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On the Enumeration of Lattices of Determinant One

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The integral lattices of determinant 1 and dimension not exceeding 20 are enumerated. Siegel's mass formula provides a check that the list is complete. The mass formula is also used to verify that Niemeier's list of even lattices of dimension 24 is complete.

1. INTRODUCTION

The problem of classifying integral lattices (or quadratic forms) has been studied by Siegel, Witt, Kneser, Niemeier and others [2; 7; 9; 10, Sect. 106; 11–14; 16; 17]. The 8- and 16-dimensional even lattices of determinant 1 were found by Witt [17], who also investigated the 24-dimensional problem, remarking: "Bei dem Versuch, eine Form aus einer solchen Klasse wirklich anzugeben, fand ich mehr als 10 verschiedene Klassen in Γ_{24}. Die Bestimmung von h_{24} scheint nicht ganz leicht zu sein." The lattices of determinant 1 and dimension not exceeding 16 were found by Kneser [7], and the even lattices of determinant 1 and dimension 24 were finally enumerated by Niemeier [9]. In this paper we extend Kneser's enumeration to dimension 20. Each lattice has a "Witt decomposition" into "components" held together by "glue." This enables us to find the automorphism group of each lattice, and then use Siegel's mass formula to verify that our enumeration is complete. In the same way we have calculated the groups of Niemeier's 24-dimensional lattices and have verified that his list is complete. The method has also been used to enumerate self-dual error-correcting codes—see, for example, [4, 5].

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TABLE I

The Component Lattices

Name	Det.	Norm 2 vectors	Number	The glue: Vectors	The glue: Norm	Glue group		g_0	g_1
$\lvert d\rvert_1$	d	None	0	$\lvert i\rvert = \left(\frac{i}{\sqrt{d}}\right)$	$\frac{i^2}{d}$	$\lvert i\rvert + \lvert j\rvert = \lvert i+j\rvert$	C_d	1	2: $\lvert i\rvert \leftrightarrow \lvert d-i\rvert$
A_n	$n+1$	$(1, -1, O^{n-2})$	$n(n+1)$	$\lvert i\rvert = \left(\frac{i}{n+1}\right)^j \left(\frac{-j}{n+1}\right)^i$	$\frac{ij}{n+1}$	$\lvert i\rvert + \lvert j\rvert = \lvert i+j\rvert$	C_{n+1}	$(n+1)!$	2: $\lvert i\rvert \leftrightarrow \lvert j\rvert$
				$i+j=n+1$					$(n>2)$
D_4	4	$(\pm 1^2 O^2)$	24	$\lvert 1\rvert = \frac{1}{2}\frac{1}{2}\frac{1}{2}\frac{1}{2}$	1	$\lvert i\rvert + \lvert i\rvert = 0$	V_4	192	3!: All perms
				$\lvert 2\rvert = 0001$	1	$\lvert i\rvert + \lvert j\rvert = \lvert k\rvert$			
				$\lvert 3\rvert = \frac{1}{2}\frac{1}{2}\frac{1}{2} - \frac{1}{2}$	1				
D_n	4	$(\pm 1^2 O^{n-2})$	$2n(n-1)$	$\lvert 1\rvert = \left(\frac{1}{2}\right)^n$	$\frac{n}{4}$	$\lvert i\rvert + \lvert i\rvert = 0$	V_4	$2^{n-1}\cdot n!$	2: $\lvert 1\rvert \leftrightarrow \lvert 3\rvert$
n even				$\lvert 2\rvert = O^{n-1}1$	1	$\lvert i\rvert + \lvert j\rvert = \lvert k\rvert$			
$n>4$				$\lvert 3\rvert = \left(\frac{1}{2}\right)^{n-1}\left(-\frac{1}{2}\right)$	$\frac{n}{4}$				

