S_1, \ldots, S_k$ be a set of spheres such that S_i touches S_j for all $i \neq j$. Suppose there are further spheres $T_1, \ldots, T_M$ each of which touches all the S_i. Then after rescaling the centers of $T_1, \ldots, T_M$ form an $(n - k + 1, M, 1/(k + 1))$ spherical code.*

Example 2. In the E_8 lattice packing in $\mathbf{R}^8$ there are 240 spheres touching each sphere, 56 that touch each pair of touching spheres, 27 that touch each triple of mutually touching spheres, and so on. From Theorem 1 the centers of these sets of spheres give rise to (8, 240, 1/2), (7, 56, 1/3), (6, 27, 1/4), (5, 16, 1/5), (4, 10, 1/6) and (3, 6, 1/7) spherical codes.

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Example 3. Similarly the Leech lattice in $\mathbf{R}^{24}$ ([**5**], [**14**], [**16**], [**26**]) gives rise to (24, 196560, 1/2), (23, 4600, 1/3), (22, 891, 1/4), (21, 336, 1/5), (20, 170, 1/6). . . spherical codes.

If C is an (n, M, s) spherical code and $\mathbf{u} \in C$ the *distance distribution of* C *with respect to* $\mathbf{u}$ is the set of numbers $\{A_t(u), -1 \leqq t \leqq 1\}$, where

$$A_t(\mathbf{u}) = |\{\mathbf{v} \in C: (\mathbf{u}, \mathbf{v}) = t\}|,$$

and the *distance distribution of* C is the set of numbers $\{A_t, -1 \leqq t \leqq 1\}$, where

$$A_t = \frac{1}{M} \sum_{\mathbf{u} \in C} A_t(\mathbf{u}).$$

Then the A_t satisfy

$$A_1 = 1,$$

$$A_t = 0 \quad \text{for } s < t < 1,$$

$$\sum_{-1 \leqq t \leqq s} A_t = M - 1,$$

and

$$\sum_{-1 \leqq t \leqq s} A_t P_k(t) \geqq -P_k(1), \quad \text{for } k = 1, 2, 3, \ldots,$$

where $P_k(x) = P_k^{(n-3)/2,(n-3)/2}(x)$ is a Jacobi polynomial in the notation of [**1**, Chapter 2]. For a proof of the last inequality see [**9**], [**12**], [**16**] or [**20**]. For a specified value of s an upper bound to M is therefore given by the following linear programming problem.

($P1$) Choose $\{A_t, -1 \leqq t \leqq s\}$ so as to maximize

$$\sum_{-1 \leqq t \leqq s} A_t$$

subject to the inequalities

$$A_t \geqq 0,$$

$$\sum_{-1 \leqq t \leqq s} A_t P_k(t) \geqq -P_k(1), \quad \text{for } k = 1, 2, 3, \ldots. \tag{1}$$

The dual problem may be stated as follows (compare the argument in [**18**, Chapter 17, § 4]).

($P2$) Choose an integer N and a polynomial $f(t)$ of degree N, say

$$f(t) = \sum_{k=0}^{N} f_k P_k(t),$$

so as to minimize $f(1)/f_0$ subject to the inequalities

$$f_0 > 0, f_k \geqq 0 \quad \text{for } k = 1, 2, \ldots, N, \tag{2}$$

$$f(t) \leqq 0 \quad \text{for } -1 \leqq t \leqq s. \tag{3}$$

Since any feasible solution to the dual problem is an upper bound to the optimal solution of the primal problem, we have

(4) $\quad M \leqq f(1)/f_0$

for any polynomial $f(t)$ satisfying (2) and (3).

2. Uniqueness of the code of size 240 in Ω_8.

THEOREM 4 ([**20**]). *If C is an $(8, M, 1/2)$ code then $M \leqq 240$.*

Proof. Consider the polynomial

$$f(t) = \frac{320}{3}(t+1)\left(t+\frac{1}{2}\right)^2 t^2\left(t-\frac{1}{2}\right)$$

$$= P_0 + \frac{16}{7}P_1 + \frac{200}{63}P_2 + \frac{832}{231}P_3 + \frac{1216}{429}P_4 + \frac{5120}{3003}P_4 + \frac{2560}{4641}P_6,$$

where P_k stands for $P_k^{2.5,2.5}(t)$. This satisfies (2) and (3) with $s = 1/2$, so from (4) we have $M \leqq f(1)/f_0 = 240$.

THEOREM 5. *If* (a) *C is an $(8, 240, 1/2)$ code then* (b) *C is a tight spherical 7-design in Ω_8 (cf.* [**9**], [**10**]), (c) *C carries a 4-class association scheme (cf.* [**7**], [**26**]), (d) *the intersection numbers of this association scheme are uniquely determined, and* (e) *the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by*

$$A_1(\mathbf{u}) = A_{-1}(\mathbf{u}) = 1,$$

(6) $$A_{1/2}(\mathbf{u}) = A_{-1/2}(\mathbf{u}) = 56,$$

$$A_0(\mathbf{u}) = 126.$$

Proof. Let $\{A_t\}$ be the distance distribution of C. Then $\{A_t\}$ is an optimal solution to the primal problem ($P1$), and the polynomial $f(t)$ in (5) is an optimal solution to the dual problem ($P2$). The dual variables $f_1, \ldots, f_6$ are nonzero, so by the theorem of complementary slackness [**23**] the primal constraints (1) must hold with equality for $k = 1, \ldots, 6$.

