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The Nonexistence of a Certain Steiner System $S(3, 12, 112)$

N. J. A. SLOANE

Bell Laboratories, Murray Hill, New Jersey 07974

AND

J. G. THOMPSON

Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Cambridge, England

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Although the automorphism group of a projective plane of order 10, if one exists, must be very small, such a plane could be the derived design of a Steiner system $S(3, 12, 112)$ with a larger group. There are several reasons why the Frobenius group of order 56 is a promising candidate for the latter group. However, in this paper it is shown that there is no $S(3, 12, 112)$ which is fixed by this Frobenius group.

1. INTRODUCTION

It is not known if a projective plane of order 10 exists; however, if one does exist its automorphism group must have order 1 or 3 [1, 17]. A plane with such a small group is difficult to analyze. It is possible, however, that this plane arises as the derived design of a Steiner system $S(3, 12, 112)$ having a larger group, i.e., that it is a cross section of a nicer object. The possible orders for the automorphism group of a Steiner system $S(3, 12, 112)$ are

$$2^i, \qquad 3 \cdot 2^i \qquad \text{or} \qquad 7 \cdot 2^i,$$

for $0 \leqslant i \leqslant 4$. In particular this group is solvable.

There are several reasons for trying to find an $S(3, 12, 112)$ which is fixed by the Frobenius group of order 56. (i) This group is a promising candidate to try since any Desarguesian affine plane of order n is fixed by a Frobenius group (namely, the group of order $n(n-1)$ consisting of the mappings

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$x \mapsto ax + b$, $a \neq 0$ [16, p. 52]). (ii) The 112 points of the Steiner system can be identified in a natural way with two copies of the Frobenius group, as we shall see, thus giving an auspicious start to the construction. (iii) Once this case is eliminated it can be assumed that the Sylow 7-subgroup of the automorphism group of any $S(3, 12, 112)$ is a normal subgroup.

Our main result is the following.

THEOREM 1. *There is no Steiner system* $S(3, 12, 112)$ *which is fixed by the Frobenius group of order* 56.

Several segments of the proof (those assertions stated as propositions) were carried out by computer. In all cases the programs used were simple, and the total computing time (on a Honeywell 6000) was less than 5 hr. The reader would encounter no difficulty in repeating these computations.

In a sequel [15] it will be shown that there is no $S(3, 12, 112)$ with an automorphism of order 3. Thus the possible orders for the automorphism group of an $S(3, 12, 112)$ are now 2^i or $7 \cdot 2^i$, with $0 \leqslant i \leqslant 4$, and any Sylow 7-subgroup is normal.

We remark that in 1973 Guza [9] showed that no $S(3, 12, 112)$ exists with $PGL(2, 7)$ acting transitively on the points. Her result is implied by Theorem 1.

2. Definitions and Notation Used in the Proof

We prove Theorem 1 by assuming that such a Steiner system exists and arriving at a contradiction. We first set the stage.

2.1. *The Steiner System S*

Let $\Omega = \{1, 2, ..., 112\}$, and let S be a *Steiner system* $S(3, 12, 112)$ on Ω; that is, S consists of 1036 12-subsets of Ω, called *blocks*, with the property that any three distinct points of Ω are contained in a unique block (cf. [5–7]).

The symmetric group on Ω will be denoted by Σ_{112}. We write permutations on the right, so that if $a \in \Omega$ and $g \in \Sigma_{112}$ then the image of a under g is ag. If A is a subset of Ω and H is a subgroup of Σ_{112} then AH denotes the set of images of A by elements of H, while H_A is the largest subgroup of H which fixes A. The *automorphism group* of S consists of all $g \in \Sigma_{112}$ such that $Bg \in S$ for all blocks $B \in S$.

The *block intersection numbers* of S are defined as follows. Let $\{P_1, ..., P_{12}\}$ be a block of S. Then λ_{ij} is the number of blocks of S which contain $P_1, ..., P_i$ and no not contain $P_{i+1}, ..., P_{i+j}$, for $0 \leqslant i + j \leqslant 12$. These numbers do not depend on which block is chosen, and are shown in Table I.

TABLE I

Block Intersection Numbers λ_{ij} for $S(3, 12, 112)$

$i+j$ \ i	0	1	2	3	4	5	6	7	8	9	10	11	12
0	1036												
1	925	111											
2	825	100	11										
3	735	90	10	1									
4	655	80	10	0	1								
5	585	70	10	0	0	1							
6	525	60	10	0	0	0	1						
7	475	50	10	0	0	0	0	1					
8	435	40	10	0	0	0	0	0	1				
9	405	30	10	0	0	0	0	0	0	1			
10	385	20	10	0	0	0	0	0	0	0	1		
11	375	10	10	0	0	0	0	0	0	0	0	1	
12	375	0	10	0	0	0	0	0	0	0	0	0	1

Note. λ_{ij} lies at the intersection of the $(i+j)$th row and the ith antidiagonal.

The last row of the table has the following interpretation.

THEOREM 2 (The 0, 2, 12)-Intersection Property). *Any two blocks of S meet in* 0, 2 *or* 12 *points.*

2.2. *The Code K*

Let K be the binary error-correcting code of length 112 generated by the blocks of S (cf. [2, 12, 13]). Then $K \subset K^{\perp}$, by Theorem 2. In fact much more is known about K.

THEOREM 3. *The code K has the following properties*:

(a) *K is self-dual* $(K = K^{\perp})$ *and the weight of every codeword is a multiple of* 4,

(b) *the minimum nonzero weight in K is* 12,

(c) *K contains no codewords of weight* 16,

(d) *K contains* 1036 *codewords of weight* 12, *and*

(e) *the weight distribution of K is given in Table II.*

Proof. For (a), (b), (c) see [13]; (c) is confirmed by [3, 4]. (d) The blocks of S give 1036 codewords of weight 12, and there are no others. For suppose M is a codeword of weight 12 which is not a block of S. Take three points in M. By definition of the Steiner system there is a unique block

TABLE II

Weight Distribution of the Code K Generated by the Blocks of $S(3, 12, 112)$

i	Number of codewords of weight i
0, 112	1
12, 100	1036
20, 92	868560
24, 88	111965910
28, 84	10847119360
32, 80	581085136170
36, 76	15631795001900
40, 72	219372154900360
44, 68	1662571548245160
48, 64	6958514212873685
52, 60	16330986833984592
56	21682256857734468

$B \in S$ containing these three points. Thus $|B \cap M| \geqslant 3$. From (a), $|B \cap M| = 4$, 6, 8 or 10. But $|B \cap M| = 4$ implies that $B + M$ has weight 16, violating (c). Similarly $|B \cap M| \geqslant 8$ violates (b). Hence $|B \cap M| = 6$. Thus any three points of M are contained in a unique 6-set $A = B \cap M$; and so these 6-sets form a Steiner system $S(3, 6, 12)$. But no such Steiner system exists [5–7]. (e) The weight distribution of K now follows from (a)–(d) using a theorem of Gleason [12, p. 602]. Q.E.D.

2.3. *The Frobenius Group G*

A transitive permutation group Γ on a set X is called a *Frobenius group* if only the identity fixes more than one point of X, and the subgroup fixing a point of X is nontrivial [8, p. 37; 11, p. 140; 14, p. 57; 18, p. 10]. Frobenius showed that those permutations which displace all the points of X, together with the identity, form a normal subgroup of Γ (now called the *Frobenius kernel* of Γ). The subgroup of Γ fixing a point of X is called a *Frobenius complement* of Γ.

THEOREM 4. *There is a unique Frobenius group G of order* 56, *consisting of the elements*

$$\zeta^i \alpha^j \beta^k \gamma^l \qquad \text{with } 0 \leqslant i \leqslant 6, \qquad j, k, l = 0 \text{ or } 1,$$

where

$$\alpha^2 = \beta^2 = \gamma^2 = \zeta^7 = 1, \qquad \alpha\beta = \beta\alpha, \qquad \alpha\gamma = \gamma\alpha, \qquad \beta\gamma = \gamma\beta,$$

$$\alpha^\zeta = \beta, \qquad \beta^\zeta = \gamma, \qquad \gamma^\zeta = \alpha\beta,$$

and $\alpha^{\zeta} = \zeta^{-1}\alpha\zeta$, *etc. The elementary abelian group* E *of order* 8 *generated by* α, β *and* γ *is the Frobenius kernel of* G, *and the cyclic group of order* 7 *generated by* ζ *is a Frobenius complement of* G.

