Voronoi Regions of Lattices, Second Moments of Polytopes, and Quantization

J. H. CONWAY AND N. J. A. SLOANE, FELLOW, IEEE

Abstract—If a point is picked at random inside a regular simplex, octahedron, 600-cell, or other polytope, what is its average squared distance from the centroid? In n-dimensional space, what is the average squared distance of a random point from the closest point of the lattice $A_n$ (or $D_n, E_n, A'_n$ or $D'_n$)? The answers are given here, together with a description of the Voronoi (or nearest neighbor) regions of these lattices. The results have applications to quantization and to the design of signals for the Gaussian channel. For example, a quantizer based on the eight-dimensional lattice $E_8$ has a mean-squared error per symbol of 0.0717 ... when applied to uniformly distributed data, compared with 0.08333 ... for the best one-dimensional quantizer.

I. QUANTIZATION; CODES FOR GAUSSIAN CHANNEL

A. Introduction

THE MOTIVATION for this work comes from block quantization and from the design of signals for the Gaussian channel. Let us call a finite set of points $y_1, \ldots, y_M$ in n-dimensional Euclidean space $\mathbb{R}^n$ a Euclidean code. An n-dimensional quantizer with outputs $y_1, \ldots, y_M$ is the function $Q: \mathbb{R}^n \to \mathbb{R}^n$ which sends each point $x \in \mathbb{R}^n$ into $Q(x) = y_j$ (in case of a tie, pick that $y_j$ with the smallest subscript). If $x$ has probability density function $p(x)$, the mean-squared error per symbol of this quantizer is

$$E(n, M, p, \{ y_j \}) = \frac{1}{n} \sum_{i=1}^{M} \int_{V(y_i)} \| x - Q(x) \|^2 p(x) \, dx,$$

where $\| x \| = (x \cdot x)^{1/2}$. Around each codepoint $y_j$, its Voronoi region $V(y_j)$ (see [49]), consisting of all points of the underlying space which are closer to that codepoint than to any other. More precisely, we define $V(y_j)$ to be the closed set

$$V(y_j) = \{ x \in \mathbb{R}^n : \| x - y_j \| \leq \| x - y_i \| \text{ for all } j \neq i \}.$$

(Voronoi regions are also called Dirichlet regions, Brillouin zones, Wigner–Seitz cells, or nearest neighbor regions.) If $x$ is an interior point of $V(y_j)$, the quantizer replaces $x$ by $Q(x) = y_j$. Then we may write

$$E(n, M, p, \{ y_j \}) = \frac{1}{n} \sum_{i=1}^{M} \int_{V(y_i)} \| x - y_j \|^2 p(x) \, dx.$$

Given $n, M,$ and $p(x)$ one wishes to find the infimum

$$E(n, M, p) = \inf_{\{y_j\}} E(n, M, p, \{ y_j \})$$

over all choices of $y_1, \ldots, y_M$. Zador ([53]; see also [6], [7], [24], [52]) showed under quite general assumptions about $p(x)$ that

$$\lim_{M \to \infty} M^{2/n} E(n, M, p) = G_n \left( \int_{\mathbb{R}^n} p(x)^{n/(n+2)} \, dx \right)^{(n+2)/n}$$

where $G_n$ depends only on $n$. Zador also showed that

$$\frac{1}{(n + 2)\pi} \Gamma\left( \frac{n}{2} + 1 \right)^{2/n} \leq G_n \leq \frac{1}{n\pi} \Gamma\left( \frac{n}{2} + 1 \right)^{2/n} \Gamma\left( 1 + \frac{2}{n} \right).$$

Asymptotically the upper and lower bounds in (3) agree, giving

$$G_n \to \frac{1}{2\pi e} = 0.0585498 \ldots \text{ as } n \to \infty.$$

Since the probability density function $p(x)$ only appears in the last term of (2), we may choose any convenient $p(x)$ when attempting to find $G_n$. From now on we assume that the input $x$ is uniformly distributed over a large region in $n$-dimensional space, and we can usually avoid edge effects by passing to a limiting situation with infinitely many $y_j$. With this assumption the mean-squared error is minimized if each codepoint $y_j$ lies at the centroid of the corresponding Voronoi region $V(y_j)$ (see [24]). It is known that, for an optimal one-dimensional quantizer with a uniform input distribution, the points $y_j$ should be uniformly spaced along the real line; correspondingly

$$G_1 = \frac{1}{12} = 0.08333 \ldots.$$

Similarly for the optimal two-dimensional quantizer it is known that the points $y_j$ should form the hexagonal lattice.
$A_2$ (described in Section III-A); correspondingly
\[ G_2 = \frac{5}{36\sqrt{3}} = 0.0801875 \ldots, \] (6)

(see [21], [23], [24], [37]).

In three dimensions Gersho [24] conjectures that the optimal quantizer is based on the body-centered cubic lattice $A_3^*$, and that
\[ G_3 = \frac{19}{192\sqrt{3}} = 0.0185433 \ldots \] (7)

Similarly in four dimensions he conjectures that the optimal quantizer is based on the lattice $D_4$, and that
\[ G_4 = 0.076602 \] (8)

(obtained by Monte Carlo integration). Furthermore, he conjectures that for large $M$ the Voronoi regions $V(y_j)$ are all congruent, to some polytope $P$ say. For such a quantizer we obtain, from (1) and (2),
\[ G_n = \frac{1}{n} \int_P \| x - \hat{x} \|^2 dx, \] (9)

where $\hat{x}$ is the centroid of $P$. The expression on the right makes sense for any polytope and will be denoted by $G(P)$: we refer to it as the dimensionless second moment of $P$. It is also convenient to have symbols for the volume, unnormalized second moment, and normalized second moment of $P$; these are
\[ \text{vol}(P) = \int_P dx, \]
\[ U(P) = \int_P \| x - \hat{x} \|^2 dx, \]
\[ I(P) = \frac{U(P)}{\text{vol}(P)}, \]

respectively. Then
\[ G(P) = \frac{1}{n} \frac{U(P)}{\text{vol}(P)^{n+2/n}} = \frac{1}{n} \frac{I(P)}{\text{vol}(P)^{2/n}}. \]

If Gersho's conjecture is correct then $G_n$ may be determined from
\[ G_n = \min_P G(P), \] (10)

taken over all $n$-dimensional space-filling polytopes. Whether or not the conjecture is true, any value of $G(P)$ for a space-filling polytope is an upper bound to $G_n$, Furthermore (1), (2) and (9) allow us to interpret $G_n$ and $G(P)$ as mean-squared quantization errors per symbol, assuming a uniform input distribution to the quantizer.