The dual constraints (3) do not hold with equality except for $t = -1$, $\pm 1/2$ and 0. Therefore the primal variables must vanish everywhere except perhaps for A_{-1}, $A_{\pm 1/2}$ and A_0. From (1) these numbers satisfy the equations

(7) $$A_{-1}P_k(-1) + A_{-1/2}P_k(-\tfrac{1}{2}) + A_0P_k(0) + A_{1/2}P_k(\tfrac{1}{2}) = -P_k(1),$$

for $k = 1, 2, \ldots, 6$. Thus

$$
\tag{8}
\begin{bmatrix}
1 & 1 & 1 & 1 \\
-\frac{7}{2} & -\frac{7}{4} & 0 & \frac{7}{4} \\
\frac{63}{8} & \frac{9}{8} & -\frac{9}{8} & \frac{9}{8} \\
-\frac{231}{16} & \frac{33}{64} & 0 & -\frac{33}{64} \\
\frac{3003}{128} & -\frac{429}{256} & \frac{143}{128} & -\frac{429}{256} \\
-\frac{9009}{256} & \frac{1287}{1024} & 0 & -\frac{1287}{1024} \\
\frac{51051}{1024} & \frac{663}{2048} & -\frac{1105}{1024} & \frac{663}{2048}
\end{bmatrix}
\begin{bmatrix}
A_{-1} \\ A_{-1/2} \\ A_0 \\ A_{1/2}
\end{bmatrix}
=
\begin{bmatrix}
239 \\ -\frac{7}{2} \\ -\frac{63}{8} \\ -\frac{231}{16} \\ -\frac{3003}{128} \\ -\frac{9009}{256} \\ -\frac{51051}{1024}
\end{bmatrix}
$$

The unique solution is

(9) $\quad A_{-1} = 1, A_{-1/2} = A_{1/2} = 56, A_0 = 126.$

Since $A_{-1}(\mathbf{u}) \leqq 1$ and $A_{-1} = 1$, we have $A_{-1}(\mathbf{u}) = 1$ for all $\mathbf{u} \in C$, and so the code is antipodal [**9**, p. 373]. Therefore (7) also holds for $k = 7$ and by [**9**, Theorem 5.5] C is a spherical 7-design. By [**9**, Definition 5.13] the design is tight, since $|C| = 2\binom{10}{3}$. By [**9**, Theorem 7.5] C carries a 4-class association scheme. Therefore $A_t(\mathbf{u}) = A_t$ is independent of $\mathbf{u}$ for all t. This proves (b), (c) and (e). The numbers (9) are the valencies of the association scheme, and by [**9**, Theorem 7.4] determine all the intersection numbers. This proves (d).

THEOREM 6. *If condition* (b) *of Theorem* 5 *holds then so do* (a), (c), (d) *and* (e).

Proof. By definition $|C| = 2\binom{10}{3}$. From [**9**, Theorem 5.12] the inner products between the members of C are ± 1 and the zeros of

$$C_3(x) = 160(x + \tfrac{1}{2})x(x - \tfrac{1}{2}).$$

Thus all the A_t are zero except perhaps for $A_{\pm 1}$, $A_{\pm 1/2}$ and A_0. From [**9**, Theorem 5.5] Eq. (7) holds for $k = 1, 2, \ldots, 7$. The rest of the proof is the same as for Theorem 5.

In Example 2 we saw that the minimal vectors in the E_8 lattice form an (8, 240, 1/2) code. Thus conditions (a)–(e) of Theorem 5 apply to this code. Conversely we have:

THEOREM 7. *If C is a tight spherical 7-design in Ω_8 there is an orthogonal transformation mapping C onto the minimal vectors of the E_8 lattice.*

Proof. From Theorem 6 the possible inner products in C are 0, $\pm 1/2$, ± 1. Let $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_{240}\}$ and let L be the lattice in $\mathbf{R}^8$ consisting of the vectors

$$\sum_{i=1}^{240} a_i \cdot \sqrt{2}\mathbf{u}_i, \quad a_i \in \mathbf{Z}.$$

Then L is an even integral lattice (cf. [**19**]). All such lattices have been classified (see [**13**], [**19**]), and are direct sums of the lattices $A_n (n \geqq 1)$, $D_n (n \geqq 4)$ and $E_n (n = 6, 7, 8)$. The only lattice of this type with at least 240 minimal vectors is E_8, so L is isometric to E_8 and C is isometric to the minimal vectors in E_8.

By combining Theorems 5 and 7 we obtain:

THEOREM 8. *There is a unique way (up to isometry) of arranging 240 nonoverlapping unit spheres in $\mathbf{R}^8$ so that they all touch another unit sphere.*

3. Uniqueness of the code of size 56 in Ω_7.

THEOREM 9. *If C is a $(7, M, 1/3)$ code then $M \leqq 56$.*

Proof. The proof here is parallel to the proof of Theorem 4, using the polynomial

$$f(t) = (t + 1)(t + 1/3)^2(t - 1/3).$$

THEOREM 10. *If* (a) *C is a $(7, 56, 1/3)$ code then* (b) *C is a tight spherical 5-design in Ω_7,* (c) *C carries a 3-class association scheme,* (d) *the intersection numbers of this association scheme are uniquely determined, and* (e) *the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by*

$$A_1(\mathbf{u}) = A_{-1}(\mathbf{u}) = 1,$$

$$A_{1/3}(\mathbf{u}) = A_{-1/3}(\mathbf{u}) = 27. \tag{10}$$

Conversely (b) *implies* (a), (c), (d) *and* (e).