Proof. Let G be a Frobenius group of order 56 with Frobenius kernel E and Frobenius complement H, where $|E| = m$, $|H| = n$ and $mn = 56$. Now m and n are relatively prime and m divides $n-1$ [14, pp. 60, 182], so $n = 8$ and $m = 7$. Choose elements $\alpha \in E$ and $\zeta \in H$ of orders 2 and 7, respectively. Then

$$1, \alpha, \beta := \alpha^{\zeta} = \zeta^{-1}\alpha\zeta,$$
$$\gamma := \beta^{\zeta} = \zeta^{2}, \alpha^{\zeta^3}, \alpha^{\zeta^4}, \alpha^{\zeta^5}, \alpha^{\zeta^6}$$

belong to E (since E is normal in G) and are distinct and so form all of E (for if, say, $\alpha^{\zeta^i} = \alpha$ then $\alpha\zeta^i\alpha^{-1} = \zeta^{-i}$ and $\alpha H\alpha^{-1} \cap H \neq \{1\}$, violating a basic property of Frobenius groups [14, p. 181, Condition ii]). Thus the nonidentity elements of E have order 2, and E is an elementary abelian group. It remains to express γ^{ζ} in terms of α, β and γ. The only possibilities are $\gamma^{\zeta} = \alpha\beta$ or $\gamma^{\zeta} = \alpha\gamma$. If $\gamma^{\zeta} = \alpha\gamma$, set $\zeta_1 = \zeta^{-1}$, $\alpha_1 = \alpha$, $\beta_1 = \alpha_1^{\zeta_1}$, $\gamma_1 = \beta_1^{\zeta_1}$, and then $\gamma_1^{\zeta_1} = \alpha_1\beta_1$. Thus we may assume that $\gamma^{\zeta} = \alpha\beta$, and G is fully determined. Q.E.D.

For future reference we record two other properties of G.

THEOREM 5. (a) *There is a single conjugacy class of involutions in* G, *consisting of the elements of* $E - \{1\}$. (b) *The map* $\tau: G \to G$ *determined by* $\tau(\zeta) = \zeta^2$, $\tau(\alpha) = \alpha$, $\tau(\beta) = \gamma$, $\tau(\gamma) = \beta\gamma$ *is an automorphism of* G.

From now on we assume that S is a Steiner system $S(3, 12, 112)$ on $\Omega = \{1, 2, ..., 112\}$, with an automorphism group which contains a group isomorphic to the Frobenius group G defined in Theorem 4. We use the same letter G to describe this permutation group on Ω.

2.4. *New Names for the* 112 *Points*

No element of G except the identity can fix a point of Ω, for otherwise the blocks containing that point would form a projective plane of order 10 with an automorphism of order 2 or 7, which is known to be impossible. Thus G is a semiregular group [18, p. 8] of order 56 and degree 112, and so has two orbits on Ω, by [18, Propositions 4.1, 4.2].

If $u \in \Omega$ and $v \in \Omega$ are in distinct orbits of G, then each point of Ω is uniquely of the form ug or vg for $g \in G$. We may write $\Omega = \Omega_1 \cup \Omega_2$, where

$$\Omega_1 = uG$$
$$= uE \cup u\zeta E \cup u\zeta^2 E \cup \cdots \cup u\zeta^6 E,$$
$$\Omega_2 = vG.$$

2.5. *Signatures*

For each subset A of Ω_1 let

$$\mathrm{sig}_1(A) = \{i \in \mathbb{Z}/7\mathbb{Z} \mid A \cap u\zeta^i E \neq \varnothing\}$$

be the *signature* of A with respect to Ω_1. $\mathrm{sig}_2(A)$ is defined similarly. Note that if $\mathrm{sig}_1(A) = \{a_1, ..., a_k\}$ then $\mathrm{sig}_1(A\zeta^i) = \{a_1 + i, ..., a_k + i\}$, for $0 \leqslant i \leqslant 6$.

2.6. *The Centralizer of G*

We shall make extensive use of the centralizer C and normalizer N of G in Σ_{112}, and we now describe these groups. For each $g \in G$ there is an element $L_1(g) \in C$ which fixes each point of Ω_2, and acts on Ω_1 by $(ux)\,L_1(g) = u(gx)$. Similarly $L_2(g) \in C$ is 1 on Ω_1 and $(vx)\,L_2(g) = v(gx)$. Let $G_i = \langle L_i(g) \mid g \in G\rangle$, for $i = 1, 2$, so that $\langle G_1, G_2\rangle = G_1 \times G_2$ and $G_i \cong G$. Finally C contains the involution $\sigma \in \Sigma_{112}$ defined by $(ux)\sigma = vx$ and $(vx)\sigma = ux$, for $x \in G$. Then the centralizer of G is

$$C = \langle G_1, G_2, \sigma\rangle,$$

and is isomorphic to the wreath product of G by a cyclic group of order 2. (We are using the primordial fact that the centralizer of the right regular representation is the left regular representation [10, p. 29].)

2.7. *The Normalizer of G*

Corresponding to the map τ of Theorem 5 there is an element in Σ_{112} (also designated by τ) given by

$$(ug)\tau = u \cdot \tau(g), \qquad (vg)\tau = v \cdot \tau(g),$$

and τ normalizes G and C. Then the normalizer of G in Σ_{112} is

$$N = \langle G, G_1, G_2, \sigma, \tau\rangle,$$

of order $56^3 \cdot 6$.

3. The Main Part of the Proof

We shall make use of the normalizer N as follows: by hypothesis, $S = Sg$ for all $g \in G$. If $x \in N$, then Sx is also an $S(3, 12, 112)$ fixed by G. So we shall use N to change S until we have enough control over it to put reasonable questions to the computer.

3.1. *The Action of G on the Blocks*

For each block $B \in S$, the stabilizer G_B acts semiregularly on B, so $|G_B|$ divides 12. Since $(12, 56) = 4$,

$$|G_B| = 1, 2 \text{ or } 4$$

or equivalently the number of distinct images of B under G is

$$|BG| = 56, 28 \text{ or } 14.$$

Let

$$S_0 = \{B \in S \mid |G_B| = 1\},$$
$$S_1 = \{B \in S \mid |G_B| = 2 \text{ or } 4\},$$

so that $S = S_0 \cup S_1$. Now $|S| = 1036 = 18\frac{1}{2} \times 56$, and S_0 is a union of G-orbits each of cardinal 56. Therefore

$$|S_1| \equiv 28 \pmod{56}. \tag{1}$$

Suppose S_1 contains a G-orbits of cardinal 28 and b of cardinal 14. Then (1) gives

$$28a + 14b \equiv 28 \pmod{56},$$

which implies

$$b \text{ is even (say, } b = 2c) \qquad \text{and} \qquad a + c \text{ is odd}. \tag{2}$$

Furthermore the blocks have the (0, 2, 12)-intersection property stated in Theorem 2.

So we are led to the consideration of two families $\mathscr{F}_{14}$ and $\mathscr{F}_{28}$ of 12-sets of Ω, namely,

$$\mathscr{F}_{14} = \{F \mid F \text{ is a } G\text{-orbit of 12-sets of } \Omega, \text{ of cardinal 14, such that if } f_1, f_2 \in F \text{ then } |f_1 \cap f_2| = 0, 2 \text{ or } 12\},$$

$$\mathscr{F}_{28} = \{F \mid F \text{ is a } G\text{-orbit of 12-sets of } \Omega, \text{ of cardinal 28, such that if } f_1, f_2 \in F \text{ then } |f_1 \cap f_2| = 0, 2 \text{ or } 12\}.$$

Every G-orbit on S_1 must belong to $\mathscr{F}_{14}$ or $\mathscr{F}_{28}$.

Now N acts on $\mathscr{F}_{14}$ and on $\mathscr{F}_{28}$. In fact since G fixes each member of $\mathscr{F}_i$, $\bar{N} := N/G$ acts on $\mathscr{F}_i$. The first step in our analysis is to determine the orbits of $\bar{N}$ on $\mathscr{F}_{14}$ and $\mathscr{F}_{28}$, and to list representatives for these orbits.

3.2. *Classification of the Members of* $\mathscr{F}_{28}$

Suppose $F = \{f_1, ..., f_{28}\} \in \mathscr{F}_{28}$, where each f_i is a 12-set of Ω, and G_{f_1} has order 2. By Theorem 5(a) we may assume that

$$G_{f_1} = \langle \alpha \rangle.$$

LEMMA 6. *If* $i \in \text{sig}_1(f_1)$ *then there are precisely two elements* x *of* $\zeta^i E$ *such that* $ux \in f_1$.

Proof. Let $X = \{x_1, ..., x_r\}$ be all the elements x of $\zeta^i E$ such that $ux \in f_1$. By hypothesis, $r \geqslant 1$. Since $f_1 = f_1 \alpha$ and $\alpha \in E$, it follows that if $ux \in f_1$ then $ux\alpha \in f_1$, so that $x\alpha \in X$. Thus $r = 2s$ is even, and we may write

$$X = \{x_1, x_1\alpha, x_2, x_2\alpha, ..., x_s, x_s\alpha\}.$$

Suppose $s \geqslant 2$. Then $x_2 = x_1\xi$ for some $\xi \in E - \langle \alpha \rangle$, $x_1 = x_2\xi$, and $\langle \alpha, \xi \rangle$ is a 4-group. But then

$$f_1 \cap f_1\xi \supseteq \{ux_1, ux_1\alpha, ux_2, ux_2\alpha\}.$$

By definition of $\mathscr{F}_{28}$ we have $f_1 = f_1\xi$. This mans that G_{f_1} has order >2, a contradiction. Thus $r = 2$. Q.E.D.

LEMMA 7. $|\text{sig}_1(f_1) \cap \text{sig}_1(f_1\zeta)| \leqslant 4$.