In the second application, the same Euclidean code $y_1, \ldots, y_M$ is used as a code for the Gaussian channel. Now the Voronoi regions are the decoding regions: all points $x$ in the interior of $V(y_j)$ are decoded as $y_j$. If the codewords are equally likely and all the Voronoi regions $V(y_j)$ are congruent to a polytope $P$, the probability of correct decoding is proportional to
\[ \int_P e^{-x \cdot x} dx. \]

The description of the Voronoi regions given in Section III thus makes it possible to calculate this probability exactly for many Euclidean codes. These results will be described elsewhere.

B. Summary of Results

In Sections II and III we compute $G(P)$ for a number of important polytopes (not just space-filling ones), including all regular polytopes (see Theorem 4). The three- and four-dimensional polytopes are compared in Tables I and II. The chief tools are Dirichlet's integral (Theorem 1), an explicit formula for the second moment of an $n$-simplex (Theorem 2), and a recursion formula giving the second moment of a polytope in terms of its cells (Theorem 3).

In Section III we study lattices, in particular the root lattices $A_n$, $D_n$, $E_6$, $E_7$, $E_8$, and their duals (defined in Section III-A). We determine the Voronoi regions for these lattices, and their second moments. The second moment gives the average squared distance of a point from the lattice. The maximum distance of any point of the underlying space from the lattice is its covering radius. The covering radii of these lattices were mostly already known (see for example [2], [4], [31]), but for completeness we rederive them. The final section (Section IV) compares the quantization errors of the different lattices—see Table V and Fig. 20. $E_8$ is the clear winner.

It is worth mentioning that for most of these lattices there are very fast algorithms for finding the closest lattice point to an arbitrary point; these are described in a companion paper [12]. The sizes of the spherical codes obtained from these lattices have been tabulated in [47].

Although we have tried to keep this paper as self-contained as possible, some familiarity with Coxeter's book [16] will be helpful to the reader. $\mathbb{Z}$ denotes the integers, $\mathbb{Q}$ the rationals and $\mathbb{R}$ the reals.

II. Second Moments of Polytopes

In this section we compute the second moments of a number of fairly simple polytopes; many others will be analyzed in Section III. The methods used are described in Theorems 1–3. A polytope in this paper means a convex region of $\mathbb{R}^n$ enclosed by a finite number of hyperplanes (cf. [16, p. 126]). The part of the polytope that lies in one of the hyperplanes is called a cell. We usually denote the edge-length of our polytopes by $2l$. The main source for information about polytopes is Coxeter [16], but there is an extensive literature, particularly for low-dimensional figures (see for example [1], [13]–[20], [22], [23], [26], [28], [29], [33], [34], [36], [40], [48], and [50]). Although second mo-
ments about an axis are tabulated for many simple polyhedra in standard engineering handbooks (see also [43]), the results given here appear to be new.

A. Dirichlet’s Integral

A few special figures can be handled using Dirichlet’s integral.

\[ \text{Theorem 1 ([51, §12.5]):} \quad \text{Let } f \text{ be continuous and } a_1, \ldots, a_n > 0. \text{ Then} \]
\[ \int f(x_1 + \cdots + x_n) x_1^{a_1} \cdots x_n^{a_n} \frac{dx_1}{x_1} \cdots \frac{dx_n}{x_n} = \frac{\Gamma(a_1) \cdots \Gamma(a_n)}{\Gamma(a_1 + \cdots + a_n)} \int_0^1 \int_0^1 \cdots \int_0^1 \tau^{2a_1-1} d\tau, \]
where the integral on the left is taken over the region bounded by \( x_1 \geq 0, \ldots, x_n \geq 0 \) and \( x_1 + \cdots + x_n \leq 1 \).

B. Generalized Octahedron or Crosspolytope

Consider for example the \( n \)-dimensional generalized octahedron or crosspolytope \( \beta_n \) [16, p. 121] of edge-length 2. Taking \( f = 1, a_i = 1 \) (to get the volume) or \( a_i = 3 \) (to get the second moment) and \( a_i = 1 \) for \( i \geq 2 \) in Theorem 1 we find
\[ \text{vol}(\beta_n) = \frac{2^n/2}{(2!)^n} = \frac{n}{n+1}(n+2), \]
\[ G(\beta_n) = \frac{1}{2(n+1)(n+2)} \]
\[ \to \frac{1}{2e^2} = 0.0676676 \cdots \quad \text{as } n \to \infty. \]

C. The \( n \)-Sphere

As a second application, for the \( n \)-dimensional (solid) sphere \( S_n \) of radius \( \rho \) we find
\[ \text{vol}(S_n) = \frac{\pi^{n/2}}{\Gamma(n/2+1)} \frac{I(S_n)}{\rho^2} = \frac{n}{n+2}, \]
\[ G(S_n) = \frac{1}{2\pi \rho} \frac{(n/2+1)^{2/n}}{n+2} \]
\[ \to \frac{1}{2\pi e} = 0.0585498 \cdots \quad \text{as } n \to \infty. \]

D. \( n \)-Dimensional Simplexes

The next result makes it possible to find the second moment of any figure provided it can be decomposed into simplexes.

\[ \text{Theorem 2: Let } P \text{ be an arbitrary simplex in } \mathbb{R}^n \text{ with vertices } v_i = (v_{i1}, \ldots, v_{in}) \text{ for } 0 \leq i \leq n. \text{ Then} \]
\[ a) \quad \text{the centroid of } P \text{ is at the barycenter} \]
\[ \hat{v} = \frac{1}{n+1}(v_0 + \cdots + v_n) \]
\[ \text{vol}(P) = \frac{1}{n!} \det \begin{bmatrix} 1 & v_{01} & \cdots & v_{0n} \\ v_{11} & 1 & \cdots & v_{1n} \\ \vdots & \vdots & \ddots & \vdots \\ v_{n1} & \cdots & v_{0n} & 1 \end{bmatrix}, \]
\[ \text{and} \]
\[ c) \quad \text{the normalized second moment about the origin } 0 \text{ is} \]
\[ I = \frac{1}{n+2} \|\hat{v}\|^2 + \frac{1}{n+1}(n+2) \sum_{i=0}^n \|v_i\|^2. \]

E. Regular Simplex

For example if \( P \) is a regular \( n \)-simplex of edge-length 2 then
\[ \text{vol}(P) = \frac{\sqrt{n+1}}{n!} \frac{n}{(\sqrt{2})^n}, \]
\[ I(P) = \frac{n}{(n+1)(n+2)} \]
\[ \to e^{-2} = 0.135335 \cdots \quad \text{as } n \to \infty. \]

F. Volume and Second Moment of a Polytope in Terms of its Cells

Instead of decomposing a figure into simplexes one may proceed by induction, expressing the volume and second moment of a polytope in terms of the volume and second moment of its cells, then in terms of its \((n-2)\)-dimensional faces, and so on. Theorem 3 is the basis for this procedure.