Proof. The proof is parallel to the proofs of Theorems 5 and 6.

For example the $(7, 56, 1/3)$ code given in Example 2 has properties (a)–(e). Conversely we have:

THEOREM 11. *If C is a tight spherical 5-design in Ω_7 there is an orthogonal transformation mapping C onto the $(7, 56, 1/3)$ code obtained from the E_8 lattice.*

Proof. Let C consist of the points $\mathbf{u}_1, \ldots, \mathbf{u}_{56}$ lying on a unit sphere $\mathbf{R}^7$ centered at $\mathbf{P}$. Choose a point $\mathbf{O}$ (in $\mathbf{R}^8$) so that $\measuredangle\, \mathbf{u}_i\mathbf{O}\mathbf{P} = \pi/3$ for all i, and thus

$$\cos \measuredangle\, \mathbf{u}_i\mathbf{O}\mathbf{u}_j = (1 + 3 \cos \measuredangle\, \mathbf{u}_i\mathbf{P}\mathbf{u}_j)/4$$

for all i, j. Let $\mathbf{v}$ be a unit vector along $\mathbf{OP}$ (see Fig. 1). From Theorem 10 $\cos \angle \mathbf{u}_i\mathbf{P}\mathbf{u}_j$ takes the values ± 1 and $\pm 1/3$, so $\cos \measuredangle \mathbf{u}_i\mathbf{O}\mathbf{u}_j$ takes the values 0, $\pm 1/2$ and 1. It follows that the vectors $\sqrt{3/2}\,\mathbf{Ou}_i$ $(1 \leqq i \leqq 56)$ span an even integral lattice, containing at least $2(56 + 1) = 114$ minimal vectors (corresponding to $\pm C$, $\pm \mathbf{v}$). This lattice must therefore be either E_8 or $E_7 \oplus A_1$, and the latter is incompatible with (10).

By combining Theorems 10 and 11 we obtain:

THEOREM 12. *There is a unique way (up to isometry) of arranging* 56 *nonoverlapping unit spheres in* $\mathbf{R}^8$ *so that they all touch two further, touching, unit spheres.*

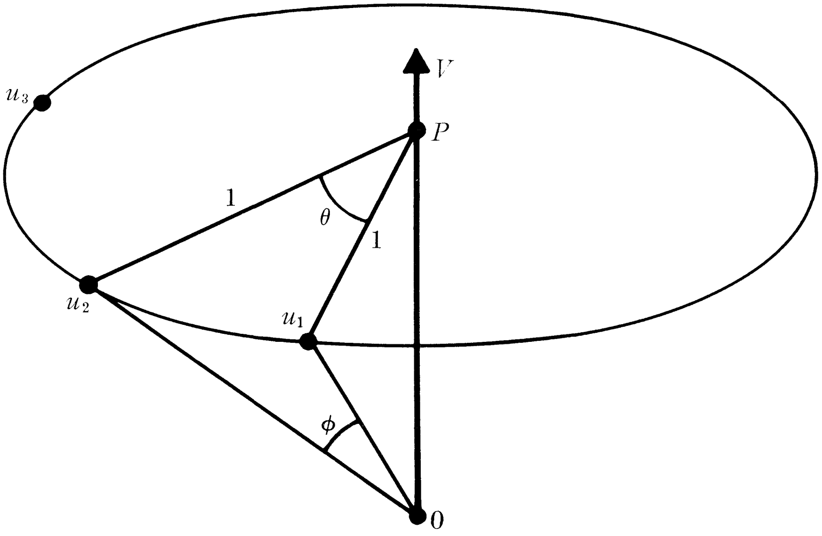

FIGURE 1. The construction used in the proof of Theorem 11: $\measuredangle \mathbf{u}_i\mathbf{OP} = \pi/3$ for all i, $|\mathbf{OP}| = 1/\sqrt{3}$, $|\mathbf{Ou}_1| = |\mathbf{Ou}_2| = 2/\sqrt{3}$, and $\cos\phi = (1 + 3\cos\theta)/4$

4. Uniqueness of the code of size 196560 in Ω_{24}.

THEOREM 13 ([**20**]). *If C is a* (24, M, 1/2) *code then* $M \leqq 196560$.

Proof. This parallels that of Theorem 4, using the polynomial

$$f(t) = (t+1)(t+\tfrac{1}{2})^2(t+\tfrac{1}{4})^2t^2(t-\tfrac{1}{4})^2(t-\tfrac{1}{2}).$$

THEOREM 14. *If* (a) *C is a* (24, 196560, 1/2) *code then* (b) *C is a tight spherical* 11*-design in* Ω_{24}, (c) *C carries a* 6*-class association scheme,* (d) *the intersection numbers of this association scheme are uniquely determined, and*

(e) *the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by*

$$
(11) \quad
\begin{aligned}
A_1(\mathbf{u}) &= A_{-1}(\mathbf{u}) = 1,\\
A_{1/2}(\mathbf{u}) &= A_{-1/2}(\mathbf{u}) = 4600,\\
A_{1/4}(\mathbf{u}) &= A_{-1/4}(\mathbf{u}) = 47104,\\
A_0(\mathbf{u}) &= 93150.
\end{aligned}
$$

Conversely (b) *implies* (a), (c), (d) *and* (e).