D_n	4	$(\pm 1^2 O^{n-2})$	$2n(n-1)$	$[1] = \left(\frac{1}{2}\right)^n$	$\frac{n}{4}$	$[i] + [j] = [i+j]$	C_4	$2^{n-1} \cdot n!$	$2: [1] \leftrightarrow [3]$
n odd				$[2] = O^{n-1}1$	1				
				$[3] = \left(\frac{1}{2}\right)^{n-1}\left(-\frac{1}{2}\right)$	$\frac{n}{4}$				
E_6	3	$(1, -1, O^4 \mid O^2)$	30	$[1] = \left(\frac{1}{3}^4 - \frac{2}{3}^2 \mid O^2\right)$	$\frac{4}{3}$		C_3	$72 \cdot 6!$	$2: [1] \leftrightarrow [2]$
		$\left(\frac{1}{2}^3, -\frac{1}{2}^3 \mid \frac{1}{2}, -\frac{1}{2}\right)$	40	$[2] = \left(-\frac{1}{3}^4 \frac{2}{3}^2 \mid O^2\right)$	$\frac{4}{3}$				
		$(O^6 \mid 1, -1)$	2						
		Total	72						
E_7	2	$(1, -1, O^6)$	56	$[1] = \left(\frac{1}{4}^6, -\frac{3}{4}^2\right)$	$\frac{3}{2}$		C_2	$8 \cdot 9!$	1
		$\left(\frac{1}{2}^4 - \frac{1}{2}^4\right)$	70						
		Total	126						
E_8	1	$(\pm 1^2 O^6)$	112	None	—		C_1	$192 \cdot 10!$	1
		$\left(\pm\frac{1}{2}^8\right)_{\text{Even}}$	128						
		Total	240						

2. The Component Lattices

The component lattices from which many of the others are built are shown in Table I. We establish our notation by explaining the second line of the table in detail. This describes the n-dimensional lattice A_n (the subscript gives the dimension), which has determinant $n+1$ and contains $n(n+1)$ vectors of norm (i.e., squared length) 2, namely, all permutations of $(1, -1, 0^{n-2})$. The *glue vectors* for A_n are a set of coset representatives for A_n in its dual A_n^*. The index of A_n in A_n^* is the determinant, $n+1$, and as glue vectors we choose

$$[i] = \left(\frac{i}{n+1}, \frac{i}{n+1}, ..., \frac{i}{n+1}, \frac{-j}{n+1}, \frac{-j}{n+1}, ..., \frac{-j}{n+1}\right)$$

$$= \left(\left(\frac{i}{n+1}\right)^j, \left(\frac{-j}{n+1}\right)^i\right),$$

where $i+j=n+1$, for $0 \leqslant i \leqslant n$. The glue vectors are usually chosen to have minimum length in their coset. In this case the norm of $[i]$ is $ij/(n+1)$. The additive structure of the glue vectors modulo the lattice, that is, of the abelian group A_n^*/A_n, is a cyclic group C_{n+1} of order $n+1$ generated by $[1]$.

Before defining g_0 and g_1 we must describe how the component lattices are glued together. First, an integral lattice Λ containing vectors of norm 1 is decomposable (provided the dimension exceeds 1), and may be written as

$$\Lambda = \mathbf{Z}^r \oplus M,$$

where $\mathbf{Z}^r$ is the r-dimensional cubic lattice and M is a lattice of smaller dimension than Λ. Thus we may assume that the minimum norm of Λ is at least 2.

The only indecomposable lattices which are *generated* by vectors of norm 2 are A_n $(n \geqslant 1)$, D_n $(n \geqslant 4)$, E_6, E_7 and E_8. So if Λ is a lattice with minimum norm 2, the sublattice Λ' generated by the vectors of norm 2 is a direct sum

$$\Lambda_1 \oplus \Lambda_2 \oplus \cdots \oplus \Lambda_k,$$

where each of the sublattices Λ_i (the *components*) is a copy of an A_n, D_n, or E_n. Then Λ is generated by Λ' together with certain *glue words*

$$y = (y_1, y_2, ..., y_k),$$

in which y_i is a glue vector for Λ_i.

For example, the first 18-dimensional lattice in Table II, the one called A_9^2, contains a sublattice $\Lambda' = A_9 \oplus A_9$, and is generated by Λ' and the single glue word

$$[1,3] = \left(\left(\frac{1}{10}\right)^9 \left(\frac{-9}{10}\right)^1, \left(\frac{3}{10}\right)^7 \left(\frac{-7}{10}\right)^3 \right)$$

obtained by concatenating the glue vectors $y_1 = [1]$ and $y_2 = [3]$ for A_9.

The automorphism group $G(\Lambda)$ of an n-dimensional lattice Λ is the subgroup of the orthogonal group $O(n)$ that fixes Λ. Just as when studying