Proof. Suppose $\text{sig}_1(f_1) \cap \text{sig}_1(f_1\zeta) = \{b_1, ..., b_l\}$. Then $f_1\zeta = e \cup e'$, where $\text{sig}_1(e) = \{b_1, ..., b_l\}$, $\text{sig}_1(e') \cap \{b_1, ..., b_l\} = \varnothing$, and $|e| = 2l$ by Lemma 6. Consider the four 12-sets

$$f_1, \quad f_1\beta, \quad f_1\gamma, \quad f_1\beta\gamma.$$

If ξ, η are distinct elements of $\langle \beta, \gamma \rangle$ then $f_1\xi \cap f_1\eta = \varnothing$, and since $E = \langle \alpha \rangle \times \langle \beta, \gamma \rangle$ it follows that each element of e is contained in precisely one of $f_1, f_1\beta, f_1\gamma, f_1\beta\gamma$. Thus

$$2l = |e \cap f_1| + |e \cap f_1\beta| + |e \cap f_1\gamma| + |e \cap f_1\beta\gamma|.$$

Each summand is 0 or 2 since $f_1\zeta$ is distinct from $f_1\xi$ for all $\xi \in E$. So $l \leqslant 4$, as asserted. Q.E.D.

Let $f_1' = f_1 \cap \Omega_1$, $f_1'' = f_1 \cap \Omega_2$. By Lemma 6, $|f_1'|$ and $|f_1''|$ are even. Replacing F by $F\sigma$ if necessary, we may assume that $|f_1'| \geqslant |f_1''|$. Thus the possible values of $(|f_1'|, |f_1''|)$ are (12, 0), (10, 2), (8, 4) and (6, 6).

3.3. *The Case* $|f_1'| = 12$

In this case $f_1 = f_1'$ and $\mathrm{sig}_1(f_1)$ has cardinal 6: $\mathrm{sig}_1(f_1) = \mathbb{Z}/7\mathbb{Z} - \{i\}$ for some i, implying $|\mathrm{sig}_1(f_1) \cap \mathrm{sig}_1(f_1 \zeta)| = 5$. But this contradicts Lemma 7. Thus $|f_1 \cap \Omega_1| \neq 12$.

3.4. *The Case* $|f_1'| = 10$

If $|f_1'| = 10$ then $|\mathrm{sig}_1(f_1)| = 5$, and there is an $h \in G$ such that $\mathrm{sig}_1(f_1 L_1(h)) = \mathbb{Z}/7\mathbb{Z} - \{0, i\}$, where $i \in \{1, 2, 4\}$. Then, for some $t \in \langle \tau \rangle$, $\mathrm{sig}_1(f_1 L_1(h) t) = \mathbb{Z}/7\mathbb{Z} - \{0, 1\}$. Replacing F by $FL_1(h)t$ we may assume at the outset that $\mathrm{sig}_1(f_1) = \mathbb{Z}/7\mathbb{Z} - \{0, 1\}$. Replacing f_1 by $f_1 \xi$ for a suitable $\xi \in \langle \beta, \gamma \rangle$ we may assume that

$$f_1' = \{u\pi \mid \pi \in \{\zeta^2, \zeta^2\alpha, \zeta^3\mu_3, \zeta^3\mu_3\alpha, \zeta^4\mu_4, \zeta^4\mu_4\alpha, \zeta^5\mu_5, \zeta^5\mu_5\alpha, \zeta^6\mu_6, \zeta^6\mu_6\alpha\}$$
$$\text{for suitable } \mu_3, \ldots, \mu_6 \in \langle \beta, \gamma \rangle\}.$$

We next choose $h' \in G$ such that $f_1'' L_2(h') = \{v, v\alpha\}$. Replacing F by $FL_2(h')$ we may assume that $f_1'' = \{v, v\alpha\}$. Since $(\alpha\beta\gamma)^{\zeta^2} = \alpha$ and $(\alpha\beta\gamma)^{\zeta^3} = \beta$, if we replace f_1 by $f_1 L_1(\alpha\beta\gamma)$ we replace μ_3 by $\mu_3\beta$. Thus we may assume that $\mu_3 \in \langle \gamma \rangle$. So there are $2 \cdot 4^3$ choices for f_1 (namely, 2 choices for μ_3 and 4 choices each for μ_4, μ_5, μ_6). These were tested by the computer to see if the 28 blocks of $f_1 G$ satisfy the $(0, 2, 12)$-intersection condition. None passed the test, establishing:

PROPOSITION 8. *If* $f_1 \in F \in \mathscr{F}_{28}$ *then* $|f_1 \cap \Omega_1| \neq 10$.

3.5. *The Case* $|f_1'| = 8$

By using G_1 and $\langle \tau \rangle$ we may assume that $\mathrm{sig}_1(f_1)$ is one of

$$0123, \qquad 0356, \qquad 0124,$$

and using G_2 we may assume that $\mathrm{sig}_2(f_1)$ is one of 01, 02, 03. Using $L_1(E)$ and $L_2(E)$ we may also assume that

$$\{u, u\alpha\} \subseteq f_1', \qquad \{v, v\alpha\} \subseteq f_1''.$$

Thus there are $3 \cdot 4^3 \cdot 3 \cdot 4$ choices for f_1. The computer showed that none of the resulting $f_1 G$ satisfy the $(0, 2, 12)$-condition, proving:

PROPOSITION 9. *If* $f_1 \in F \in \mathscr{F}_{28}$ *then* $|f_1 \cap \Omega_1| \neq 8$.

3.6. *The Case* $|f_1'| = 6$

We are left with the case where $\mathrm{sig}_1(f_1)$ and $\mathrm{sig}_2(f_1)$ both have cardinal 3. Using G_1 and G_2 we may assume that $\mathrm{sig}_1(f_1)$ and $\mathrm{sig}_2(f_1)$ both belong to

$$\{012, 013, 014, 015, 024\},$$

and that

$$\{u, u\alpha\} \subseteq f_1', \qquad \{v, v\alpha\} \subset f_1''.$$

Finally, since $\alpha^\zeta = \beta$ and $\alpha^{\zeta^2} = \gamma$, we may assume that $f_1' = \{ux \mid x \in X\}$, where X is one of the following:

$$\{1, \alpha, \zeta\mu_1, \zeta\mu_1\alpha, \zeta^2\mu_2, \zeta^2\mu_2\alpha\} \quad \text{for} \quad \mu_1 \in \langle\gamma\rangle, \mu_2 \in \langle\beta, \gamma\rangle,$$
$$\{1, \alpha, \zeta\mu_1, \zeta\mu_1\alpha, \zeta^3\mu_3, \zeta^3\mu_3\alpha\} \quad \text{for} \quad \mu_1 \in \langle\gamma\rangle, \mu_3 \in \langle\beta, \gamma\rangle,$$
$$\{1, \alpha, \zeta\mu_1, \zeta\mu_1\alpha, \zeta^4\mu_4, \zeta^4\mu_4\alpha\} \quad \text{for} \quad \mu_1 \in \langle\gamma\rangle, \mu_4 \in \langle\beta, \gamma\rangle,$$
$$\{1, \alpha, \zeta\mu_1, \zeta\mu_1\alpha, \zeta^5\mu_5, \zeta^5\mu_5\alpha\} \quad \text{for} \quad \mu_1 \in \langle\gamma\rangle, \mu_5 \in \langle\beta, \gamma\rangle,$$
$$\{1, \alpha, \zeta^2\mu_2, \zeta^2\mu_2\alpha, \zeta^4\mu_4, \zeta^4\mu_4\alpha\} \quad \text{for} \quad \mu_2 \in \langle\beta\rangle, \mu_4 \in \langle\beta, \gamma\rangle.$$

Thus there are 40 choices for f_1', and another 40 for f_1'', giving a total of 1600 choice for f_1. The computer then established:

PROPOSITION 10. *There are 52 f_1's that satisfy the* (0, 2, 12)-*condition and are inequivalent under the centralizer C. They are shown in Table* III, *where we write $f_1 = uX \cup vY$, with $X = X'\langle\alpha\rangle$ and $Y = Y'\langle\alpha\rangle$.*

Some of these cases can be eliminated at once using Theorem 3(c). For if $\text{sig}_1(f_1) = \text{sig}_2(f_1) = \{0, 1, 2\}$, then

$$(f_1 + f_1\beta + f_1\gamma + f_1\beta\gamma)(1 + \zeta^2 + \zeta^3 + \zeta^5 + \zeta^6) = uE + vE$$

TABLE III

The 52 Solutions for f_1 Described in Proposition 10

	$X' = Y'$				$X' = Y'$		
1	1	ζ	$\zeta^2\gamma$	15	1	$\zeta\gamma$	ζ^4
2	1	ζ	$\zeta^2\beta\gamma$	16	1	$\zeta\gamma$	$\zeta^4\gamma$
3	1	$\zeta\gamma$	ζ^2	17	1	ζ	ζ^5
4	1	$\zeta\gamma$	$\zeta^2\beta$	18	1	ζ	$\zeta^5\gamma$
5	1	ζ	ζ^3	19	1	ζ	$\zeta^5\beta$
6	1	ζ	$\zeta^3\gamma$	20	1	ζ	$\zeta^5\beta\gamma$
7	1	ζ	$\zeta^3\beta$	21	1	$\zeta\gamma$	ζ^5
8	1	ζ	$\zeta^3\beta\gamma$	22	1	$\zeta\gamma$	$\zeta^5\gamma$
9	1	$\zeta\gamma$	ζ^3	23	1	$\zeta\gamma$	$\zeta^5\beta$
10	1	$\zeta\gamma$	$\zeta^3\gamma$	24	1	$\zeta\gamma$	$\zeta^5\beta\gamma$
11	1	$\zeta\gamma$	$\zeta^3\beta$	25	1	ζ^2	$\zeta^4\beta$
12	1	$\zeta\gamma$	$\zeta^3\beta\gamma$	26	1	ζ^2	$\zeta^4\beta\gamma$
13	1	ζ	$\zeta^4\beta$	27	1	$\zeta^2\beta$	$\zeta^4\beta$
14	1	ζ	$\zeta^4\beta\gamma$	28	1	$\zeta^2\beta$	$\zeta^4\beta\gamma$