Proof. The proof here is parallel to those of Theorems 5 and 6.

In Example 3 we saw that the minimal vectors in the Leech lattice when suitably scaled form a (24, 196560, 1/2) code. We shall require an explicit description of this code, and take Λ to consist of the vectors

$$(\mathbf{O} + 2\mathbf{c} + 4\mathbf{x})/\sqrt{8}$$

and

$$(\mathbf{1} + 2\mathbf{c} + 4\mathbf{y})/\sqrt{8},$$

where $\mathbf{O} = 00\ldots0$, $\mathbf{1} = 11\ldots1$, $\mathbf{c}$ is any codeword in the binary Golay code g_{24} (cf. [**18**]) $\mathbf{x}, \mathbf{y} \in \mathbf{Z}^{24}$, and $\sum x_i$ is even, $\sum y_i$ odd. The minimal vectors in Λ consist of

$$
\begin{aligned}
&759 \cdot 2^7 \text{ with components } ((\pm 2)^8 0^{16})/\sqrt{8},\\
&2^2 \cdot \binom{24}{2} \text{ with components } ((\pm 4)^2 0^{22})/\sqrt{8},\\
(12) \quad &24 \cdot 2^{12} \text{ with components } ((\pm 1)^{23} (\mp 3)^1)/\sqrt{8}
\end{aligned}
$$

and have norm $(x, x) = 4$.

This set of 196560 vectors will be denoted by Λ_4. Then $\frac{1}{2}\Lambda_4$ is a (24, 196560, 1/2) code to which conditions (a)–(e) of Theorem 14 apply. Conversely we have:

THEOREM 15. *If C is a tight spherical 11-design in Ω_{24} there is an orthogonal transformation mapping C onto $\frac{1}{2}\Lambda_4$.*

Proof. From Theorem 14 the distance distribution of C with respect to any $\mathbf{u} \in C$ is given by (11), and in particular the inner products in C are $0, \pm\frac{1}{4}, \pm\frac{1}{2}, \pm 1$. Let $C = \{\mathbf{u}_1, \mathbf{u}_2, \ldots, \mathbf{u}_{196560}\}$, and let L be the lattice in $\mathbf{R}^{24}$ consisting of the vectors

$$\sum_{i=1}^{196560} a_i \cdot 2\mathbf{u}_i, \quad a_i \in \mathbf{Z}.$$

Then

$$(13) \quad (2\mathbf{u}_i, 2\mathbf{u}_j) \in \{0, \pm 1, \pm 2, \pm 4\}$$

and L is an even integral lattice. We shall establish Theorem 15 by showing that there is an orthogonal transformation mapping L onto 2Λ and C onto $\frac{1}{2}\Lambda_4$.

LEMMA 16. *The minimal norm* $(\mathbf{v}, \mathbf{v})$ *for* $\mathbf{v} \in L$, $\mathbf{v} \neq \mathbf{O}$, *is* 4.

Proof. The minimal norm is even, so suppose it is 2, with $(\mathbf{v}, \mathbf{v}) = 2$, $\mathbf{v} \in L$. For $\mathbf{u} \in 2C$ we have

$$|(\mathbf{u}, \mathbf{v})| = |\mathbf{u}| \cdot |\mathbf{v}| \cdot |\cos \measuredangle\, (\mathbf{u}, \mathbf{v})| \leqq 2\sqrt{2},$$

so $(\mathbf{u}, \mathbf{v}) \in \{0, \pm 1, \pm 2\}$ since L is integral. Suppose $(\mathbf{u}, \mathbf{v}) = 0$ for α choices of $\mathbf{u}$, $(\mathbf{u}, \mathbf{v}) = 1$ for β choices, and $(\mathbf{u}, \mathbf{v}) = 2$ for γ choices, with $\alpha + 2\beta + 2\gamma = 196560$. Without loss of generality we may assume $\mathbf{v} = (\sqrt{2}, 0, 0, \ldots, 0)$.

Since C is an 11-design,

$$(14) \quad \frac{1}{196560} \sum_{i=1}^{196560} f(\mathbf{u}_i) = \frac{1}{\omega_{24}} \int_{\Omega_{24}} f(\xi) d\omega(\xi)$$

holds for any homogeneous polynomial $f(\xi_1, \xi_2, \ldots, \xi_{24})$ of total degree $\leqq 11$, where ω_{24} is the surface area of Ω_{24} [**9**, p. 372]. Let us choose $f = f_k = \xi_1^{\,k}$, for $k = 2$ and 4, so that

$$f_k(\mathbf{u}_i) = 2^{-k/2}((\mathbf{u}_i, \mathbf{v}))^k.$$

The right hand side of (14) can be evaluated from

$$\frac{1}{\omega_{24}} \int_{\Omega_{24}} f_k(\xi) d\omega(\xi) = \frac{1}{196560} \sum_{u \in 1/2\Lambda_4} f_k(\mathbf{u})$$

$$= \frac{8190}{196560} \quad \text{if } k = 2, \quad \text{or} \quad \frac{945}{196560} \quad \text{if } k = 4,$$

using (12). The equations (14) now read

$$2\beta \cdot \frac{1^2}{8} + 2\gamma \cdot \frac{2^2}{8} = 8190,$$

$$2\beta \cdot \frac{1^4}{64} + 2\gamma \cdot \frac{2^4}{64} = 945,$$

which imply $\beta = 33600$, $\gamma = -210$, an impossibility.