Suppose \( P \) is an \( n \)-dimensional polytope with \( N_1 \) congruent cells \( F_1, F'_1, \ldots, N_2 \) congruent cells \( F_2, F'_2, \ldots \), and so on. Suppose also that \( P \) contains a point \( 0 \) such that
all of the generalized pyramids \(0F_1, 0F_2, \ldots\) are congruent, all of \(0F_2, 0F_3, \ldots\) are congruent, \ldots. Let \(a_i \in F_i\) be the foot of the perpendicular from 0 to \(E_i\), let \(h_i = ||0a_i||\), and let \(V_{n-1}(i)\) be the volume of \(F_i\) and \(U_{n-1}(i)\) the unnormalized second moment of \(F_i\) about \(a_i\).

**Theorem 3:** The volume and unnormalized second moment about 0 of \(P\) are given by

\[
\text{vol}(P) = \sum_i \frac{N_i d_i}{n} V_{n-1}(i),
\]

\[
U(P) = \sum_i \frac{N_i d_i}{n+2} [h_i^2 V_{n-1}(i) + U_{n-1}(i)].
\]

**Proof:** Follows from elementary calculus by dividing each generalized pyramid \(OF_i\) into slabs parallel to the cell \(F_i\).

---

**G. Truncated Octahedron**

For example let \(P\) be the truncated octahedron with vertices consisting of all permutations of \(\sqrt{2}(0, \pm 1, \pm 2)\). \(P\) has \(N_1 = 6\) square cells and \(N_2 = 8\) cells which are regular hexagons, all with edge-length \(2l\). The second moments of these cells can be calculated directly, or else found in Section II-1 below. Then from the theorem we find that

\[
\text{vol}(P) = \frac{6 \cdot \sqrt{8}}{3} 4l^2 + \frac{8 \cdot \sqrt{6}}{3} 6\sqrt{3} l^2 = 64\sqrt{2} l^3,
\]

\[
U(P) = \frac{6 \cdot \sqrt{8}}{5} \left[ 8l^2 \cdot 4l^2 + \frac{8l^4}{3} \right]
\]

\[
+ \frac{8 \cdot \sqrt{6}}{5} \left[ 6l^2 \cdot 6\sqrt{3} l^2 + 10\sqrt{3} l^4 \right]
\]

\[
= 304\sqrt{2} l^3,
\]

hence

\[
I(P) = \frac{U(P)}{\text{vol}(P)} = \frac{19l^4}{4},
\]

\[
G(P) = \frac{1}{3} \frac{I(P)}{\text{vol}(P)^{2/3}} = \frac{19}{192\sqrt{2}} = 0.0785433 \ldots,
\]

(17)

---

**H. Second Moment of Regular Polytopes**

The next theorem gives an explicit formula for the second moment of any regular polytope. Suppose \(P\) is an \(n\)-dimensional regular polytope [16]. For \(0 \leq j \leq n\) choose a \(j\)-dimensional face \(F_j\) of \(P\) so that \(F_0 \subseteq F_1 \subseteq \ldots \subseteq F_n = P\), and let 0 be the center of \(F_j\), \(R_j = ||0a_j||\), and for \(j \geq 1\) let \(r_j = ||00_j||\). Thus \(r_j\) is the inradius of \(F_j\) measured from 0, and \(r_j^2 = R_j^2 - R_j^2\). Let \(N_{n-j-1}\) be the number of \((j-1)\)-dimensional cells of \(F_j\). Then it is known that the symmetry group of \(P\) has order

\[
g = N_{n,n-1} N_{n-1,n-2} \ldots N_{2,1} N_{1,0}
\]

(see [16, p. 130]), and that the volume of \(P\) is

\[
\text{vol}(P) = \prod_{i=0}^{n} \frac{r_i}{p!}
\]

(18)

(see [16, p. 137]).

**Theorem 4:** The second moment of any \(n\)-dimensional regular polytope \(P\) about its center 0, is given by

\[
I(P) = \frac{2}{(n+1)(n+2)}
\]

\[
\cdot \left( R_0^2 + 2 R_1^2 + 3 R_2^2 + \cdots + n R_{n-1}^2 \right)
\]

(19)

or equivalently

\[
I(P) = \frac{2}{(n+1)(n+2)}
\]

\[
\cdot \left( r_1^2 + 3 r_2^2 + 6 r_3^2 + \cdots + n(n+1) r_n^2 \right)
\]

(20)

**Proof:** The proof is by induction, the one-dimensional case being immediate. From Theorem 3 we have

\[
U(P) = \frac{N_{n,n-1} r_{n-1}}{n+2} [r_{n-1}^2 V_{n-1}(P) + U_{n-1}(P)],
\]

where (from (18) and the induction hypothesis)

\[
V_{n-1}(P) = N_{n-1,n-2} \ldots N_{2,1} N_{1,0} \frac{r_1 r_2 \cdots r_{n-1}}{(n-1)!},
\]

\[
U_{n-1}(P) = V_{n-1}(P)
\]

\[
\cdot \frac{2}{n(n+1)} \left[ r_1^2 + \cdots + n(n-1) r_{n-1}^2 \right].
\]

Then

\[
I(P) = \frac{U(P)}{\text{vol}(P)} = \frac{n}{n+2} \left[ \frac{r_1^2 + \cdots + n(n-1) r_{n-1}^2}{2} \right]
\]

\[
\cdot \left( r_1^2 + \cdots + n(n-1) r_{n-1}^2 \right)
\]

which simplifies to (20).

The values of \(g\), \(\text{vol}(P)\) and \(R_j\) are tabulated for all regular polytopes in [16, table I, pp. 292–295]. We have already dealt with the simplex and generalized octahedron. For an \(n\)-dimensional cube \(G(P) = 1/12\) for all \(n\) (since the cube is a direct product of line segments). We now treat the remaining regular polytopes.