TABLE III—*Continued*

	X'			Y'		
29	1	ζ	ζ^2	1	$\zeta\gamma$	$\zeta^2\gamma$
30	1	ζ	$\zeta^2\gamma$	1	$\zeta\gamma$	ζ^2
31	1	ζ	$\zeta^2\beta$	1	$\zeta\gamma$	$\zeta^2\beta\gamma$
32	1	ζ	$\zeta^2\beta\gamma$	1	$\zeta\gamma$	$\zeta^2\beta$
33	1	ζ	ζ^3	1	ζ	ζ^5
34	1	ζ	ζ^3	1	ζ	$\zeta^5\beta\gamma$
35	1	ζ	$\zeta^3\beta\gamma$	1	ζ	$\zeta^5\gamma$
36	1	ζ	$\zeta^3\beta\gamma$	1	ζ	$\zeta^5\beta$
37	1	$\zeta\gamma$	ζ^3	1	$\zeta\gamma$	$\zeta^5\gamma$
38	1	$\zeta\gamma$	ζ^3	1	$\zeta\gamma$	$\zeta^5\beta$
39	1	$\zeta\gamma$	$\zeta^3\beta\gamma$	1	$\zeta\gamma$	ζ^5
40	1	$\zeta\gamma$	$\zeta^3\beta\gamma$	1	$\zeta\gamma$	$\zeta^5\beta\gamma$
41	1	ζ	ζ^4	1	ζ	$\zeta^4\gamma$
42	1	ζ	$\zeta^4\beta$	1	ζ	$\zeta^4\beta\gamma$
43	1	$\zeta\gamma$	ζ^4	1	$\zeta\gamma$	$\zeta^4\gamma$
44	1	$\zeta\gamma$	$\zeta^4\beta$	1	$\zeta\gamma$	$\zeta^4\beta\gamma$
45	1	ζ	ζ^5	1	ζ	$\zeta^5\beta\gamma$
46	1	ζ	$\zeta^5\gamma$	1	ζ	$\zeta^5\beta$
47	1	$\zeta\gamma$	ζ^5	1	$\zeta\gamma$	$\zeta^5\beta\gamma$
48	1	$\zeta\gamma$	$\zeta^5\gamma$	1	$\zeta\gamma$	$\zeta^5\beta$
49	1	ζ^2	ζ^4	1	$\zeta^2\beta$	ζ^4
50	1	ζ^2	$\zeta^4\gamma$	1	$\zeta^2\beta$	$\zeta^4\gamma$
51	1	ζ^2	$\zeta^4\beta$	1	$\zeta^2\beta$	$\zeta^4\beta$
52	1	ζ^2	$\zeta^4\beta\gamma$	1	$\zeta^2\beta$	$\zeta^4\beta\gamma$

is a 16-set. This excludes cases 1, 2, 3, 4 and 29, 30, 31, 32. Since $\mathrm{sig}_i(f_1\tau) = 2 \cdot \mathrm{sig}_i(f_1)$ we can also exclude those cases where $\mathrm{sig}_1(f_1) = \mathrm{sig}_2(f_1) = \{0, 2, 4\}$ or $\{0, 4, 1\}$; that is, we eliminate cases 13, 14, 15, 16, 25, 26, 27, 28, 41, 42, 43, 44, 49, 50, 51, 52, leaving 28 possible f_1's.

PROPOSITION 11. *These 28 sets $f_1 G$ fall into 8 orbits under the action of $\bar{N} = N/G$. As representatives for these orbits, we may take the sets $f_1 = uX \cup vY$, where $X = X'\langle\alpha\rangle$, $Y = Y'\langle\alpha\rangle$, and X' and Y' are shown in Table* IV.

We define $F_i = f_i G$, where $1 \leqslant i \leqslant 8$ and f_i is the ith 12-set described in Table IV.

3.7. *Classification of the Members of $\mathscr{F}_{14}$*

Since the analysis of $\mathscr{F}_{14}$ is much easier than that of $\mathscr{F}_{28}$ we merely state the result.

TABLE IV

The Eight Inequivalent Solutions for f_1 Described in Proposition 11

	X'			Y'		
1.	1	ζ	ζ^3	1	ζ	ζ^3
2.	1	ζ	$\zeta^3\gamma$	1	ζ	$\zeta^3\gamma$
3.	1	$\zeta\gamma$	$\zeta^3\gamma$	1	$\zeta\gamma$	$\zeta^3\gamma$
4.	1	ζ	ζ^5	1	ζ	ζ^5
5.	1	ζ	$\zeta^5\gamma$	1	ζ	$\zeta^5\gamma$
6.	1	ζ	ζ^3	1	ζ	ζ^5
7.	1	ζ	ζ^3	1	ζ	$\zeta^5\beta\gamma$
8.	1	ζ	ζ^5	1	ζ	$\zeta^5\beta\gamma$

PROPOSITION 12. *$\bar{N}$ has 3 orbits on $\mathscr{F}_{14}$, with representatives $\tilde{F}_1 = \tilde{f}_1 G$, $\tilde{F}_2 = \tilde{f}_2 G$, $\tilde{F}_3 = \tilde{f}_3 G$, where $\tilde{f}_i = uX_i \cup vY_i$, $X_i = X'_i\langle\alpha, \beta\rangle$, $Y_i = Y'_i\langle\alpha, \beta\rangle$ and the X'_i, Y'_i are as follows.*

i	X'_i	Y'_i
1.	$1, \zeta, \zeta^3$	$\emptyset$
2.	$1, \zeta, \zeta^5$	$\emptyset$
3.	$1, \zeta$	1

Since $|\bar{N}| = 56^2 \cdot 6$, we conclude that there are at most $8 \cdot 56^2 \cdot 6$ members of $\mathscr{F}_{28}$ to be considered (given in Proposition 11 and Table IV) and $3 \cdot 56^2 \cdot 6$ members of $\mathscr{F}_{14}$ (given in Proposition 12). We must now investigate how these G-orbits may be fitted together.

3.8. *The Oddness Condition*

A further condition on the blocks can be obtained from the fact that every 2-subset A of Ω is contained in precisely 11 blocks (see Table I). If $A = A\alpha$ we conclude that an odd number of blocks are fixed by α and contain A.

LEMMA 13 (The Oddness Condition). *For each orbit A of $\langle\alpha\rangle$ on Ω, the number of blocks of S_1 which are fixed by α and contain A is odd.*

The following observations make it relatively easy to decide if a candidate for S_1 satisfies the oddness condition. Note that the orbits of $\langle\alpha\rangle$ on Ω are the sets $\{u\zeta^i\mu, u\zeta^i\mu\alpha\}$ and $\{v\zeta^i\mu, v\zeta^i\mu\alpha\}$ for $i \in \mathbb{Z}/7\mathbb{Z}$, $\mu \in \langle\beta, \gamma\rangle$.

First, consider a G-orbit $F \in \mathscr{F}_{28}$, and a block $f \in F$ with $f = f\alpha$. Then f, $f\beta$, $f\gamma$, $f\beta\gamma$ are the only members of F which are fixed by α, and if $\mathrm{sig}_1(f) = \{a_1, ..., a_k\}$, there is just one member of F which is fixed by α and contains

$\{u\zeta^{a_v}, u\zeta^{a_v}\alpha\}$, for $v = 1,..., k$, while if $i \notin \{a_1,..., a_k\}$, then no member of F is fixed by α and contains $\{u\zeta^i, u\zeta^i\alpha\}$.

Second, consider $F \in \mathscr{F}_{14}$ and a block $f \in F$ with $f = f\alpha = f\beta$. There are precisely two members of F which are fixed by $\langle \alpha, \beta \rangle$, namely, f and $f\gamma$. There are 6 members of F fixed by α, namely, f, $f\gamma$, $f\zeta^{-1}$, $f\gamma\zeta^{-1}$, $f\zeta^{-3}$ and $f\gamma\zeta^{-3}$. Let $\text{sig}_1(f) = \{b_1,..., b_k\}$. Then $\text{sig}_1(f\zeta^{-1}) = \{b_1 - 1,..., b_k - 1\}$ and $\text{sig}_1(f\zeta^{-3}) = \{b_1 - 3,..., b_k - 3\}$. The number of members of F which are fixed by α and contain $\{u\zeta^i, u\zeta^i\alpha\}$ is $\delta_0 + \delta_{-1} + \delta_{-3}$, where $\delta_j = 0$ if $i \notin \{b_1 + j,..., b_k + j\}$ and $\delta_j = 1$ if $i \in \{b_1 + j,..., b_k + j\}$.