LEMMA 17. *The set* L_4 *of vectors of norm* 4 *in* L *coincides with* $2C$.

Proof. By construction L_4 contains $2C$. Conversely take $\mathbf{u}, \mathbf{v} \in L_4$. Then $(\mathbf{u}, \mathbf{v}) \neq 3$, or else

$$(\mathbf{u} - \mathbf{v}, \mathbf{u} - \mathbf{v}) = (\mathbf{u}, \mathbf{u}) - 2(\mathbf{u}, \mathbf{v}) + (\mathbf{v}, \mathbf{v}) = 2,$$

contradicting Lemma 16. Similarly $(\mathbf{u}, \mathbf{v}) \neq -3$. Therefore $(\mathbf{u}, \mathbf{v}) \in \{0, \pm 1, \pm 2, \pm 4\}$ and $\measuredangle\, (\mathbf{u}, \mathbf{v}) \geqq \pi/3$ for $\mathbf{u} \neq \mathbf{v}$. From Theorem **13**

$$|L_4| \leqq 196560 = |2C|.$$

Therefore $L_4 = 2C$.

For $n \geqq 3$ let D_n be the lattice in $\mathbf{R}^n$ spanned by the vectors

$$(15)\quad \mathbf{g}_1 = \sqrt{2}(\mathbf{e}_1 + \mathbf{e}_2), \mathbf{g}_2 = \sqrt{2}(\mathbf{e}_1 - \mathbf{e}_2),$$
$$\mathbf{g}_3 = \sqrt{2}(\mathbf{e}_2 - \mathbf{e}_3), \dots, \mathbf{g}_n = \sqrt{2}(\mathbf{e}_{n-1} - \mathbf{e}_n),$$

with respect to an orthonormal basis $\{\mathbf{e}_1, \dots, \mathbf{e}_n\}$ for $\mathbf{R}^n$ ([**4**], [**19**]). There are $2n(n - 1)$ minimal vectors $((\pm\sqrt{2})^2 0^{n-2})$ in D_n. These lattices are nested: $D_3 \subseteq D_4 \subseteq \dots$.

LEMMA 18. (i) *For any pair of vectors* $\mathbf{u}, \mathbf{v}$ *in* Λ_4 *with* $\measuredangle(\mathbf{u}, \mathbf{v}) = \pi/2$ *there are* 44 *vectors* $\mathbf{w}$ *in* Λ_4 *with* $\measuredangle(\mathbf{u}, \mathbf{w}) = \measuredangle(\mathbf{v}, \mathbf{w}) = \pi/3$. (ii) *The same statement holds with* Λ_4 *replaced by* $L_4 = 2C$. (iii) *There are* $2n - 4$ *minimal vectors* $\mathbf{w}$ *in* D_n *such that* $\measuredangle(g_1, w) = \measuredangle(g_2, w) = \pi/3$.

Proof. (i) and (iii) are straightforward, and (ii) follows from (i) since Λ_4 and $2C$ are association schemes with the same parameters (Theorem 14).

LEMMA 19. *L contains a sublattice isometric to* D_3.

Proof. For the generators $\mathbf{g}_1$, $\mathbf{g}_2$, $\mathbf{g}_3$ of D_3 we can take any triple $\mathbf{u}, \mathbf{v}, \mathbf{w} \in L_4$ with $\measuredangle(u, v) = \pi/2$, $\measuredangle(\mathbf{u}, \mathbf{w}) = \measuredangle(\mathbf{v}, \mathbf{w}) = \pi/3$. Such a triple exists by Lemma 18(ii).

LEMMA 20. *L contains a sublattice isometric to D_n, for* $n = 3, 4 \dots, 24$.

Proof. We proceed by induction on n. Suppose the assertion holds for $n \geqq 3$. By choosing a suitable orthonormal basis $\mathbf{e}_1, \dots, \mathbf{e}_n$ L_4 contains vectors $\mathbf{g}_1, \dots, \mathbf{g}_n$ given by (15) which span D_n. By Lemma 18 (ii) there are 44 vectors $\mathbf{w}$ in L_4 with $\measuredangle(\mathbf{g}_1, w) = \measuredangle(\mathbf{g}_2, \mathbf{w}) = \pi/3$. By Lemma 18 (iii) at least one of these is not a minimal vector of D_n. Then this vector $\mathbf{w}$ is not in $\mathbf{R}D_n$. (For suppose $\mathbf{w} = w_1\mathbf{e}_1 + \dots + w_n\mathbf{e}_n$. Since $\measuredangle(\mathbf{g}_1, \mathbf{w}) = \measuredangle(\mathbf{g}_2, \mathbf{w}) = \pi/3$, $w_1 = \sqrt{2}$ and $w_2 = 0$. For $3 \leqq i \leqq n$,

$$\sqrt{2}(\mathbf{e}_1 \pm \mathbf{e}_i) \in L_4 \cap D_n \subseteq 2C,$$

and therefore

$$(\mathbf{w}, \sqrt{2}(\mathbf{e}_1 \pm \mathbf{e}_i)) \in \{0, \pm 1, \pm 2\}$$

from (13). This implies $w_3 = w_4 = \dots = w_n = 0$, and contradicts $(\mathbf{w}, \mathbf{w}) = 4$.) Choose $\mathbf{e}_{n+1}$ so that $\{\mathbf{e}_1, \dots, \mathbf{e}_{n+1}\}$ is an orthonormal basis for $\mathbf{R}\langle D_n, \mathbf{w}\rangle$, and suppose

$$\mathbf{w} = w_1\mathbf{e}_1 + \dots + w_n\mathbf{e}_n + w_{n+1}\mathbf{e}_{n+1}.$$

The above argument shows that $w_1 = \sqrt{2}$, $w_2 = \dots = w_n = 0$, and $w_{n+1} = \pm\sqrt{2}$. Therefore $\langle D_n, \mathbf{w}\rangle = D_{n+1} \subseteq L$.

