On exceptions of integral quadratic forms

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Let f_n denote an even unimodular positive definite quadratic form of dimension n. (This implies that n is of the form $8m$.) Let T_n (the *exceptions* for n) be the set of natural numbers a such that some f_n does not represent $2a$.

It has been known for a long time (see [8]) that $T_8 = T_{16} = \emptyset$. In [8] Peters used Deligne's estimates for cusp forms to show

$$(1)\qquad \begin{aligned} &T_{24} = \{1\},\ T_{32} \subseteqq \{1, 2\},\ T_{40} \subseteqq \{1, 2, 3\},\ T_{48} \subseteqq \{1, 2, \ldots, 96\},\\ &T_{56} \subseteqq \{1, 2, \ldots, 133\}, \text{ and } T_{64} \subseteqq \{1, 2, \ldots, 210\}. \end{aligned}$$

We shall extend his work to prove that $T_{32} = T_{40} = \{1\}$, $T_{48} \subseteqq \{1, 2, 3\}$, $T_{56} \subseteqq \{1, 2, 4\}$, $T_{64} \subseteqq \{1, 2, 5\}$ and $T_{72} \subseteqq \{1, 2, 3, 4, 6\}$. Similar results could be obtained for the weight enumerators of even self-dual codes (cf. [4], [10]). In dimensions $n \geqq 48$ we do not determine the exceptions completely. However, if for example $4 \in T_{56}$, then as will be explained below any quadratic form f_{56} that fails to represent 8 must be very special; in particular it must represent $2a$ for every nonnegative $a \neq 4$.

The theta series of f_n is

$$\theta(q) = \sum_{\underset{\sim}{x}} q^{\frac{1}{2} f_n(\underset{\sim}{x})},$$

the sum extending over all integral $\underset{\sim}{x} = (x_1, \ldots, x_n)$. We may also write

$$\theta(q) = \sum_{m=0}^{\infty} \alpha_m q^m,$$

where α_m is the number of times f_n represents $2m$. Of course $\alpha_m \geqq 0$, and by a theorem of Witt (see [7]) α_1 cannot be too large:

$$(2)\qquad \alpha_1 \leqq 2n(n-1), \text{ if } n \geqq 16.$$

Furthermore there is a unique quadratic form, D_n, for which equality holds in (2).

The proofs of our results rely on the fact that the theta series of f_n is a modular form of weight $n/2$ (see [9]), and so the coefficient vector $(\alpha_0, \alpha_1, \ldots)$ belongs to a finite-dimensional vector space. The conditions that the α_m be nonnegative integers, that $\alpha_0 = 1$, and that α_1 satisfy (2) restrict the possible choices severely, and enable us to determine which of the α_m can be zero. For example, when $n = 56$ it is possible to choose the α_m to

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satisfy all of the required conditions and have $\alpha_4 = 0$, but then all the other α_m (for $m \neq 4$) must be positive. Whether there actually exist quadratic forms with such theta series is an interesting open question.

The theta series of f_n can be expressed ([9], Theorem 6. 1. 2) in terms of the Eisenstein series

$$E_k(q) = 1 + (-1)^k \frac{4k}{|B_{2k}|} \sum_{m=1}^{\infty} \sigma_{2k-1}(m) q^m, \quad k = 2, 3, 4, \ldots,$$

and the cusp form

$$\Delta(q) = q \prod_{m=1}^{\infty} (1 - q^m)^{24},$$

where B_{2k} is a Bernoulli number and

$$\sigma_{2k-1}(m) = \sum_{d|m} d^{2k-1}.$$

Thus we may write

$$\theta(q) = E_{n/4}(q) + \sum_{i=1}^{[n/24]} A_i E_{n/4 - 6i}(q)\, \Delta(q)^i$$

for appropriate coefficients A_i. Alternatively we may write

$$\theta(q) = E_2(q)^{n/8} + \sum_{i=1}^{[n/24]} B_i E_2(q)^{n/8 - 3i} \Delta(q)^i.$$

Since $E_2(q)$ and $\Delta(q)$ have integer coefficients in their q-expansions, and leading coefficient 1, the second expression above shows that if the coefficients of $q^0, q^1, \ldots, q^{[n/24]}$ in $\theta(q)$ are integers then so are all the coefficients. In each dimension n there is a unique theta series beginning

$$1 + 0.q + 0.q^2 + \cdots + 0.q^{[n/24]} + \cdots,$$

the so-called *extremal* theta series. For large n this always contains a negative coefficient, but for small n (certainly for $n \leqq 48$) there exist quadratic forms corresponding to these extremal theta series ([1], [4], [10]).