---

**I. Regular Polygons**

If \(P\) is a regular \(p\)-gon of edge-length \(2l\) then from Theorem 4 we find

\[
\text{vol}(P) = p l^2 \cot^\frac{\pi}{p},
\]

\[
I(P) = \frac{l^2}{6} \left( 1 + 3 \cot^2 \frac{\pi}{p} \right),
\]

\[
G(P) = \frac{1}{6p} \left( \csc 2\frac{\pi}{p} + \cot \frac{\pi}{p} \right)
\]

(21)

For \(p = 3, 4,\) and 6, \(G(P) = 1/6\sqrt{3}, 1/12,\) and \(5/36\sqrt{3} .\)
J. Icosahedron and Dodecahedron

For the icosahedron
\[
\text{vol} = \frac{20l^3\tau^2}{3}, \quad I = \frac{3l^2\tau^2}{5},
\]
\[
G = \frac{1}{20} \left( \frac{6\tau}{5} \right)^{2/3} = 0.0778185 \ldots, \quad (22)
\]
where \( \tau = (\sqrt{5} + 1)/2 \), and for the dodecahedron
\[
\text{vol} = \frac{4\sqrt{5}l^3\tau^4}{15}, \quad I = \frac{l^2}{25} \left( 39\tau + 28 \right),
\]
\[
G = \frac{11\tau + 17}{300} \left( \frac{2}{\tau^2} \right)^{2/3} = 0.0781285 \ldots. \quad (23)
\]

K. The Exceptional Four-Dimensional Polytopes

There are three “exceptional” regular polytopes in four dimensions, the 24-cell, the 120-cell, and the 600-cell (the prefix giving the number of cells—see [16]). For the 24-cell
\[
\text{vol} = 32l^4, \quad I = \frac{26l^2/15},
\]
\[
G = \frac{13}{120/2} = 0.0766032 \ldots; \quad (24)
\]
for the 120-cell
\[
\text{vol} = 120\sqrt{5}l^4\tau^6, \quad I = \frac{2l^2}{15\sqrt{5}} \left( 282 + 127\sqrt{5} \right),
\]
\[
G = \frac{43\tau + 13}{300\sqrt{5}^1/4} = 0.0751470 \ldots; \quad (25)
\]
and for the 600-cell
\[
\text{vol} = 100l^4\tau^3, \quad I = \frac{4l^2/15}{12 + 5\sqrt{5}},
\]
\[
G = \frac{(3\tau + 4)l^{1/2}}{150} = 0.0750839 \ldots. \quad (26)
\]

L. Comparisons

The three- and four-dimensional polytopes that have been considered are compared in Tables I and II.

III. Voronoi Regions of Lattices and the Mean-Square Error of Lattice Quantizers

In this section we determine the Voronoi regions of various n-dimensional lattices, and their volumes and second moments. Since these lattices can be used to construct n-dimensional quantizers for uniformly distributed inputs, the dimensionless second moments give upper bounds to \( G \), (see (10) and Section IV). The lattices considered are the root lattices\(^1\) and their duals, namely \( A_n, A_n^*, D_n, D_n^*, E_6, E_7, \) and \( E_8 \) (we shall not consider \( E_6^* \) or \( E_7^* \) here, while \( E_8^* = E_8 \)).

\(^1\)A root lattice is a lattice spanned by the root system of a Lie algebra—see [5], [30].

For general information about lattices see for example [4], [5], [8], [11], [14]–[17], [28], [32], [38], [39], and [45]–[47]. In particular if \( \Lambda \) is a lattice in \( \mathbb{R}^n \) the dual (or reciprocal or polar) lattice \( \Lambda^* \) consists of all points \( x \) in the span \( \mathcal{Q}\Lambda \) such that \( x \cdot y \in \mathbb{Z} \) for all \( y \in \Lambda \). Since all the points in a lattice are equivalent, it is enough to find the Voronoi region around the origin, i.e., the closed set
\[
P(0) = \{ x \in \mathbb{R}^n \mid \|x\| = \|x - u\| \text{ for all nonzero } u \in \Lambda \}.
\]
The volume of the Voronoi region can be written down immediately from the other standard parameters of the lattice:
\[
V(0) = \frac{V_{n} \rho^n}{\Delta} - \frac{\rho^n}{\delta} - \sqrt{d}, \quad (27)
\]
where \( V_{n} \) is the volume of a unit sphere in \( \mathbb{R}^n \), \( 2\rho \) is the minimum distance between the points of \( \Lambda \), and \( \Delta, \delta \) and \( d \) are respectively the density, center density, and determinant of \( \Lambda \). The covering radius will be denoted by \( R_c \).
A. Definition of the Root Lattices

For \( n \geq 1 \), \( A_n \) is the \( n \)-dimensional lattice consisting of the points \((x_0, x_1, \ldots, x_n)\) in \( \mathbb{Z}^{n+1} \) with \( \sum x_i = 0 \). The dual \( A_n^* \) consists of the union of \( n + 1 \) cosets of \( A_n \):

\[
A_n^* = \bigcup_{i=0}^{n} (i] + A_n),
\]

where

\[
[i] = \left( \frac{-j}{n+1}, \frac{-j}{n+1}, \ldots, \frac{-j}{n+1}, \frac{i}{n+1}, \ldots, \frac{i}{n+1} \right)
\]

\[
= \left( \frac{-j}{n+1}, \left( \frac{i}{n+1} \right)^j \right)
\]

and \( i + j = n + 1 \). For \( n = 1 \) and 2, \( A_1^* = A_1 \) (i.e., they differ only by a rotation and change of scale).

For \( n \geq 2 \), \( D_n \) consists of the points \((x_0, x_1, \ldots, x_n)\) in \( \mathbb{Z}^n \) with \( x_i \) even. In other words, if we color the integer lattice points alternately red and blue in a checkerboard coloring, \( D_n \) consists of the red points. The dual \( D_n^* \) is the union of four cosets of \( D_n \):

\[
D_n^* = \bigcup_{j=0}^{3} ([i] + D_n),
\]

where

\[
[0] = (0^n), \quad [1] = (1^n).
\]

\[
[2] = (0^{n-1}, 1), \quad [3] = (0^{n-1}, -1).
\]

Also \( D_2 \approx A_2 \oplus A_1 \), \( D_3 \approx A_3 \), and \( D_4^* \approx D_4 \). Equivalently, \( D_n \) may be obtained by applying Construction A of [32] or [45] to the even weight code of length \( n \). Similarly \( D_n^* \) is obtained by applying Construction A to the dual code \( \{0^n, 1^n\} \).

There are many possible definitions of the lattices \( E_6, E_7 \), and \( E_8 \) (see the references given at the beginning of this section). We shall use the following: \( E_6 \) is the union of \( D_6 \) and the coset

\[
\left( \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2} \right) + D_6.
\]

In other words \( E_6 \) consists of the points \((x_1, \ldots, x_6)\) with \( x_i \in \mathbb{Z} \) and \( \sum x_i \) even, together with the points \((y_1, \ldots, y_6)\) with \( y_i \in \mathbb{Z} + \frac{1}{2} \) and \( \sum y_i \) even. \( E_7 \) is a subspace of dimension 7 in \( E_8 \), consisting of the points \((u_1, \ldots, u_7) \in E_8 \) with \( u_1 = -u_7 \), \( E_6 \) is a subspace of dimension 6 in \( E_8 \), consisting of the points \((u_1, \ldots, u_6) \in E_8 \) with \( u_6 = u_7 = -u_8 \).