LEMMA 21. *L is isometric to* Λ.

Proof. From Lemma 20 we may choose an orthonormal basis $\mathbf{e}_1, \ldots, \mathbf{e}_{24}$ so that $2C$ contains the vectors $(\pm\sqrt{2})^2\, 0^{22}$. Let $\mathbf{u} = (u_1, \ldots, u_{24})/\sqrt{8}$ be any vector in $2C$. From (13) the inner products of $\mathbf{u}$ with the vectors $(\pm\sqrt{2})^2\, 0^{22}$ are $0, \pm 1, \pm 2, \pm 4$. By considering the inner products with $(\sqrt{2}, \pm\sqrt{2}, 0, \ldots, 0)$ we obtain

$$u_1^2 + u_2^2 + \ldots + u_{24}^2 = 32,$$

$$\tfrac{1}{2}(u_1 \pm u_2) \in \{0, \pm 1, \pm 2, \pm 4\},$$

$$u_1, u_2, \ldots \in \{0, \pm 1, \pm 2, \pm 3, \pm 4, \pm 5\}.$$

Suppose $u_1 = \pm 5$. Then another u_i, say u_2, is zero. The inner product of $\mathbf{u}$ with $(\sqrt{2}, \sqrt{2}, 0, \ldots, 0)$ is $5/2$, a contradiction. Proceeding in this way it is not difficult to show that the only possibilities for the components of $\mathbf{u}$ are

$$((\pm 2)^8 0^{16})/\sqrt{8},\ ((\pm 4)^2 0^{22})/\sqrt{8}, \text{ and } ((\pm 1)^{23}(\pm 3)^1)/\sqrt{8}.$$

In particular $u_1, \ldots, u_{24}$ are integers with the same parity.

It remains to show that these vectors are the same as those in Λ_4 (see (12)). To see this we define a binary linear code $\mathscr{C}$ of length 24 by taking as codewords all binary vectors $\mathbf{c}$ such that there is a vector $\mathbf{u} \in L$ with

$$\mathbf{u} = (\mathbf{0} + 2\mathbf{c} + 4\mathbf{x})/\sqrt{8}$$

for some $\mathbf{x} \in \mathbf{Z}^{24}$. Then as in [**5**, p. 139] it follows that $\mathrm{wt}(\mathbf{c}) \geqq 8$ for $\mathbf{c} \neq 0$, and that there are at most 759 codewords of weight 8. Therefore $|\mathscr{C}| \leqq 2^{12}$ (see for example [**18**, Fig. 1, p. 674]). The argument on page 140 of [**5**] now shows that the only way that $2\mathscr{C}$ can contain 196560 vectors $\mathbf{u}$ is for these vectors to coincide with the minimal vectors (12) in Λ_4.

This completes the proof of Theorem 15. By combining Theorems 14 and 15 we obtain:

THEOREM 22. *There is a unique way (up to isometry) of arranging 196560 nonoverlapping unit spheres in* $\mathbf{R}^{24}$ *so that they all touch another unit sphere.*

5. Uniqueness of the code of size 4600 in Ω_{23}.

THEOREM 23. *If C is a* $(23, M, 1/3)$ *code then* $M \leqq 4600$.

Proof. Use $f(t) = (t + 1)(t + 1/3)^2 t^2 (t - 1/3)$.

THEOREM 24. *If* (a) *C is a* $(23{,}4600,\ 1/3)$ *code then* (b) *C is a tight spherical 7-design in* Ω_{23}, (c) *C carries a 4-class association scheme,* (d) *the intersection numbers of this association scheme are uniquely determined,* *and* (e) *the distance distribution of* C *with respect to any* $\mathbf{u} \in C$ *is given by*

$$A_1(\mathbf{u}) = A_{-1}(\mathbf{u}) = 1,$$
$$A_{1/3}(\mathbf{u}) = A_{-1/3}(\mathbf{u}) = 891,$$
$$A_0(\mathbf{u}) = 2816.$$

Conversely (b) *implies* (a), (c), (d) *and* (e).

For example the (23, 4600, 1/3) code given in Example 3 has properties (a)–(e). Conversely we have:

THEOREM 25. *If* C *is a tight spherical* 7*-design in* Ω_{23} *there is an orthogonal transformation mapping* C *onto the* (23, 4600, 1/3) *code obtained from the Leech lattice.*

Proof. As in the proof of Theorem 11 we embed $C = \{\mathbf{u}_1, \ldots, \mathbf{u}_{4600}\}$ in $\mathbf{R}^{24}$, choosing $\mathbf{0}$ so that $\measuredangle\, \mathbf{u}_i\mathbf{OP} = \pi/3$ for all i (cf. Fig. 1). Then

$$\cos \measuredangle\, \mathbf{u}_i\mathbf{O}\mathbf{u}_j \in \{-\tfrac{1}{2}, 0, \tfrac{1}{4}, \tfrac{1}{2}, 1\}.$$

Let L be the even integral lattice in $\mathbf{R}^{24}$ spanned by the vectors $\sqrt{3}\,\mathbf{O}\mathbf{u}_i$. For convenience we set $\mathbf{U}_i = \sqrt{3}\,\mathbf{O}\mathbf{u}_i$.