Dimension 32. The theta series of f_{32} has the form

$$\theta(q) = E_8(q) - A E_2(q)\, \Delta(q),$$

where A is a rational number. Choosing A to make $\alpha_1 = 0$ leads to the extremal theta series, for which quadratic forms exist ([2], [8]). Choosing A instead so that $\alpha_2 = 0$ makes α_1 negative. Therefore from (1) we have

$$T_{32} = \{1\}.$$

Dimension 40. Similar arguments show that the only exception arises from the extremal theta series, and since a lattice with this theta function exists [6] we have

$$T_{40} = \{1\}.$$

Dimension 48. The theta series of f_{48} may be written in the form

$$\theta(q)=E_{12}(q)-(A_0-l)\,E_6(q)\,\Delta(q)+B\Delta(q)^2, \tag{3}$$

where

$$A_0=\frac{48}{|B_{24}|}=\frac{131040}{236364091},$$

l is an integer and B is a rational number. The coefficient of q in (3) is l, and from (2)

$$0\leqq l\leqq 4512.$$

Lemma. *If* (3) *has nonnegative coefficients in its q-expansion then the point* (l, B) *lies in or on the region PQRS shown in Figure* 1. *Furthermore the only coefficients that can vanish are* α_1 *(on PQ),* α_2 *(on QR) and* α_3 *(on SP).*

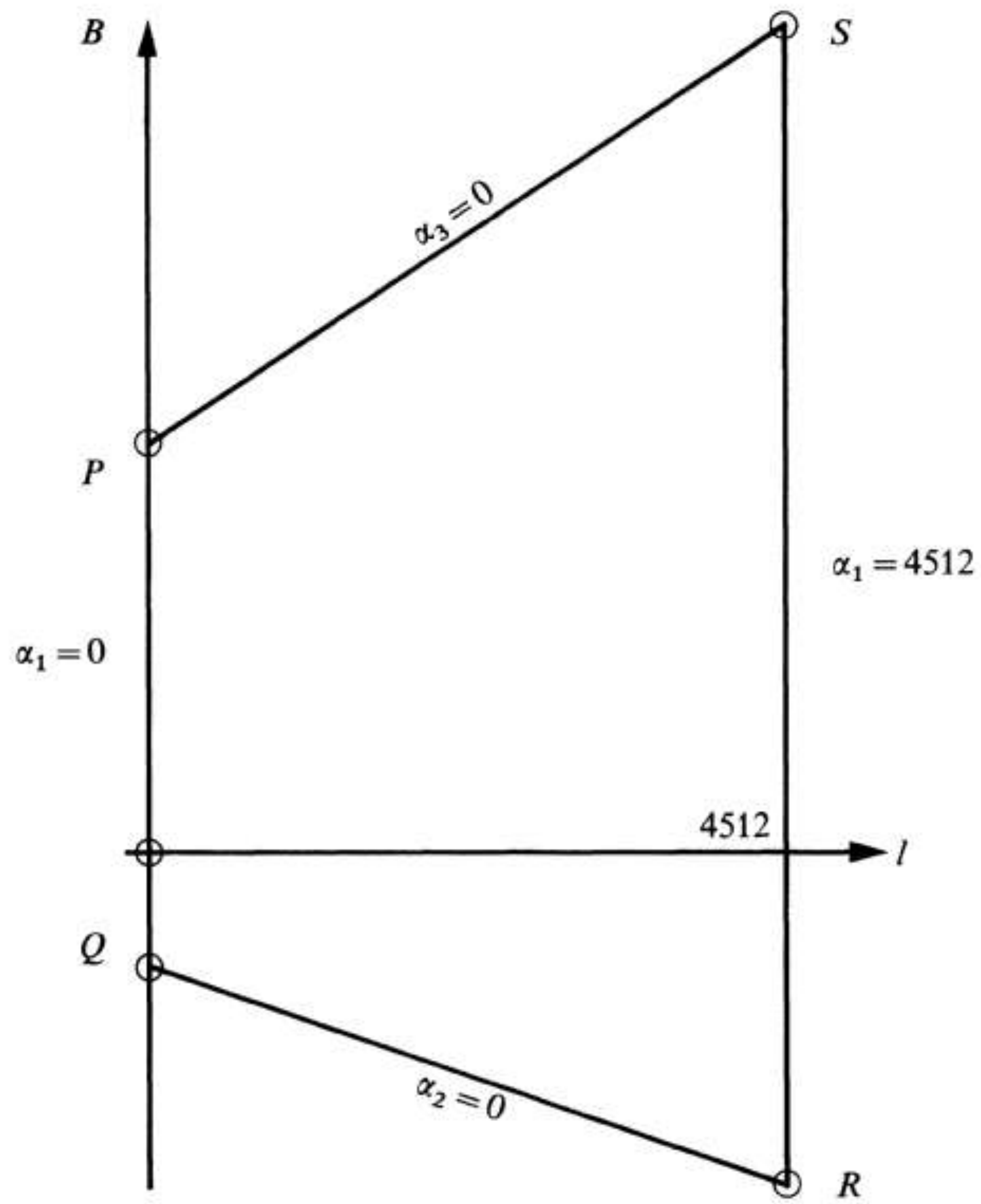

Figure 1. The theta series (3) has nonnegative coefficients if and only if (l, B) lies in the closure of $PQRS$, where $P\approx(0, 1.087\times 10^6)$, $Q\approx(0, -4651)$, $R\approx(4512, -324200)$, $S\approx(4512, 1.916\times 10^7)$. The diagram is not drawn to scale.