TABLE II

Lattices of Determinant 1 With No Vectors of Norm 1

Dimension	Components	Type	Glue words	g_1	g_2
0	0	II	—	1	1
8	E_8	II	—	1	1
12	D_{12}	I	[1]	1	1
14	E_7^2	I	[11]	1	2
15	A_{15}	I	[4]	2	1
16	D_{16}	II	[1]	1	1
16	E_8^2	II	—	1	2
16	D_8^2	I	[12], [21]	1	2
17	$A_{11}E_6$	I	[21]	2	1
18	A_9^2	I	[13]	2	2
18	D_6^3	I	[(013)]	1	6
18	$A_{17}A_1$	I	[31]	2	1
18	$D_{10}E_7A_1$	I	[110], [301]	1	1
19	$E_6^3[3]$	I	[(012) 1]	2	6
19	$A_{11}D_7[3]$	I	[111]	2	1
19	$A_7^2D_5$	I	[111], [042]	2	2
20	D_{20}	I	[1]	1	1
20	$A_{15}D_5$	I	[21]	2	1
20	$D_{12}D_8$	I	[12], [21]	1	1
20	$D_{12}E_8$	I	[10]	1	1
20	$A_{11}E_6A_3$	I	[111]	2	1
20	$A_9^2A_1^2$	I	[1300], [5010], [0501]	2	2
20	$D_8^2D_4$	I	[101], [013], [330]	1	2
20	$E_7^2D_6$	I	[101], [013]	1	2
20	$A_7^2D_5[4]$	I	[1112], [1721]	2	2
20	$D_6^3A_1^2$	I	[(123) 00], [11110], [33301]	1	6
20	A_5^4	I	[2(024)], [3300], [0033]	2	8
20	D_4^5	I	[11111], [(02332)]	1	5!

codes [4, 5], decomposing a lattice into components and glue words makes it much easier to find the automorphism group. It is essential for this that every automorphism of Λ takes the set of component lattices $\Lambda_1, ..., \Lambda_k$ to itself. We shall always choose the components so that this is true.

This being the case, any automorphism in $G(\Lambda)$ will effect some permutation of the Λ_i, so that $G(\Lambda)$ will have a normal subgroup G' consisting of just those automorphisms for which this permutation is trivial. The group of all permutations of the Λ_i that arise in this way we call $G_2(\Lambda)$—it is isomorphic to the quotient group $G(\Lambda)/G'$.

Let $G_0(\Lambda)$ be the normal subgroup of G' consisting of those automorphisms which, for every i, send each glue vector y_i into a vector in the same coset $y_i + \Lambda_i$, i.e., which fix the glue words modulo the components. Then $G'/G_0(\Lambda)$ is isomorphic to a permutation group acting on the glue vectors of each component: we call this permutation group $G_1(\Lambda)$. The order of the full group $G(\Lambda)$ is therefore the product of the orders of $G_0(\Lambda)$, $G_1(\Lambda)$, and $G_2(\Lambda)$. We denote these numbers by $g(\Lambda)$, $g_0(\Lambda)$, $g_1(\Lambda)$, and $g_2(\Lambda)$, respectively. Also $G_0(\Lambda)$ is the direct product of the groups $G_0(\Lambda_i)$. But in general $G_1(\Lambda)$ is only a subgroup of the direct product of the $G_1(\Lambda_i)$ and therefore must be computed separately for each Λ.

The last two columns in Table I give the orders $g_0(\Lambda)$ and $g_1(\Lambda)$. For example, $G_0(A_n)$ is the symmetric group $\sum_{n+1}$, and $G_1(A_n)$ contains the identity and the permutation which exchanges the glue vectors $[i]$ and $[n+1-i]$ for all i. Thus $g_0(A_n) = (n+1)!$, $g_1(A_n) = 2$.

Finally, we observe that the cubic lattice $\mathbf{Z}^r$ is self-orthogonal, has no glue vectors, and $g_0(\mathbf{Z}^r) = 2^r \cdot r!$.

The theta-series of the lattices in Table I may be found in [15].

3. The Enumeration

Theorem 1. *The integral lattices of determinant* 1, *dimension* $\leqslant 20$, *and containing no vectors of norm* 1 *are those shown in Table II.*

Table II gives for each lattice Λ its dimension, its components Λ_i (in terms of those given in Table I), its type (II, if Λ contains only vectors of even norm; otherwise I), generators for the glue words, and the orders $g_1(\Lambda)$, $g_2(\Lambda)$, ($g_0(\Lambda)$ can be read off Table I). The glue words are expressed in terms of the glue vectors of the components, as illustrated in the previous section for the lattice A_9^2. If a glue word contains parentheses, this indicates that all vectors obtained by cyclically shifting the part of the vector inside the parentheses are also glue words. For example, the glue words for the lattice D_6^3 in Table II are described by $[(0, 1, 3)]$, indicating that the glue words are spanned by

$$[0,1,3] = (000000, \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}),$$
$$[3,0,1] = (\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}, 000000, \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}),$$
$$[1,3,0] = (\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}, \tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}\tfrac{1}{2}-\tfrac{1}{2}, 000000).$$