LEMMA 26. *The minimum norm* $(\mathbf{v}, \mathbf{v})$ *for* $\mathbf{v} \in L$, $\mathbf{v} \neq \mathbf{0}$, *is* 4.

Proof. Suppose $\mathbf{v} \in L$ with $(\mathbf{v}, \mathbf{v}) = 2$, and write $\mathbf{v} = \mathbf{v}' + \mathbf{v}''$ with $\mathbf{v}' \| \mathbf{OP}$, $\mathbf{v}'' \perp \mathbf{OP}$, $|\mathbf{v}'| = y$, $|\mathbf{v}''| = \sqrt{2 - y^2}$, and $\mathbf{U}_i = \mathbf{U}_i' + \mathbf{U}_i''$ with $\mathbf{U}_i' \| \mathbf{OP}$, $\mathbf{U}_i' \perp \mathbf{OP}$, $|\mathbf{U}_i'| = 1$, $|\mathbf{U}''| = \sqrt{3}$. Then

$$(\mathbf{U}_i, \mathbf{v}) = (\mathbf{U}_i', \mathbf{v}') + (\mathbf{U}_i'', \mathbf{v}'') \in \{0, \pm 1, \pm 2\},$$

$$\cos \measuredangle\, (\mathbf{U}_i'', \mathbf{v}'') \in \frac{\{0, \pm 1, \pm 2\} - y}{\sqrt{3}\sqrt{2 - y^2}}.$$

Since C is a tight 7-design, the set $\{\cos \measuredangle\, (\mathbf{U}_i'', \mathbf{v}''): 1 \leqq i \leqq 4600\}$ is symmetric about 0. Therefore $y \in \{0, \pm\frac{1}{2}, \pm 1\}$. First suppose $y = 0$. Then

$$\cos \measuredangle\, (\mathbf{U}_i'', \mathbf{v}'') \in \left\{-\frac{2}{\sqrt{6}}, -\frac{1}{\sqrt{6}}, 0, \frac{1}{\sqrt{6}}, \frac{2}{\sqrt{6}}\right\}.$$

Let these values occur $\gamma, \beta, \alpha, \beta, \gamma$ times respectively. Then by evaluating the 0th, 2nd and 4th moments of C with respect to $\mathbf{v}''$, as in the proof of Lemma 16, we obtain the equations

$$\alpha + 2\beta + 2\gamma = 4600$$
$$\beta/3 + 4\gamma/3 = 200$$
$$\beta/8 + 8\gamma/9 = 24,$$

which imply $\gamma = -14$, an impossibility. Similarly for the other values of y.

LEMMA 27. *L contains a sublattice isometric to D_n, for $n = 3, 4, \ldots, 24$.*

Proof. This is similar to the proof of Lemma 20, starting from the fact that if we take $\mathbf{u}_1, \mathbf{u}_2 \in C$ with $\measuredangle \mathbf{u}_1\mathbf{O}\mathbf{u}_2 = \pi/2$, there are 42 vectors $\mathbf{u}_i \in C$ with

$$\measuredangle \mathbf{u}_1\mathbf{O}\mathbf{u}_i = \measuredangle \mathbf{u}_2\mathbf{O}\mathbf{u}_i = \pi/3.$$

Furthermore the vector $\mathbf{v} = 2\mathbf{OP} \in L$ also satisfies

$$\measuredangle \mathbf{u}_1\mathbf{O}\mathbf{v} = \measuredangle \mathbf{u}_2\mathbf{O}\mathbf{v} = \pi/3.$$

LEMMA 28. *L is isometric to Λ, and C is isometric to the* (23, 4600, 1/3) *code obtained from the Leech lattice.*

Proof. Let L_4 denote the set of minimal vectors in L. From Lemma 27 we may assume that L_4 contains all the vectors $((\pm 4^2 0^{22}))/\sqrt{8}$, and that $\mathbf{v} = 2\mathbf{OP}$ is $(440\ldots0)/\sqrt{8}$. As in Lemma 21 it follows that the vectors in L_4 have the form $((\pm2)^8 0^{16})/\sqrt{8}$, $((\pm 4^2 0^{22})/\sqrt{8}$, and $((\pm1)^{23}(\pm3)^1)/\sqrt{8}$. Furthermore the vectors U_i begin $(22\ldots)/\sqrt{8}$, $(40\ldots)/\sqrt{8}$, $(04\ldots)/\sqrt{8}$, $(31\ldots)/\sqrt{8}$, or $(13\ldots)/\sqrt{8}$. The code $\mathscr{C}$ is defined as in Lemma 21: it is a linear code of minimum distance 8 containing at most 2^{12} codewords. The zero codeword corresponds to the vectors $\mathbf{U}_i$ beginning $(40\ldots)/\sqrt{8}$ or $(04\ldots)/\sqrt{8}$, and there are at most $2\cdot2\cdot22$ of them. The codewords of weight 8 beginning $11\ldots$ correspond to the vectors $\mathbf{U}_i$ beginning $(22\ldots)/\sqrt{8}$. The number of such codewords is at most 77 ([**18**, Fig. 3, p. 688]), and there are at most $2^5\cdot77$ corresponding $\mathbf{U}_i$. The remaining $\mathbf{U}_i$ come from codewords beginning $10\ldots$ or $01\ldots$, and there are at most $2\cdot2^{10}$ of them ([**18**, Fig. 1, p. 674]). Since $2\cdot2\cdot22 + 2^5\cdot77 + 2\cdot2^{10} = 4600$, all the inequalities in the argument must be exact. In particular the codewords of weight 8 beginning $11\ldots$ must form the unique Steiner system S(3, 6, 22) (cf. [**28**]), and hence L must be the Leech lattice.