B. Voronoi Region of a Root Lattice

In this section we give a uniform method for finding the Voronoi region of any root lattice \( \Lambda \) (the dual lattices must be handled differently). The method is based on finding a fundamental simplex for the affine Weyl group of the lattice (cf. [3], [5], [16]).

The (ordinary) Weyl group \( W(\Lambda) \) of an \( n \)-dimensional root lattice \( \Lambda \) is a certain finite group of orthogonal transformations of \( \mathbb{R}^n \) which sends \( \Lambda \) to itself (for the precise definition see [5, p. 143] or [30, p. 43]). Similarly the affine Weyl group \( W(\Lambda) \) is a certain infinite group of isometries of \( \mathbb{R}^n \) which sends \( \Lambda \) to itself (see [5, p. 173]); and \( W(\Lambda) \) is the subgroup of \( W(\Lambda) \) fixing the origin. The affine Weyl group is described by the extended Coxeter-Dynkin diagram shown in Figs. 1–3.

This diagram can be read in at least three different ways (see [3], [5], [16], [27]). First, it provides a presentation for \( W(\Lambda) \), defining the group in terms of generators and relations; however we shall not make use of this interpretation here. Second, it can be used to specify a fundamental simplex \( S \) for \( W(\Lambda) \). This is an \( n \) dimensional closed simplex whose images under the action of \( W(\Lambda) \) are distinct and tile \( \mathbb{R}^n \). In other words we can write

\[
\mathbb{R}^n = \bigcup_{g \in W(\Lambda)} g(S),
\]

where (except for the boundaries of \( g(S) \), a set of measure zero) each point \( x \in \mathbb{R}^n \) belongs to a unique \( g(S) \). In this interpretation the nodes of the diagram represent the hyperplanes which are the walls of the fundamental simplex [16, p. 191]. The angle between two walls or hyperplanes is indicated by the branch of the diagram joining the corresponding nodes. If the hyperplanes are at an angle of \( \pi/3 \) the nodes are joined by a single branch, if the angle is \( \pi/4 \) they are joined by a double branch (see Fig. 4), if the angle is \( \pi/p \) with \( p > 4 \) they are joined by a branch labeled \( p \), and finally if the hyperplanes are perpendicular the nodes \( \Lambda \) is generated by the reflections in the hyperplanes through the origin perpendicular to the minimal vectors of the lattice. Alternatively, in the terminology of [11], \( W(\Lambda) \) is the group \( G_0(\Lambda) \), while the full group of orthogonal transformations sending \( \Lambda \) to itself is a split extension of \( G_0(\Lambda) \) by \( G_1(\Lambda) \).

We may think of \( \Lambda \) itself as being an Abelian group of translations of \( \mathbb{R}^n \), which sends \( \Lambda \) to \( \Lambda \). Then \( W(\Lambda) \) is a split extension of \( \Lambda \) by \( W(\Lambda) \).
CONWAY AND SLOANE: LATTICES, POLYTOPES, AND QUANTIZATION

Fig. 4. Coxeter-Dynkin diagrams for the spherical simplexes of (a) $W(A_n)$ and (b) $W(C_n)$. (The labeling of the nodes is for convenience only and has no geometrical significance.)

(a) $\begin{array}{cccccc}
W_6(E_6) \\
\begin{array}{cccccc}
& x_1 & x_3 & x_5 & x_7 & \\
& x_2 & & x_6 & & \\
& & x_4 & & & \\
x_1 & x_5 & x_6 & x_7 & \\
& x_2 & & & & \\
& & & & & \\
\end{array}
\end{array}$

(b) $\begin{array}{cccccc}
W_6(E_6) \\
\begin{array}{cccccc}
& x_1 & x_3 & x_5 & x_7 & \\
& x_2 & & x_6 & & \\
& & x_4 & & & \\
x_1 & x_5 & x_6 & x_7 & \\
& x_2 & & & & \\
& & & & & \\
\end{array}
\end{array}$

(c) $\begin{array}{cccccc}
W_6(E_6) \\
\begin{array}{cccccc}
& x_1 & x_3 & x_5 & x_7 & \\
& x_2 & & x_6 & & \\
& & x_4 & & & \\
x_1 & x_5 & x_6 & x_7 & \\
& x_2 & & & & \\
& & & & & \\
\end{array}
\end{array}$

Fig. 5. Extended Coxeter-Dynkin diagrams for (a) $W_6(E_6)$, (b) $W_6(E_7)$, and (c) $W_6(E_8)$.

are not joined by a branch. The nodes in Figs. 1–3 have been labeled with the equations to the corresponding hyperplanes.

In the third interpretation the nodes in the extended Coxeter-Dynkin diagram are taken to represent the vertices of a fundamental simplex, rather than the bounding hyperplanes. Each node represents the vertex opposite to the corresponding hyperplane (some examples are shown in Figs. 6, 8, and 9)—see [16, p. 196].

One of the nodes in the diagram is indicated by a solid circle. This is the extending node; removing it leaves a Coxeter-Dynkin diagram for the Weyl group $W(A)$. Of the $n + 1$ hyperplanes represented by the extended diagram, all except that corresponding to the extending node pass through the origin. It is helpful to think of the latter hyperplane as forming the roof of the fundamental simplex. The vertex of the fundamental simplex opposite the roof is the origin.

For later use we remark that the finite Weyl group $W(\Lambda)$ also has a fundamental domain, consisting of an infinite cone centered at the origin. A fundamental simplex for $W(\Lambda)$ is obtained by taking the finite part of the cone beneath the roof. The intersection of this cone with the roof, or more precisely with a unit sphere centered at the origin, is a spherical simplex. The Coxeter-Dynkin diagram for $W(\Lambda)$—i.e., with the extending node deleted—describes this spherical simplex in the same way as the extended diagram describes the fundamental simplex for $W_6(\Lambda)$. These spherical simplexes and (unextended) Coxeter-Dynkin diagrams can be used to define the Weyl groups of all the root systems (and not just the root lattices $A_n$, $D_n$, and $E_n$). In Sections F and G we shall require the spherical simplexes corresponding to $W(A_n)$ and $W(C_n)$, shown in Fig. 4. The Weyl group $W(A_n)$ is usually written as $[n+1]$ and is isomorphic to the symmetric group on $n + 1$ letters. $W(C_n)$ is written $[n-2, 4_1]$ and has order $2^n n!$. (See [3], [5], [16], [18], and [30].)

Lemma: The origin is the closest lattice point to any interior point of the fundamental simplex.