Outline of Proof. The assertion was verified for $\alpha_1, \ldots, \alpha_9$ by computer. For the higher coefficients we followed Peters and used Deligne's estimates for cusp forms. Since we knew the exact values of l and B at the vertices P, Q, R, S we were able to improve on Peters' estimates and show that none of $\alpha_{10}, \alpha_{11}, \ldots$ can vanish in or on $PQRS$.

Q.E.D.

The extremal theta series corresponds to the point Q and has exceptions 1 and 2. Two quadratic forms with this theta series were given in 1971 in [3], and [8] cites a 1979 paper by Venkov. As mentioned above, D_{48} is the only quadratic form on RS. It is an interesting open question to determine whether any quadratic form exists on the line SP, with 3 as an exception. There are many theta series on this line with nonnegative integral coefficients, the most interesting being that corresponding to the point P:

$$1+1092000\, q^2+40186692000\, q^4+6592596541440\, q^5+437982028848000\, q^6+\cdots.$$

In any case we have established

$$\{1, 2\} \subseteqq T_{48} \subseteqq \{1, 2, 3\}.$$

Dimension 56. Dimension 56 is parallel to 48, with one surprise: the line SP in the Lemma and Figure 1 now corresponds to $\alpha_4=0$ (instead of $\alpha_3=0$). Again it is not known if there is any quadratic form on SP. The vertex P no longer corresponds to a theta series with integral coefficients, but we can take $l=810$ for example and obtain

$$1+810\, q+10434006\, q^2+2141828952\, q^3+19518351334380\, q^5+2114964879330600\, q^6+\cdots,$$

having 4 as an exception. Again Q corresponds to the extremal theta series, which is

$$1+15590400\, q^3+36957286800\, q^4+15284192071680\, q^5+2099603881267200\, q^6+\cdots,$$

but now it is not known if a corresponding quadratic form exists. We can conclude only that

$$T_{56} \subseteqq \{1, 2, 4\}.$$

Dimension 64. Dimension 64 is similar to 48 and 56, except that the line SP corresponds to $\alpha_5=0$, containing for example

$$1+24672\, q+338215905\, q^2+147350910000\, q^3+15702471353496\, q^4 + 35296495687141164\, q^6+\cdots.$$

The extremal theta series is

$$1+2611200\, q^3+19524758400\, q^4+19715347537920\, q^5+5615943999897600\, q^6+\cdots,$$

and we have

$$T_{64} \subseteqq \{1, 2, 5\}.$$

Dimension 72. The theta series of f_{72} may be written

$$\theta(q)=E_{18}(q)-(A_0-l)\, E_{12}(q)\, \Delta(q)+BE_6(q)\, \Delta(q)^2-C\Delta(q)^3; \tag{4}$$

where $A_0=\dfrac{72}{|B_{36}|}$, l is an integer satisfying

$$0 \leqq l \leqq 10224 \tag{5}$$

and B, C are rational numbers.

Lemma. *If* (4) *has nonnegative coefficients then*

$$|B| \leqq 3.40463\times 10^8, \quad |C| \leqq 5.21241\times 10^{11}.$$

Furthermore the only coefficients that can vanish are α_1, α_2, α_3, α_4 *and* α_6.

Outline of Proof. The lemma was established by identifying the intersection of (5) and the half-planes

$$\alpha_m \geqq 0, \quad m=2, 3, \ldots, 9,$$

in (l, B, C)-space. The intersection is a thin slab bounded by (5) and the faces $\alpha_2=0$, $\alpha_3=0$, $\alpha_4=0$ and $\alpha_6=0$. Deligne's estimates were then applied to show that $\alpha_{10}, \alpha_{11}, \ldots \geqq 0$ in and on this slab. Q.E.D.

Examination of the faces of this slab shows that if $\alpha_6=0$ then at most one of α_1, α_3 and α_4 can vanish, an example being

$$1+43659000\,q+714910316160\,q^2+1962733030283070\,q^3$$
$$+1185198550852737156\,0q^5+1480801957976416700286\,0q^7+\cdots,$$

with exceptions 4 and 6. If $\alpha_6 \neq 0$ and $\alpha_4=0$ then it is possible to have $\alpha_1=0$ or $\alpha_2=0$ or both, for example

$$1+86363550\,q^3+15496315412760\,q^5+9022158250560000\,q^6+\cdots.$$

The extremal theta series is

$$1+6218175600\,q^4+15281788354560\,q^5+9026867482214400\,q^6+\cdots.$$

It is not known if quadratic forms corresponding to any of these exist. We conclude that

$$T_{72} \subseteqq \{1, 2, 3, 4, 6\}.$$

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