This completes the proof of Theorem 25. By combining Theorem 24 and 25 we obtain:

THEOREM 29. *There is a unique way (up to isometry) of arranging* 4600 *unit spheres in $\mathbf{R}^{24}$ so that they all touch two further, touching, unit spheres.*

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REFERENCES

1. M. Abramowitz and I. A. Stegun, *Handbook of mathematical functions*, National Bureau of Standards Applied Math. Series *55* (Washington, DC, U.S. Dept. of Commerce, 1972).

2. E. Bannai and R. M. Damerell, *Tight spherical designs, I*, J. Math. Soc. Japan *31* (1979), 199–207.

3. ——— *Tight spherical designs, II*, J. London Math. Soc. *21* (1980), 13–30.

4. N. Bourbaki, *Groupes et algebras de Lie*, Chapitres IV, V, VI, Actualités Scientif. et Indust. *1337* (Hermann, Paris, 1968).

5. J. H. Conway, *A characterization of Leech's lattice*, Inventiones Math. *7* (1969), 137–142.

6. R. T. Curtis, *A new combinatorial approach to* M_{24}, Math. Proc. Camb. Phil. Soc. *79* (1976), 25–41.
7. P. Delsarte, *An algebraic approach to the association schemes of coding theory*, Philips Research Reports Supplements *10* (1973).
8. P. Delsarte and J.-M. Goethals, *Unrestricted codes with the Golay parameters are unique*, Discrete Math. *12* (1975), 211–224.
9. P. Delsarte, J.-M. Goethals and J. J. Seidel, *Spherical codes and designs*, Geometriae Dedicata *6* (1977), 363–388.
10. J. M. Goethals and J. J. Seidel, *Spherical designs*, in *Relations between combinatorics and other parts of mathematics*, Proc. Symp. Pure Math. *34* (Amer. Math. Soc., Providence, Rhode Island, 1979), 255–272.
11. W. Jónsson, *On the Mathieu groups* M_{22}, M_{23}, M_{24} *and the uniqueness of the associated Steiner systems*, Math. Zeit. *125* (1972), 193–214.
12. G. A. Kabatiansky and V. I. Levenshtein, *Bounds for packings on a sphere and in space*, Problems of Information Transmission *14*, No. 1 (1978), 1–17.
13. M. Kneser, *Klassenzahlen definiter quadratischer Formen*, Archiv der Math. *8* (1957), 241–250.
14. J. Leech, *Notes on sphere packings*, Can. J. Math. *19* (1967), 251–267.
15. J. Leech and N. J. A. Sloane, *Sphere packing and error-correcting codes*, Can. J. Math. *23* (1971), 718–745.
16. S. P. Lloyd, *Hamming association schemes and codes on spheres*, SIAM J. of Math. Analysis *11* (1980), 488–505.
17. H. Lüneburg, *Transitive Erweiterungen endlicher Permutationsgruppen*, Lecture Notes in Math. *84* (Springer-Verlag, New York, 1969).
18. F. J. MacWilliams and N. J. A. Sloane, *The theory of error-correcting codes* (North-Holland, Amsterdam, and Elsevier/North-Holland, New York, 1977).
19. H.-V. Niemeier, *Definite quadratische Formen der Dimension* 24 *und Diskriminante* 1, J. Number Theory *5* (1973), 142–178.
20. A. M. Odlyzko and N. J. A. Sloane, *New bounds on the number of unit spheres that can touch a unit sphere in n dimensions*, J. Combinatorial Theory *26A* (1979), 210–214.
21. V. Pless, *On the uniqueness of the Golay codes*, J. Combinatorial Theory *5* (1968), 215–228.
22. V. Pless and N. J. A. Sloane, *On the classification and enumeration of self-dual codes*, J. Combinatorial Theory *18A* (1975), 313–335.
23. M. Simonnard, *Linear programming* (Prentice-Hall, Englewood Cliffs, NJ, 1966).
24. N. J. A. Sloane, *An introduction to association schemes and coding theory* in *Theory and application of special functions* (Academic Press, New York, 1975), 225–260.
25. ——— *Binary codes, lattices and sphere-packings* in *Combinatorial surveys* Proc. *6th* British Combinatorics Conf. (Academic Press, London and New York, 1977), 117–164.
26. ——— *Self-dual codes and lattices*, in *Relations between combinatorics and other parts of mathematics*, Proc. Symp. Pure Math. *34* (Amer. Math. Soc., Providence, Rhode Island, 1979), 273–308.
27. R. G. Stanton, *The Mathieu groups*, Can. J. Math. *3* (1951), 164–174.
28. E. Witt, *Uber Steinersche Systeme*, Abh. Math. Sem. Hamburg *12* (1938), 265–275

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