Proof: Let $u$ be the closest lattice point to $x \in S$. Suppose $u \neq 0$. Then $u \not\in S$, and $u$ and $x$ are on opposite sides of a reflecting hyperplane of $W(A)$. Let $u' \in \Lambda$ be the image of $u$ in this hyperplane, and let $y$ be the foot of the perpendicular from $x$ to the line $uu'$ (see Fig. 5). Then $|| xu' ||^2 = || xy ||^2 + || yu' ||^2 < || xy ||^2 + || yu ||^2 = || xu ||^2$, and $x$ is closer to $u'$ than to $u$, a contradiction. Therefore $u = 0$.

The connection between the fundamental simplex and the Voronoi region is given by the following basic theorem.

Theorem 5: For any root lattice $\Lambda$, the Voronoi region around the origin is the union of the images of the fundamental simplex under the Weyl group $W(\Lambda)$.

Proof: Let $x$ be any point of the Voronoi region around the origin. From (29), $x \in g(S)$ for some $g \in W_d(\Lambda)$. Suppose $x$ is an interior point of $g(S)$. By the lemma, the closest lattice point to $x$ is $g(0)$. Therefore $g(0) = 0$, $g \in W(\Lambda)$, and $x \in \bigcup_{g \in W(\Lambda)} g(S)$.

We omit the discussion of the case when $x$ is a boundary point of $g(S)$. The converse statement, that $x \in \bigcup g(S)$ implies $x$ is in the Voronoi region, follows by reversing the steps.

It follows from Theorem 5 that the Voronoi region is the union of $|W(\Lambda)|$ copies of the fundamental simplex $S$. Furthermore the cells of the Voronoi region are the images of the roof of the fundamental simplex under $W(\Lambda)$. Thus the Voronoi region is bounded by hyperplanes which are
the perpendicular bisectors of the lines joining 0 to its nearest neighbors in the lattice.

**Corollary:** The number of \((n - 1)\)-dimensional cells of the Voronoi region of a root lattice is equal to the contact number of the lattice (the number of nearest neighbors of any lattice point).

This is not true for all lattices, as we shall see in Section III-H.

The second moment of the Voronoi region can now be obtained from that of the fundamental simplex. The results are given in the following sections.

**C. Voronoi Region for \(A_n\)**

We first find the vertices \(v_0, v_1, \ldots, v_n\) of the fundamental simplex \(S\). These are found by omitting each of the hyperplanes of Fig. 1 in turn and calculating the point of intersection of the remaining \(n\) hyperplanes. The results are shown in Fig. 6, where each node is labeled with the coordinates of the vertex opposite the corresponding hyperplane. The \(i\)th vertex is

\[
v_i = \left(\left(\frac{-j}{n+1}\right)^i\left(\frac{j}{n+1}\right)^i\right)
\]

where \(i + j = n + 1\), for \(0 \leq i \leq n\), and is the same as the coset representative \([i]\) for \(A_n\) in \(A_n^*\) (see (28)). Also

\[
\|v_i\|^2 = \frac{ij}{n+1}.
\]

The barycenter of \(S\) (13) is

\[
\hat{v} = \frac{1}{n+1} \sum_{i=0}^{n} v_i
\]

\[
= \left(\frac{-n}{2n+2}, \frac{-n+2}{2n+2}, \ldots, \frac{n-2}{2n+2}, \frac{n}{2n+2}\right)
\]

and from (15) the normalized second moment about the origin is

\[
I(S) = \frac{n+1}{n+2}\|\hat{v}\|^2 + \frac{1}{(n+1)(n+2)} \sum_{i=0}^{n} \|v_i\|^2
\]

\[
= \frac{n+1}{n+2} \cdot \frac{n(n+2)}{12(n+1)} + \frac{1}{(n+1)(n+2)} \cdot \frac{n(n+2)}{6}
\]

\[
= \frac{n}{12} + \frac{n}{6(n+1)}.
\]

Now \(|W(A_n)| = (n+1)!\) and the determinant \(d\) is \(n+1\) [5, pp. 250–251]. By Theorem 5 the Voronoi region around the origin, \(V(0)\), is the union of \(|W(A_n)| = \text{copies of } S\), so

\[
I(V(0)) = \frac{U(V(0))}{\text{vol}(V(0))}
\]

\[
= \frac{|W(A_n)| \cdot U(S)}{|W(A_n)| \cdot \text{vol}(S)}
\]

\[
= \frac{U(S)}{\text{vol}(S)} = I(S).
\]

Also \(\text{vol}(V(0)) = \sqrt{n+1}\) from (27). Therefore the dimensionless second moment of the Voronoi region of \(A_n\) is

\[
G(A_n) = \frac{1}{n} \frac{I(V(0))}{\text{vol}(V(0))^{2/n}}
\]

\[
= \frac{1}{(n+1)^{2/n}} \left[ \frac{1}{12} + \frac{1}{6(n+1)} \right]
\]

\[
= \frac{1}{12} \quad \text{as } n \to \infty.
\]

Once the Voronoi region has been found we can also determine the points in \(\mathbb{R}^n\) at maximum distance from the lattice, since these are necessarily vertices of the Voronoi regions. From (30) it follows that the covering radius of \(A_n\) (the maximum distance of any point in \(\mathbb{R}^n\) from \(A_n\)) is

\[
R_c = \sqrt{\frac{ab}{n+1}} - \rho \sqrt{\frac{2ab}{n+1}},
\]

where \(\rho\) is the packing radius (see (27)) and \(a = [(n+1)/2], b = n+1-a\). Typical points at this distance from \(A_n\) are the vertex \(v_0\) of \(V(0)\) and its images under \(W(A_n)\).

The lattice \(A_n\) consists of equally spaced points on the real line, and \(G(A_n) = 1/12\). The lattice \(A_2\) is the hexagonal lattice, the fundamental simplex is an equilateral triangle, the Voronoi region is a hexagon, and \(G(A_2) = 5/36\sqrt{3}\) (compare Section II-I). The lattice \(A_3\) is the face centered cubic lattice, the densest known sphere packing in \(\mathbb{R}^3\), the Voronoi region is a rhombic dodecahedron (Fig. 7; see also [33, p. 130] and [22, p. 294 and anaglyph XI]), and \(G(A_3) = 2^{-11/3} = 0.0787451\ldots\).