3.9. *Combining Small Orbits: The Case $b = 0$*

Recall from 3.1 that S_1 contains a members of $\mathscr{F}_{28}$ and $b = 2c$ members of $\mathscr{F}_{14}$, with $a + c$ odd. First suppose $b = 0$. Then $S_1 = E_1 \cup \cdots \cup E_a$, with $E_i \in \mathscr{F}_{28}$ and a odd. We check that no single F_i of Table IV satisfies the oddness condition, and so

$$a \geqslant 3.$$

We may assume that $\{E_1,..., E_a\} \cap \{F_1,..., F_8\} \neq \varnothing$ and that if j is the smallest integer such that $F_j \in \{E_1,..., E_a\}$ then

$$\{E_1,..., E_a\} \subseteq F_j\bar{N} \cup F_{j+1}\bar{N} \cup \cdots \cup F_8\bar{N}.$$

The computer was used to take each F_j in turn and to list all the members of $F_j\bar{N},..., F_8\bar{N}$ all of whose blocks meet those of F_j in 0 or 2 points. Once this had been done it was easy to establish:

PROPOSITION 14. *$\mathscr{F}_{28}$ does not contain 5 pairwise compatible G-orbits, so $a < 5$.*

We conclude that $a = 3$. It was then not difficult to check that to within $\bar{N}$-equivalence there are just 3 choices for $\{E_1, E_2, E_3\}$ which satisfy the oddness condition. The computer then eliminated these cases by proving:

PROPOSITION 15. *In each of these three cases the code generated by $\{E_1, E_2, E_3\}$ contains a vector of weight 8 or 16.*

(Since there are 84 vectors in $\{E_1, E_2, E_3\}$ it was not possible to look at all vectors in the code they generate. Instead a generator matrix for the code was obtained in canonical form, consisting of some permutation of the columns of $[I \mid D]$, where I is an $r \times r$ identity matrix, r is the dimension of the code, and D is an $r \times (112\text{-}r)$ matrix. Then the rows of the generator matrix were taken 1, 2, 3,... at a time until a codeword of weight 4, 8 or 16 was found. The same algorithm was used to establish Propositions 19 and 21 below. In most cases one of the rows of the generator matrix itself had weight 8 or 16.)

Since Proposition 15 violates Theorem 3 we conclude that $b \neq 0$.

3.10. *The Case* $b = 2$, $a = 0$

Next, suppose S_1 consists of two G-orbits of size 14, say, E_1 and E_2. The (0, 2, 12)-condition between E_1 and E_2 may be used as follows. Choose $e_i \in E_i$ with $e_i = e_i\alpha = e_i\beta$. Then

$$\operatorname{sig}_j(e_1) \cap \operatorname{sig}_j(e_2) = \varnothing \qquad (j = 1, 2), \tag{3}$$

and

$$|\operatorname{sig}_1(e_1) \cap \operatorname{sig}_1(e_2 g)| + |\operatorname{sig}_2(e_1) \cap \operatorname{sig}_2(e_2 g)| \leqslant 1 \quad \text{for all } g \in G. \tag{4}$$

This makes it easy to handle the cases $E_1 = \tilde{F}_1$ and $E_1 = \tilde{F}_2$.

LEMMA 16. *If* $S_1 = E_1 \cup E_2$ *with* $E_1 \in \{\tilde{F}_1, \tilde{F}_2\}$, *then* $E_1 = e_1 G$, $E_2 = e_2 G$ *with*

$$e_1 = uX_1 \cup vY_1, \qquad e_2 = uX_2 \cup vY_2,$$

where

$$X_i = X_i'\langle \alpha, \beta \rangle, \qquad Y_i = Y_i'\langle \alpha, \beta \rangle$$

and X_i' *and* Y_i' *are one of the* 9 *possibilities given in Table* V.

TABLE V

The Nine Solutions for $S_1 = E_1 \cup E_2$ Described in Lemma 16

	X_1'	Y_1'	X_2'	Y_2'
1.	$1, \zeta, \zeta^3$	$\varnothing$	$\varnothing$	$1, \zeta, \zeta^3$
2.	$1, \zeta, \zeta^3$	$\varnothing$	$\varnothing$	$1, \zeta, \zeta^5$
3.	$1, \zeta, \zeta^5$	$\varnothing$	$\varnothing$	$1, \zeta, \zeta^5$
4.	$1, \zeta, \zeta^3$	$\varnothing$	ζ^2	$1, \zeta^3$
5.	$1, \zeta, \zeta^3$	$\varnothing$	ζ^4	$1, \zeta^2$
6.	$1, \zeta, \zeta^3$	$\varnothing$	ζ^5	$1, \zeta$
7.	$1, \zeta, \zeta^5$	$\varnothing$	ζ^3	$1, \zeta$
8.	$1, \zeta, \zeta^5$	$\varnothing$	ζ^4	$1, \zeta^2$
9.	$1, \zeta, \zeta^5$	$\varnothing$	ζ^6	$1, \zeta^3$

The remaining case is when $E_1 = \tilde{F}_3$ and $E_2 \in \tilde{F}_3\bar{N}$. Pick $e \in E_2$ with $e = e\alpha = e\beta$.

(i) First, assume that $\text{sig}_1(e)$ has cardinal 2. Then $\text{sig}_2(e) = \{k\}$ and, by (3), $k \neq 0$. Choose $\lambda \in N$ such that $E\lambda = \tilde{F}_3$. Then $\tilde{F}_3\lambda = E\lambda^2$ and $\text{sig}_2(e\lambda^2) = \{-k\}$. So we may assume that

$$k \in \{1, 2, 3\}.$$

By (4) it follows that $(\text{sig}_1(e), \text{sig}_2(e))$ is one of:

$\text{sig}_1(e)$	$\text{sig}_2(e)$
3, 5	1
3, 6	1
4, 6	1
4, 6	2
2, 5	3
2, 6	3

Now $u\langle\alpha, \beta\rangle \cup u\zeta\langle\alpha, \beta\rangle \cup v\langle\alpha, \beta\rangle = f \in \tilde{F}_3$ is fixed by $L_1(\alpha)$, $L_2(\alpha)$ and $L_2(\beta)$, while γ fixes $\tilde{F}_3$. Therefore, given $(\text{sig}_1(e), \text{sig}_2(e))$, the 8 possible choices of e are permuted by $\langle L_1(\alpha), L_2(\alpha), L_2(\beta), \gamma\rangle$. From this we obtain:

LEMMA 17. *Suppose* $S_1 = E_1 \cup E_2$ *with* $E_1 = \tilde{F}_3$ *and* $E_2 = eG \in \tilde{F}_3\bar{N}$, *where* $e = e\alpha = e\beta$. *If* $|\text{sig}_1(e)| = 2$, *there are* 9 $\bar{N}$*-inequivalent choices for e. They are given by* $e = uX \cup vY$, *where* $X = X'\langle\alpha, \beta\rangle$, $Y = Y'\langle\alpha, \beta\rangle$, *and the* X', Y' *are shown in Table* VI.

(ii) This leaves the case where $\text{sig}_1(e)$ has cardinal 1. In this case, (3) and (4) imply that the possibilities for $(\text{sig}_1(e), \text{sig}_2(e))$ are as shown in Table VII.

Now $\langle L_1(\alpha), L_2(\alpha), L_2(\beta), L_1(\beta\gamma) L_2(\beta\gamma), \gamma\rangle = H$ fixes $\tilde{F}_3$, and for each choice of $(\text{sig}_1(e), \text{sig}_2(e))$ the 8 possible choices for e are permuted transitively by H. Thus we have proved:

LEMMA 18. *Suppose* $S_1 = E_1 \cup E_2$ *with* $E_1 = \tilde{F}_3$ *and* $E_2 = eG \in \tilde{F}_3\bar{N}$, *where* $e = e\alpha = e\beta$. *If* $|\text{sig}_1(e)| = 1$ *there are* 17 $\bar{N}$*-inequivalent choices for e, as shown in Table* VII.

Collecting the results from Lemmas 16–18 we see that $\bar{N}$ has 35 orbits on compatible pairs of members of $\mathscr{F}_{14}$. However, out of these 35 pairs, only two satisfy the oddness condition. These are the first,

$$\{\tilde{F}_1, \tilde{F}_1\sigma\},$$

TABLE VI

The Nine Solutions for $S_1 = E_1 \cup E_2$ Described in Lemma 17

	X'	Y'
10.	ζ^3, ζ^5	ζ
11.	ζ^3, ζ^6	ζ
12.	ζ^4, ζ^6	ζ
13.	$\zeta^4, \zeta^6\gamma$	ζ
14.	ζ^4, ζ^6	ζ^2
15.	$\zeta^4, \zeta^6\gamma$	ζ^2
16.	ζ^2, ζ^5	ζ^3
17.	$\zeta^2, \zeta^5\gamma$	ζ^3
18.	ζ^2, ζ^6	ζ^3

TABLE VII

The Possible Signatures $(\mathrm{sig}_1(e), \mathrm{sig}_2(e))$ in Case (ii)

	$\mathrm{sig}_1(e)$	$\mathrm{sig}_2(e)$
19.	2	3, 4
20.	2	3, 5
21.	2	4, 5
22.	2	4, 6
23.	2	5, 6
24.	3	4, 1
25.	3	4, 5
26.	3	4, 6
27.	3	1, 5
28.	3	6, 1
29.	4	1, 2
30.	4	1, 5
31.	4	6, 1
32.	5	1, 2
33.	5	2, 3
34.	6	1, 2
35.	6	4, 1

and the 25th,

$$\{\tilde{F}_3, eG\},$$

where $e = u\zeta^3\langle\alpha, \beta\rangle \cup v\zeta^4\langle\alpha, \beta\rangle \cup v\zeta^5\langle\alpha, \beta\rangle$. Furthermore the computer eliminated the latter possibility.