The proof of Theorem 1 will use Siegel's mass formula. For type II lattices this states that [11, p. 54; 12; 13, Eq. (15); 14]

$$\sum_{\Lambda} \frac{1}{g(\Lambda)} = \frac{B_{4k}}{8k} \prod_{j=1}^{4k-1} \frac{B_{2j}}{4j}, \tag{1}$$

where the sum is taken over all inequivalent type II lattices of determinant 1 and dimension $n = 8k$, and $B_0 = 1$, $B_1 = -1/2$, $B_2 = 1/6$, $B_4 = -1/30, \ldots$ are the Bernoulli numbers [1, p. 810]. The formula is unfortunately stated incorrectly in [14, Theorem 9.2.2]. The right-hand side of (1) is given in Table III for $n = 8$, 16, and 24.

For type I lattices the mass formula states that

$$\sum_{\Lambda} \frac{1}{g(\Lambda)} = M(n), \tag{2}$$

where the sum is taken over all inequivalent type I lattices of determinant 1 and dimension n, and $M(n)$ is a constant which may be found in [14, Theorem 9.2.1]. Table IV gives some values of $M(n)$.

Proof of Theorem 1. It is enough to verify that (1) and (2) hold. To do this we write down all lattices of dimension n that can be obtained as direct

TABLE III

The Mass for Type II Lattices of Dimension n

n	Right-hand side of (1)
8	$\dfrac{1}{696729600}$
16	$\dfrac{691}{277667181515243520000}$
24	$\dfrac{1027637932586061520960267}{129477933340026851560636148613120000000}$

TABLE IV

The Mass for Type I Lattices of Dimension n

n	$M(n)$ = Right-hand side of (2)
0	1
1	1/2
2	1/8
3	1/48
...	...
15	$\frac{29713}{385648863215616000}$
...	...
19	$\frac{8003636403977}{774891356798220396134400000}$
20	$\frac{248112728523287}{619913085438576316907520000}$

sums of the lattices in Table II and copies of the cubic lattice $\mathbf{Z}^r$. For example, when $n = 3$ there is just $\mathbf{Z}^3$ itself, for which $g(\mathbf{Z}^3) = 2^3 \cdot 3!$. Indeed from Table IV

$$M(3) = \frac{1}{2^3 \cdot 3!},$$

verifying that $\mathbf{Z}^3$ is the unique 3-dimensional type I lattice.

Dimension 15 is a more typical example. According to Table II there are five type I lattices, namely,

$$\mathbf{Z}^{15}, \quad \mathbf{Z}^7 \oplus E_8, \quad \mathbf{Z}^3 \oplus D_{12}, \quad \mathbf{Z} \oplus E_7^2, \quad A_{15}.$$

The corresponding group orders $g(\Lambda)$ are

$$2^{15} \cdot 15!, \quad (2^7 \cdot 7!) \cdot (192 \cdot 10!), \quad (2^3 \cdot 3!) \cdot (2^4 \cdot 12!),$$
$$(2^1 \cdot 1!) \cdot (8 \cdot 9!)^2 \cdot 2, \quad 16! \cdot 2$$

(obtained by taking the product of the g_0's from Table I with g_1 and g_2 from Table II). Adding the reciprocals of these numbers we get

$$\frac{29713}{385648863215616000},$$

in agreement with $M(15)$ in Table IV.

Similar verifications are carried out in dimensions 1–20 for type I lattices, and dimensions 8 and 16 for type II lattices. This completes the proof of Theorem 1.

Remark. One advantage of this method of proof is that it is not necessary to explain how the lattices in Table II were obtained. They were in fact found as follows. Those of dimension $\leqslant 16$ were enumerated by Kneser [5]. Those of dimension $24-n$ for $n=4, 5, 6, 7$ were found using the fact that if we append vectors of D_n or its coset $D_n+[2]$ to vectors of even or odd norm, respectively, we get a Niemeier lattice from which the desired lattice can be recovered by the "removal" of the D_n. The following observations simplify this process. (a) The only inclusions among the A_n, D_n, and E_n lattices are those shown in (3.2) of [9]. (b) All D_n's contained in an E_m are equivalent, for $n \geqslant 2$ and $m=6, 7, 8$. (c) The "removal" of a D_n from a D_m with $m>n>4$ leaves a lattice with a vector of norm 1, and need not be considered. For example, to find all lattices of dimension 17 we look for a