For \(n = 1\) and \(2\) it is known that \(A_n\) is the optimal quantizer (see (5) and (6)), but for \(n = 3\) the dual lattice \(A_n^*\) is better. The values of \(G(A_n)\) for \(n \leq 9\) are plotted in Fig. 20. \(G(A_n)\) decreases to its minimum value of 0.0773907\ldots at \(n = 8\) and then slowly increases to 1/12 as \(n \to \infty\).
D. Voronoi Region for \( D_n \) (\( n \geq 4 \))

\( D_n \) (for \( n \geq 4 \)), \( E_6 \), \( E_7 \), and \( E_8 \) are handled in the same way as \( A_n \) and our treatment will be brief. The vertices \( v_0, \ldots, v_n \) of a fundamental simplex for \( D_n \), \( n > 4 \), are shown in Fig. 8. Their barycenter is

\[
\delta = \frac{1}{2(n+1)} (0, 2, 3, \ldots, n - 2, n - 1, n + 1),
\]

\[ |W(D_n)| = 2^{n-1} \cdot n! \] and the determinant \( d = 4 \). For the Voronoi region \( V(0) \) we find

\[
I(V(0)) = \frac{n}{12} + \frac{1}{2(n+1)},
\]

\[
\text{vol}(V(0)) = 2,
\]

and

\[
G(D_n) = \frac{1}{2^n} \left( \frac{1}{12} + \frac{1}{2n(n+1)} \right) \to \frac{1}{12} \quad \text{as } n \to \infty.
\]

The covering radius of \( D_n \) (for \( n \geq 4 \)) is

\[
R_c = \sqrt{\frac{n}{4}} = \rho \sqrt{\frac{n}{2}},
\]

as illustrated by the vertex \( v_0 = (\frac{1}{2^n}) \). For \( n = 4 \) and \( 5 \), \( D_n \) is the densest known sphere packing in \( R^n \) (cf. [37]). For \( n = 4 \) the Voronoi region is a 24-cell (see for example [16, p. 156]), and \( G(D_4) = 13/120/2 = 0.0766032 \cdots \), in agreement with (24). This number also agrees closely with the value (8) that Gersho obtained for this region by Monte Carlo integration. The values of \( G(D_n) \) for \( n \leq 9 \) are plotted in Fig. 20. \( G(D_n) \) takes its minimum value of 0.0755905 \cdots at \( n = 6 \) and then slowly increases to 1/12 as \( n \to \infty \).

E. Voronoi Regions for \( E_6, E_7, E_8 \)

The vertices of fundamental simplexes for \( E_6 \), \( E_7 \), and \( E_8 \) are shown in Fig. 9. For \( E_6 \)

\[
|W(E_6)| = 2^4 \cdot 3^5 \cdot 5^2 \cdot 7 = 696729600, \quad d = 1, \]

\[
\delta = \frac{1}{1080} (5, 35, 55, 79, 109, 149, 209, 751),
\]

\[
I(V(0)) = \frac{929}{1620},
\]

\[
\text{vol}(V(0)) = 1,
\]

\[
G(E_6) = \frac{929}{12960} = 0.0716821 \cdots
\]

The Voronoi region \( V(0) \) is an eight-dimensional polytope which is the reciprocal\(^5\) to the Gosset polytope \( 4_{21} \) [16, p. 204].

Let \( N_i \) denote the number of \( i \)-dimensional faces of \( V(0) \). Then from [16, p. 204] we have \( N_0 = 19440, N_1 = 207360, N_2 = 483840, N_3 = 483840, N_4 = 241920, N_5 = 60480, N_6 = 6720, \) and \( N_7 = 240. \) The 19440 vertices consist of 2160 at distance one from \( 0 \) and 7200 at distance \( \sqrt{2}/3 \). The former are the images of the vertex \( (0, 1) \) of the fundamental simplex under \( W(E_6) \), and are at the maximum possible distance from \( E_6 \), while the latter are the images of the vertex \( (1/6, 1/6, 1/6, 1/6, 1/6, 1/6) \) under \( W(E_6) \). Thus \( R_c = 1 = \rho \sqrt{2} \).

The other seven vertices of the fundamental simplex are not vertices of the Voronoi region.

For \( E_7 \), \( |W(E_7)| = 2^{14} \cdot 3^8 \cdot 5 \cdot 7 = 2903040, d = 2, \)

\[
\delta = \frac{1}{96} (1, 5, 8, 12, 18, 30, -42, 42),
\]

\[
I(V(0)) = \frac{163}{288},
\]

\[
\text{vol}(V(0)) = \sqrt{2},
\]

\[
G(E_7) = \frac{163}{2016} \cdot 2^{-1/7} = 0.0732306 \cdots
\]

The covering radius of \( E_7 \) is \( R_c = \sqrt{3/2} = \rho \sqrt{3} \), as illustrated by the vertex \( (0, -1, 1, 1, -1, -1, -1) \).

\(^5\)It is not difficult to give a direct proof of this statement; it also follows from Theorem 8 below.
For $E_6$, $|W(E_6)| = 2^7 \cdot 3^4 \cdot 5 = 51840$, $d = 3$,
\[ \hat{v} = \frac{1}{42} (0, 3, 5, 8, 14, -14, -14, 14), \]
\[ I(V(0)) = \frac{15}{28}, \]
\[ \text{vol}(V(0)) = \sqrt{3}, \]
\[ G(E_6) = \frac{5}{56 \cdot 3^{1/6}} = 0.0743467 \ldots. \quad (36) \]
The Voronoi regions for $E_7$ and $E_8$ are the reciprocals of the Gosset polytopes $2_2$ and $1_2$ described in [16, pp. 202–203]. The covering radius of $E_6$ is $R_c = 2/\sqrt{3} = \rho \sqrt{8/3}$, as illustrated by the vertices $(0, 2/3, -2/3, 2/3)$ and $(0, 1, -1/3, -1/3, 1/3)$.

**F. Voronoi Region for $D^*_n$**

In order to determine the Voronoi regions for the dual lattices $A^*_n$ and $D^*_n$ we shall use Wythoff’s construction, as described in [13] and [16, § 11.6]. The idea is to construct new polytopes out of the spherical simplexes described in Section B, the vertices of the new polytope being indicated by drawing rings around certain nodes in the Coxeter—Dynkin diagram. More precisely, let $v_1, \ldots, v_n$ be the vertices of a spherical simplex for a Weyl group $W(\Lambda)$. If a single node of the diagram is ringed, say that corresponding to $v_i$, the vertices of the new polytope are the images of $v_i$ under the Weyl group. If two or more nodes are ringed, say those corresponding to $v_i, v_j, \ldots$, the symbol represents a polytope whose vertices are the images under $W(\Lambda)$ of some interior point of the spherical subsimplex with vertices $v_i, v_j, \ldots$. We can adjust the metrical properties of the polytope (for example, equalize its edge lengths) by choosing this interior point suitably. Some one-, two-, and three-dimensional examples are shown in Fig. 10, others may be found in [13] and [16].