PROPOSITION 19. *The code generated by* $\{\tilde{F}_3, eG\}$, *with* e *as above, contains a codeword of weight* 16.

We conclude that if $b=2$ and $a=0$ the only possibility for S_1 is $S_1=\tilde{F}_1 \cup \tilde{F}_1\sigma$. We return to this case in Section 3.13.

3.11. *The Case* $b=2$, $a \geqslant 2$

The computer was invited to take in succession each of the 35 pairs $\{E_1, E_2\}$ and to list all the pairs of elements of $\mathscr{F}_{28}$ which are compatible with each other and with E_1 and E_2. Then by hand it was (rather trivially) verified that the oddness condition of Lemma 13 is never satisfied. The cmputer results also showed that $b=2$, $a \geqslant 4$ is impossible, since for no pair $\{E_1, E_2\}$ do there exist four pairwise compatible members of $\mathscr{F}_{28}$ which are also compatible with E_1 and E_2. Hence:

PROPOSITION 20. *The case* $b=2$ *is excluded, except possibly for the case* $S_1=\tilde{F}_1 \cup \tilde{F}_1\sigma$.

3.12. *The Case* $b \geqslant 4$

Finally, suppose $b=2c \geqslant 4$. Let $\{W_1,..., W_{2c}\}$ be a $2c$-subset of $\mathscr{F}_{14}$ whose elements are pairwise compatible. For each $i=1,..., 2c$, pick $w_i \in W_i$ with $w_i=w_i\alpha=w_i\beta$. Then $\mathrm{sig}_1(w_i) \cap \mathrm{sig}_1(w_j)=\varnothing=\mathrm{sig}_2(w_i) \cap \mathrm{sig}_2(w_j)$ for $i \neq j$. Since $\mathrm{sig}_1(w_i) \cup \mathrm{sig}_2(w_i)$ lies in a 14-set (2 copies of $\mathbb{Z}/7\mathbb{Z}$), and since $|\mathrm{sig}_1(w_i) \cup \mathrm{sig}_2(w_i)|=3$, we have $2c \leqslant 4$, so

$$b=4, \qquad c=2.$$

Since $a+c$ is odd, $a \geqslant 1$. The computer printout of the members of $\mathscr{F}_{28}$ which are compatible with at least one of the 35 pairs of compatible elements of $\mathscr{F}_{14}$ showed that there are no elements of $\mathscr{F}_{28}$ which are compatible with a pair $\{E_1, E_2\}$, where E_1, E_2 are compatible elements of $\mathscr{F}_{14}$ with $E_1=\tilde{F}_2$. Thus in searching for a 4-set $\{W_1, W_2, W_3, W_4\}$ we may assume that each $W_i \in \tilde{F}_1\bar{N} \cup \tilde{F}_3\bar{N}$.

If $W_1, W_2 \in \tilde{F}_1\bar{N}$ then we may take $W_1=\tilde{F}_1$, $W_2=\tilde{F}_1\sigma$, and there are then no available choices for W_3. Thus at most one of W_1, W_2, W_3, W_4 is in $\tilde{F}_1\bar{N}$.

(i) First, suppose $W_1=\tilde{F}_1$. For $i=2, 3, 4$ let

$$\mathrm{sig}_1(W_i)=\{x_i\}, \qquad \mathrm{sig}_2(W_i)=\{y_i, z_i\}.$$

By (4), $\{y_2, z_2\}$, $\{y_3, z_3\}$ and $\{y_4, z_4\}$ are in distinct orbits under $x \mapsto x+1$, so we may assume that

$$\{y_2, z_2\} \in \{01, 12, 23, \ldots, 60\},$$
$$\{y_3, z_3\} \in \{02, 13, 24, \ldots, 61\},$$
$$\{y_4, z_4\} \in \{03, 14, 25, \ldots, 62\}.$$

By multiplying W_2, W_3 and W_4 on the right by a suitable $L_2(\zeta^k)$ we may assume that

$$\{y_2, z_2\} = \{0, 1\}.$$

Now (4) implies that the $\{y_i, z_i\}$ are one of the following:

$$\begin{bmatrix} y_2 & z_2 \\ y_3 & z_3 \\ y_4 & z_4 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 2 & 4 \\ 3 & 6 \end{bmatrix}, \quad \begin{bmatrix} 0 & 1 \\ 3 & 5 \\ 6 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & 1 \\ 4 & 6 \\ 2 & 5 \end{bmatrix}.$$

For each of these three possibilities we try to choose x_2, x_3, x_4 so that (3) and (4) are satisfied. There are three solutions:

$$\begin{bmatrix} x_2 & y_2 & z_2 \\ x_3 & y_3 & z_3 \\ x_4 & y_4 & z_4 \end{bmatrix} = \underset{\text{(a)}}{\begin{bmatrix} 5 & 0 & 1 \\ 4 & 2 & 4 \\ 2 & 3 & 6 \end{bmatrix}}, \quad \underset{\text{(b)}}{\begin{bmatrix} 5 & 0 & 1 \\ 4 & 3 & 5 \\ 2 & 6 & 2 \end{bmatrix}} \quad \text{or} \quad \underset{\text{(c)}}{\begin{bmatrix} 4 & 0 & 1 \\ 2 & 2 & 4 \\ 5 & 3 & 6 \end{bmatrix}}. \quad (5)$$

Since $L_1(\alpha)$ fixes $\tilde{F}_1$ and $u\zeta^5\langle\alpha, \beta\rangle L_1(\alpha) = u\zeta^5\langle\alpha, \beta\rangle\gamma$, if we are in case (5a) or (5b) we may assume that W_2 contains

$$\xi = u\zeta^5\langle\alpha, \beta\rangle \cup v\langle\alpha, \beta\rangle \cup v\zeta\langle\alpha, \beta\rangle.$$

The computer showed that no member of $\mathscr{F}_{28}$ is compatible with $\tilde{F}_1$ and ξG, so we must be in case (5c).

Since $L_1(\alpha)$, $L_2(\alpha)$, $L_2(\beta)$, $L_1(\beta\gamma)\, L_2(\gamma)$, γ and τ all fix $W_1 = \tilde{F}_1$, there are six possibilities for (w_2, w_3, w_4) as shown in Table VIII. We have written $w_i = uX_i \cup vY_i$, where $X_i = X_i'\langle\alpha, \beta\rangle$ and $Y_i = Y_i'\langle\alpha, \beta\rangle$.

The computer then established:

Proposition 21. *For each of the six cases shown in Table* VIII *the code generated by* $W_1 \cup W_2 \cup W_3 \cup W_4$ *contains codewords of weight* 8 *or* 16.

(ii) Suppose $W_i \in \tilde{F}_3\bar{N}$ for $i = 1, \ldots, 4$. Then (4) implies that not all four w_i satisfy

$$|\operatorname{sig}_1(w_i)| = 2.$$

TABLE VIII

The Six Possibilities for (w_2, w_3, w_4)

	X_2'	Y_2'	X_3'	Y_3'	X_4'	Y_4'
1	ζ^4	$1, \zeta$	ζ^2	ζ^2, ζ^4	ζ^5	ζ^3, ζ^6
2	ζ^4	$1, \zeta$	ζ^2	$\zeta^2, \zeta^4\gamma$	ζ^5	ζ^3, ζ^6
3	ζ^4	$1, \zeta$	ζ^2	ζ^2, ζ^4	$\zeta^5\gamma$	ζ^3, ζ^6
4	ζ^4	$1, \zeta$	ζ^2	$\zeta^2, \zeta^4\gamma$	$\zeta^5\gamma$	ζ^3, ζ^6
5	ζ^4	$1, \zeta$	ζ^2	$\zeta^2\gamma, \zeta^4$	$\zeta^5\gamma$	ζ^3, ζ^6
6	ζ^4	$1, \zeta$	ζ^2	$\zeta^2\gamma, \zeta^4\gamma$	$\zeta^5\gamma$	ζ^3, ζ^6

Similarly not all four w_i satisfy $|\mathrm{sig}_2(w_i)| = 2$. Replacing each W_i by $W_i\sigma$, if necessary, we may therefore assume that

$$\begin{bmatrix} |\mathrm{sig}_1(w_1)|, & |\mathrm{sig}_2(w_1)| \\ |\mathrm{sig}_1(w_2)|, & |\mathrm{sig}_2(w_2)| \\ |\mathrm{sig}_1(w_3)|, & |\mathrm{sig}_2(w_3)| \\ |\mathrm{sig}_1(w_4)|, & |\mathrm{sig}_2(w_4)| \end{bmatrix}$$

is either

$$\begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \\ 1 & 2 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 1 & 2 \\ 1 & 2 \end{bmatrix}.$$

A straightforward case-by-case analysis now shows that it is impossible to choose the $\mathrm{sig}_i(w_j)$ so as to satisfy (3) and (4).