TABLE V

Niemeier's 24-Dimensional Type II Lattices of Determinant 1

Components	Generators for glue words	g_1	g_2	N_2
D_{24}	[1]	1	1	1104
$D_{16}E_8$	[10]	1	1	720
E_8^3	[000]	1	6	720
A_{24}	[5]	2	1	600
D_{12}^2	[12], [21]	1	2	528
$A_{17}E_7$	[31]	2	1	432
$D_{10}E_7^2$	[110], [301]	1	2	432
$A_{15}D_9$	[21]	2	1	384
D_8^3	[(122)]	1	6	336
A_{12}^2	[15]	2	2	312
$A_{11}D_7E_6$	[111]	2	1	288
E_6^4	[1(012)]	2	24	288
$A_9^2D_6$	[240], [501], [053]	2	2	240
D_6^4	[Even perms. of {0123}]	1	24	240
A_8^3	[(114)]	2	6	216
$A_7^2D_5^2$	[1112], [1721]	2	4	192
A_6^4	[1(216)]	2	12	168
$A_5^4D_4$	[2(024)0], [33001], [30302], [30033]	2	24	144
D_4^6	[111111], [0(02332)]	3	720	144
A_4^6	[1(01441)]	2	120	120
A_3^8	[3(2001011)]	2	1344	96
A_2^{12}	[2(11211122212)]	2	$\lvert M_{12}\rvert$	72
A_1^{24}	[1(00000101001100110101111)]	1	$\lvert M_{24}\rvert$	48
Leech	—	1	1	0

Niemeier lattice containing a D_7: there is only one that need be considered, namely $E_6D_7A_{11}$, leading to the unique lattice E_6A_{11} in Table II. Further details are omitted.

4. NIEMEIER'S LATTICES

In Table V we give Niemeier's 24-dimensional type II lattices in our notation, together with the group orders g_1 and g_2, and the number N_2 of norm 2 vectors. The lattices appear in decreasing order of N_2 and in tied cases by increasing value of g_2. This ordering has the happy property that occurrences from any one of the families A_n, D_n, or E_n are in descending order of n.

In Table V the glue words for D_4^6 are isomorphic to the hexacode—the [6, 3, 4] self-dual code over $GF(4)$. In A_4^6 the group $G_2(A_4^6)$ is isomorphic to

TABLE VI

Verification That Niemeier's List Is Complete

Lattice Λ	$g(\Lambda)^{-1} \times$ Denominator of (3)
D_{24}	24877125
E_8^3	63804560820
$D_{16}E_8$	271057837050
A_{24}	4173688995840
D_{12}^2	67271626831500
$A_{17}E_7$	3483146354688000
$D_{10}E_7^2$	4134535541136000
$A_{15}D_9$	33307587016704000
D_8^3	156983146327507500
A_{12}^2	834785957117952000
E_6^4	3735033917655040000
$A_{11}D_7E_6$	8082641116053504000
D_6^4	19144966823230248000
$A_9^2D_6$	106690862731906252800
A_8^3	225800767686574080000
$A_7^2D_5^2$	2700612462901377024000
A_6^4	8361079854908571648000
D_4^6	11965604264518905000000
$A_5^4D_4$	52278522738634063872000
A_4^6	180674574584719324741632
A_3^8	437599241673834240000000
A_2^{12}	3129279325918986240000000
A_1^{24}	3152271217195900800000000
Leech	155705728523304960000
Total	10276379325860615209602670

$PGL_2(5)$ acting on $\{\infty, 0, 1, 2, 3, 4\}$. In A_3^8 the group $G_2(A_3^8)$ is isomorphic to $2^3 \cdot PGL_2(7)$ acting on the extended Hamming code of length 8 over the integers modulo 4. In A_2^{12} and A_1^{24} the groups G_2 are the Mathieu groups M_{12} and M_{24}. The last line of the table refers to the Leech lattice, for which there is an extensive literature—see [14].

THEOREM 2.

$$\sum_{\text{Niemeier's list}} \frac{1}{g(A)} = \frac{1027637932586061520960267}{129477933340026851560636148613120000000}, \quad (3)$$

and thus the completeness of Niemeier's list is verified by the mass formula (1).

The proof is given in Table VI.

Our method, which is essentially due to Kneser (see [7]), easily extends to find all classes of even lattices with determinant + dimension at most 24. We intend to describe the enumeration of unimodular lattices in dimensions up to 24 in a later paper.

ACKNOWLEDGMENTS

The mass formula calculations were carried out using the MACSYMA computer system [8].

Note added in proof. Since this work was completed we have discovered some remarkable connections between the Niemeier lattices and the Leech lattice—see [3, 6]—and have also extended our enumeration to dimension 23 (see[18]). An independent proof that Neimeier's list is complete has been given by Venkov [16], using modular forms and coding theory.

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