We conclude that

$$S_1 = \tilde{F}_1 \cup \tilde{F}_1 \sigma.$$

Further analysis is now required, for we must learn something about S_0, $\binom{112}{12}$ being too large a number for the computer.

3.13. *The "Type" of a Block of S_0*

We assume $S_1 = \tilde{F}_1 \cup \tilde{F}_1\sigma$, and proceed to classify the blocks of S_0 according to their "type." Let $B \in S_0$, and let $B' = B \cap \Omega_1$ and $B'' = B \cap \Omega_2$ be the left and right halves of B. Then

$$|f \cap B'| = 0 \text{ or } 2 \qquad \text{for all } f \in \tilde{F}_1, \tag{6}$$

since $f \subseteq \Omega_1$ if $f \in \tilde{F}_1$ (see Proposition 12). Since

$$\sum_{f \in \tilde{F}_1} = \Omega_1,$$

where the summation sign denotes the symmetric difference, it follows that

$$|B'| \text{ is even.}$$

For each $i \in \mathbb{Z}/7\mathbb{Z}$ let

$$a_i = |\{ux \in B' \mid x \in \zeta^i E\}|,$$

and call the vector $(a_0, a_1, ..., a_6)$ the *type* of B'. The type of B'' is defined similarly. Now $\tilde{F}_1$ contains

$$\tilde{f}_1 = u\langle \alpha, \beta\rangle \cup u\zeta\langle \alpha, \beta\rangle \cup u\zeta^3\langle \alpha, \beta\rangle,$$

and

$$\begin{aligned} a_0 + a_1 + a_3 &= |\tilde{f}_1 \cap B'| + |\tilde{f}_1 \gamma \cap B'| \\ &= 0, 2 \text{ or } 4 \qquad \text{from (6).} \end{aligned}$$

Therefore

$$a_i + a_{i+1} + a_{i+3} = 0, 2 \text{ or } 4 \qquad \text{for all } i, \tag{7}$$

where the subscripts are to be read modulo 7. From (7) we may deduce that

$$|B'| \neq 10, \qquad |B'| \neq 12. \tag{8}$$

To see this, first suppose $a_i \geqslant 3$ for some i. Then (7) implies $a_j \leqslant 1$ for all $j \neq i$, whence $\sum_k a_k = |B'| \leqslant 3 + 6 = 9$. On the other hand, suppose $a_i \leqslant 2$ for all i and $|B'| = 10$ or 12. Then the components of the type of B' are one of the following:

$$(2^6 0), \qquad (2^5 1^2), \qquad (2^5 0^2), \qquad (2^4 1^2 0), \qquad (2^3 1^4).$$

However, it is easily checked that in each case there is no way to choose the a_i's without violating (7). Thus (8) holds. By symmetry

$$|B''| \neq 10, \qquad |B''| \neq 12. \tag{9}$$

From (8) and (9) we deduce:

LEMMA 22. *If* $B \in S_0$ *then* $(|B'|, |B''|)$ *is one of* $(8, 4)$, $(6, 6)$ *or* $(4, 8)$.

If $C \in S_0$ and $C' = C \cap \Omega_1$, $C'' = C \cap \Omega_2$ with the type of C' equal to $(x_0, ..., x_6)$, we say that the types of B' and C' are *equivalent* if for some n

$$x_i = a_{i+n} \qquad \text{for} \quad i = 0, ..., 6.$$

We need a complete list of inequivalent types.

Case 1. $|B'| = 4$.

The components of type (B') are (40^6), (310^5), (2^20^5), (21^20^4) or (1^40^3). The second and fourth possibilities violate (7), and we find that type (B') is equivalent to one of

$$t_0 = (4000000),$$

$$t_1 = (2200000),$$

$$t_2 = (2020000),$$

$$t_3 = (2002000),$$

$$t_4 = (1001011).$$

Case 2. $|B'| = 6$.

The components of type (B') are (60^6), (510^5), (420^5), (41^20^4), (3^20^5), (3210^4), (31^30^3), (2^30^4), $(2^21^20^3)$, (21^40^2) or (1^60). Using (7) we find that type (B') is equivalent to one of

$$t_5 = (3001011),$$

$$t_6 = (1003011),$$

$$t_7 = (1001031),$$

$$t_8 = (1001013),$$

$$t_9 = (2220000),$$

$$t_{10} = (2200200),$$

$$t_{11} = (2200020),$$

$$t_{12} = (2020200),$$

$$t_{13} = (1201011),$$

$$t_{14} = (1021011),$$

$$t_{15} = (1001211).$$

Case 3. $|B'| = 8$.

Equation (7) implies $a_i \leqslant 3$ for all i, and if $a_1 = 3$ for some i then $a_j \leqslant 1$ for all $j \neq i$. The cases (31^50), $(2^31^20^2)$ and (21^6) also violate (7), leaving (2^40^3) and (2^21^40), and so type (B') is equivalent to one of

$$t_{16} = (2002022),$$

$$t_{17} = (1221011),$$

$$t_{18} = (1201211),$$

$$t_{19} = (1021211).$$

By symmetry, type (B'') is also equivalent to one of these 20 types. Thus we have proved the following result.

LEMMA 23. *If $B \in S_0$ then type (B') and type (B'') are equivalent to one of $t_0, t_1, \ldots, t_{19}$.*

We can immediately eliminate one of the possibilities.

LEMMA 24. *The type $t_0 = (4000000)$ does not occur.*

Proof. Suppose on the contrary that type $(B') = t_0$. Then type $(B'') = (b_0, \ldots, b_6)$ is equivalent to t_{16}, t_{17}, t_{18} or t_{19}. There are two values $i_0 \neq i_1$ with $b_{i_0} = b_{i_1} = 2$, and so for $\nu = 0$ and 1 we may define $\alpha_\nu, \beta_\nu \in E$ by

$$B'' \cap v\zeta^{i_\nu}E = \{v\zeta^{i_\nu}\alpha_\nu, v\zeta^{i_\nu}\beta_\nu\}.$$

Let $\gamma_\nu = \alpha_\nu \beta_\nu$. Then

$$B'' \cap B''\gamma_\nu \supseteq B'' \cap v\zeta^{i_\nu}E,$$

whence $\gamma_0 \neq \gamma_1$ (or else $|B'' \cap B''\gamma_0| \geqslant 4$). Furthermore since $|B \cap Bg| = 0$ or 2 for all $g \in G - \{1\}$ it follows that

$$B' \cap B'\gamma_\nu = \varnothing \qquad \text{for} \quad \nu \in \{0, 1\}.$$

But $B' = \{u\zeta^i\pi_j \mid 1 \leqslant j \leqslant 4\}$ for some fixed i, where $\pi_1, \pi_2, \pi_3, \pi_4$ are distinct elements of E. If we set

$$\tilde{E} = \{\pi_k\pi_l \mid 1 \leqslant k < l \leqslant 4\}$$

we see that

$$B' \cap B'\pi_k\pi_l = \{u\zeta^i\pi_k, u\zeta^i\pi_l\}.$$

Therefore $\gamma_0, \gamma_1 \notin \tilde{E}$. But this is impossible, since $|\tilde{E}| = 6$ and $|E - \{1\}| = 7$.
Q.E.D.

3.14. *The Final Step in the Proof*

The final step is to show that S_0 must contain a G-orbit of blocks of a certain kind, but that no such G-orbit satisfies the (0, 2, 12)-intersection property.

For $1 \leqslant i \leqslant 19$ let

$$X_i = \{B \in S_0 \mid \text{type } (B') \text{ is equivalent to } t_i\},$$

$$x_i = |X_i|.$$

Our first goal is to determine the x_i. For this purpose, let

$$G_3 = \text{the family of 3-element subsets of } uG,$$
$$G_3^* = \{\xi \in G_3 \mid \xi \subseteq F \text{ for some } F \in \tilde{F}_1\},$$
$$\tilde{G}_3 = G_3 - G_3^*.$$

If $\xi \in G_3$, then type (ξ) is equivalent to one of

$$u_1 = (3000000),$$
$$u_2 = (2100000),$$
$$u_3 = (2010000),$$
$$u_4 = (2001000),$$
$$u_5 = (2000100),$$
$$u_6 = (2000010),$$
$$u_7 = (2000001),$$
$$u_8 = (1110000),$$
$$u_9 = (1101000),$$
$$u_{10} = (1100100),$$
$$u_{11} = (1100010),$$
$$u_{12} = (1010100).$$

Let

$$b_i = |\{\xi \in G_3 \mid \text{type } (\xi) \text{ is equivalent to } u_i\}|,$$
$$c_i = |\{\xi \in G_3^* \mid \text{type } (\xi) \text{ is equivalent to } u_i\}|,$$
$$d_i = |\{\xi \in \tilde{G}_3 \mid \text{type } (\xi) \text{ is equivalent to } u_i\}|,$$

so that

$$d_i = b_i - c_i.$$

It is straightforward to calculate the values of b_i, c_i and hence of $e_i := d_i/56 = (b_i - c_i)/56$. The results are given in Table IX.

We now relate the x_i and d_i. For $1 \leqslant i \leqslant 19$, $1 \leqslant j \leqslant 12$, let B be any 12-element subset of Ω with type $(B') = t_i$, and set

$$a_{ij} = |\{\xi \in G_3 \mid \xi \subseteq B', \text{ type } (\xi) \text{ equivalent to } u_j\}|.$$

TABLE IX

Values of b_i, c_i, $e_i = d_i/56$

	i					
	1	2	3	4	5	6
b_i	$7\binom{8}{3}$	$56\binom{8}{2}$	$56\binom{8}{2}$	$56\binom{8}{2}$	$56\binom{8}{2}$	$56\binom{8}{2}$
c_i	$12 \cdot 14$	$24 \cdot 14$	$24 \cdot 14$	$24 \cdot 14$	$24 \cdot 14$	$24 \cdot 14$
e_i	4	22	22	22	22	22
	7	8	9	10	11	12
b_i	$56\binom{8}{2}$	$7 \cdot 8^3$	$7 \cdot 8^3$	$7 \cdot 8^3$	$7 \cdot 8^3$	$7 \cdot 8^3$
c_i	$24 \cdot 14$	0	$4^3 \cdot 14$	0	0	0
e_i	22	64	48	64	64	64

Since every 3-subset of Ω_1 is contained in a unique element of either S_0 or S_1, it follows that

$$\sum_{i=1}^{19} x_i a_{ij} = d_j, \qquad j = 1,..., 12. \tag{10}$$

G acts semiregularly on S_0, so each x_i is a multiple of 56. After dividing (10) by 56 and setting $y_i = x_i/56$, we obtain

$$\sum_{i=j}^{19} y_i a_{ij} = e_j, \qquad j = 1,..., 12. \tag{11}$$

The values of a_{ij} are also straightforward to calculate and are given in Table X.

There are three further equations. The first two involve the 3-subsets ξ of Ω which are in neither Ω_1 nor Ω_2. For such ξ, $(|\xi \cap \Omega_1|, |\xi \cap \Omega_2|)$ is $(1, 2)$ or $(2, 1)$, and there are $56\binom{56}{2}$ subsets of each kind. For $1 \leqslant i \leqslant 19$, $1 \leqslant \nu \leqslant 2$, let $u_{i\nu}$ be the number of ξ with $|\xi \cap \Omega_\nu| = 1$ such that ξ is contained in a 12-subset B of Ω with type $(B') = t_i$. Then

$$u_{i1} = u_{20-i,2} = \binom{4}{1}\binom{8}{2} = 112, \qquad 1 \leqslant i \leqslant 4,$$

$$u_{i1} = u_{i2} = \binom{6}{1}\binom{6}{2} = 90, \qquad 5 \leqslant i \leqslant 15,$$

$$u_{i1} = u_{20-i,2} = \binom{8}{1}\binom{4}{2} = 48, \qquad 16 \leqslant i \leqslant 19,$$

TABLE X

Values of a_{ij}

	j											
i	1	2	3	4	5	6	7	8	9	10	11	12
1	0	2	0	0	0	0	2	0	0	0	0	0
2	0	0	2	0	0	2	0	0	0	0	0	0
3	0	0	0	2	2	0	0	0	0	0	0	0
4	0	0	0	0	0	0	0	1	0	1	1	1
5	1	0	0	3	0	3	3	3	0	3	1	3
6	1	0	3	3	3	0	0	1	0	3	3	3
7	1	3	3	0	0	3	0	3	0	1	3	3
8	1	3	0	0	3	0	3	3	0	3	3	1
9	0	4	2	0	0	2	4	8	0	0	0	0
10	0	2	0	4	4	0	2	0	0	8	0	0
11	0	2	2	2	2	2	2	0	0	0	8	0
12	0	0	4	2	2	4	0	0	0	0	0	8
13	0	0	1	0	1	1	1	3	4	1	3	5
14	0	1	0	1	1	1	0	1	4	5	3	3
15	0	1	1	1	0	0	1	5	4	3	3	1
16	0	4	4	4	4	4	4	8	0	8	8	8
17	0	3	1	1	2	2	3	11	8	9	9	7
18	0	1	2	3	3	1	2	7	8	11	9	9
19	0	2	3	2	1	3	1	9	8	7	9	11

and

$$\sum_{i=1}^{19} y_i u_{iv} = \binom{56}{2}, \qquad v = 1, 2. \tag{12}$$

Finally $|S_0| = 1036 - 28 = 56 \cdot 18$, and so

$$\sum_{i=1}^{19} y_i = 18. \tag{13}$$

The computer then established:

PROPOSITION 25. *Equations* (11)–(13) *have a unique solution in nonnegative integers, namely,*

$y_i = 1$ *for* $1 \leqslant i \leqslant 8$ *and* $16 \leqslant i \leqslant 19$,

$y_i = 0$ *for* $9 \leqslant i \leqslant 12$,

$y_i = 2$ *for* $13 \leqslant i \leqslant 15$.

Since $y_5 = 1$, there is a G-orbit, BG say, of blocks in S_0 with type (B') equivalent to t_5. Then type (B'') is equivalent to t_{13}, t_{14} or t_{15}. For $|B'| = 6$ implies that type (B'') is equivalent to one of $t_5,..., t_{15}$; $t_9,..., t_{12}$ do not occur since $y_9,..., y_{12} = 0$; and the next lemma eliminates $t_5,..., t_8$.

LEMMA 26. *If $B \in S_0$ is a block with type $(B') = t_5$, then type (B'') does not have a component equal to* 3.

Proof. Suppose on the contrary that type (B'') is equivalent to t_5, t_6, t_7 or t_8. We represent subsets of Ω_1 or Ω_2 by the corresponding elements in the group ring $\mathbb{Z}G$. For example, B' is represented by an element $T_1 \in \mathbb{Z}G$ of the form

$$\pi_1 + \pi_2 + \pi_3 + \zeta^3\mu_3 + \zeta^5\mu_5 + \zeta^6\mu_6, \tag{14}$$

where $\pi_i, \mu_j \in E$; and B'' is represented by $T_2 h \in \mathbb{Z}G$, for some $h \in G$, where T_2 is of the form (14) if type (B'') is equivalent to t_5, or has a similar expression in the other three cases.

From Theorem 2,

$$f(B) := \sum_{g \in G} |B \cap Bg|\, g$$

is an element of $\mathbb{Z}G$ with coefficients 0, 2 and 12. We can find $f(B)$ from the observations that

$$f(B) = f(B') + f(B'')$$

and

$$f(B') = T_1^* T_1, \qquad f(B'') = h^{-1} T_2^* T_2 h,$$

where * is the map $\mathbb{Z}G \to \mathbb{Z}G$ given by

$$\left(\sum_{g \in G} a_g g\right)^* = \sum_{g \in G} a_g g^{-1}.$$

Therefore

$$T_1^* T_1 + h^{-1} T_2^* T_2 h$$

is an element of $\mathbb{Z}G$ with coefficients 0, 2 and 12. It is easy to verify by hand that if T_1 is given by (14), none of the possible choices for T_2 satisfy this condition. Q.E.D.

Thus we may assume $B = B' \cup B''$ with type $(B') = t_5$ and type (B'') equivalent to t_{13}, t_{14} or t_{15}. It is straightforward to verify that there are 64 choices for B' that satisfy the $(0, 2, 12)$-intersection property with the blocks of S_1, namely,

$$\{1, \alpha^i\beta, \alpha^j\gamma, \zeta^3\gamma(\alpha\beta)^k\alpha^\nu, \zeta^5(\alpha\beta\gamma)^l\alpha^\nu; \zeta^6\beta(\alpha\beta)^m\alpha^\nu\},$$

for $i, j, k, l, m, \nu \in \{0, 1\}$. Similarly there are $144 \cdot 56$ choices for B'' if type (B'') is equivalent to t_{13}, namely,

$$B'' = Ch, \qquad \text{for some } h \in G,$$

where C has the form of either

$$\{1, \zeta\gamma\alpha^\nu, \zeta\beta\gamma\alpha^\nu, \zeta^3(\alpha\beta)^i\alpha^\nu, \zeta^5(\alpha\beta\gamma)^j\alpha^\nu, \zeta^6(\alpha\gamma)^k\alpha^\nu\}$$

or

$$\{1, \zeta\beta^i\alpha^\theta, \zeta\gamma\beta^j\alpha^\theta, \zeta^3\gamma(\alpha\beta)^k\alpha^\nu, \zeta^5(\alpha\beta\gamma)^l\alpha^\nu, \zeta^6(\alpha\gamma)^m\alpha^{\nu+1}\}$$

for $i, j, k, l, m, \theta, \nu \in \{0, 1\}$. There are $2 \cdot 144 \cdot 56$ further choices for B'' if type (B'') is equivalent to t_{14} or t_{15}. The computer now established:

PROPOSITION 27. *None of the preceding* $64 \cdot 3 \cdot 144 \cdot 56$ *possible G-orbits has the* $(0, 2, 12)$*-intersection property among its blocks.*

Thus it is impossible to construct the required set of 56 blocks, and the proof of Theorem 1 is complete.

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