We now use this construction to find the Voronoi region for $D^*_n$, $n \geq 3$. In Section III-A we saw (using the second definition) that $D^*_n$ is the union of the sets $(2\mathbb{Z})^n$ and $(1^n) + (2\mathbb{Z})^n$. The closest points to the origin from the first set consist of $2n$ points of the form $(\pm 2, 0^{n-1})$, and the closest from the second set consist of $2^n$ points of the form $(\pm 1^n)$. The Voronoi region $V(0)$ is the intersection of the Voronoi regions determined by these two sets. The first of these, $P$ say, is a cube centered at zero with vertices $(\pm 1^n)$. The second, $Q$ say, is a generalized octahedron with vertices $(\pm n/2, 0^{n-1})$. Furthermore $Q$ can be obtained by reciprocating $P$ in a sphere of radius $\rho = \sqrt{n/2}$ centered at the origin.

Thus the Voronoi region $V(0)$ is the intersection of $P$ and a reciprocal polytope $Q = P^\ast$. In other words $V(0)$ is obtained by truncating $P$ in the manner described in [16, p. 147], and is therefore specified by ringing one or two nodes of the Coxeter—Dynkin diagram (Fig. 4(b)) for the spherical simplex of $P$ ([13], [16, § 8.1 and § 11.7]).

The radii $R_j$ (defined in Section II-H) for the cube $P$ are given by $R_j = \sqrt{n} - j$ ([16, p. 295]). If $n$ is even the radius $\rho$ of the sphere of reciprocation is equal to $R_{n/2}$, and we must ring the node labeled $n/2$ in Fig. 4(b). If $n$ is odd $\rho$ lies between $R_{(n-1)/2}$ and $R_{(n+1)/2}$ and both nodes $(n-1)/2$ and $(n+1)/2$ must be ringed. We have therefore established the following theorem.

**Theorem 6:** The Voronoi region around the origin of the lattice $D^*_n$ is the polytope defined by the diagrams in Fig. 11.

The coordinates for $\beta(n,k)$ and $\delta(n,k)$ given below show that the edge-lengths of the Voronoi regions are all equal. In Coxeter’s notation [16, p. 146] the Voronoi region for $D^*_n$ is

\[
\begin{cases}
3 & 3 \cdots 3 \\
3 & 3 \cdots 3 \quad 4
\end{cases}
\]

with $t - 1$ threes in each row, and for $D^*_{n+1}$ it is

\[
\begin{cases}
3 & 3 \cdots 3 \\
3 & 3 \cdots 3 \quad 4
\end{cases}
\]

with $t - 1$ threes in the top and bottom rows.

We shall determine the second moments of these Voronoi regions recursively, using Theorem 3. In order to do this it will be necessary to find the second moments of all the polytopes $a(n,k)$, $\beta(n,k)$, $\gamma(n,k)$, and $\delta(n,k)$ defined in Fig. 12. In this notation the Voronoi region of $D^*_n$ is (up to a scale factor) equal to $\beta(n,n/2)$ if $n$ is even and to $\delta(n, (n-1)/2)$ if $n$ is odd. Let $R_n(n,k)$, $V_n(n,k)$, and $U_n(n,k)$ denote respectively the circumradius, volume, and unnormalized second moment about the center of $a(n,k)$, with a similar notation for $\beta(n,k)$, $\gamma(n,k)$, and $\delta(n,k)$.

For the vertices of the polytope $a(n,k)$ it is convenient to take the points in $\mathbb{H}^{n+1}$ whose coordinates are all
permutations of \((0^{n-k+1}, 1^k)\) [16, pp. 157-158]. The centroid of \(\alpha(n, k)\) is the point 
\[
\frac{k}{n+1}(1^n),
\]
and so the circumradius is 
\[
R_{\alpha}(n, k) = \frac{\sqrt{k(n-k+1)}}{n+1}.
\]

Similarly 
\[
\beta(n, k) \text{ has vertices } (0^{n-k+1}, \pm 1^k), \quad R_{\beta}(n, k) = \sqrt{k},
\]
\[
\gamma(n, k) \text{ has vertices } (0^{n-k}, 1, 2^k), \quad R_{\gamma}(n, k) = \sqrt{\frac{4k(n-k)+n}{n+1}},
\]
\[
\delta(n, k) \text{ has vertices } (0^{n-k}, \pm 1, \pm 2^k), \quad R_{\delta}(n, k) = \sqrt{4k+1}.
\]

(These polytopes appear with different names in [17].) Each of these polytopes has two kinds of cells obtained by deleting either the left or the right node of its diagram [16, § 7.6, 11.6, 11.7] For example deleting the left node of the \(\alpha(n, k)\) diagram produces an \(\alpha(n-1, k-1)\), while deleting the right node produces an \(\alpha(n-1, k)\). Thus in general \(\alpha(n, k)\) has cells of type \(\alpha(n-1, k-1)\) and \(\alpha(n-1, k)\). (If \(k = 0\) the first type is absent, while if \(k = n\) the second type is absent.) The number of cells of each type is given by the ratio of the orders of the underlying Weyl groups (obtained by ignoring the rings on the diagram). Thus the number of \(\alpha(n-1, k-1)\)-type cells of an \(\alpha(n, k)\) is 
\[
\frac{[3^{n-1}]-1}{[3^{n-2}]} = \frac{(n+1)!}{n!} = n+1.
\]

This is also the number of \(\alpha(n-1, k)\)-type cells. We represent this process of finding the cells by the graph shown in Fig. 13.

We can now apply Theorem 3 to \(\alpha(n, k)\), obtaining 
\[
V_{\alpha}(n, k) = \frac{(n+1)h_L}{n}V_{\alpha}(n-1, k-1) + \frac{(n+1)h_R}{n}V_{\alpha}(n-1, k),
\]
\[
\alpha(n, k) = \alpha(n-1, k-1) + \alpha(n-1, k).
\]

For \(n \geq 2\) and \(1 \leq k \leq n\), with \(v_a(n, 0) = v_a(n, n+1) = 0\) for \(n \geq 1\), and \(v_a(1,1) = 1\). The first few values of \(v_a\) and \(u_a\) may be identified as the Eulerian numbers [41, p. 215], and are given by 
\[
v_a(n, k) = \sum_{j=0}^{k} (-1)^j \binom{n+1}{j} (k-j)^n. \tag{41}
\]

There is a more complicated formula for \(u_a(n, k)\) which we omit.

Similarly for the polytope \(\beta(n, k)\) we have the graph shown in Fig. 14, and writing 
\[
V_{\beta}(n, k) = v_{\beta}(n, k) \frac{2^n}{n!}, \tag{42}
\]
\[
U_{\beta}(n, k) = u_{\beta}(n, k) \frac{2^n}{(n+2)!}, \tag{43}
\]
TABLE III

<table>
<thead>
<tr>
<th>n</th>
<th>v_a(n,k)</th>
<th>u_a(n,k)</th>
<th>q(n,k)</th>
<th>u_b(n,k)</th>
</tr>
</thead>
<tbody>
<tr>
<td>6</td>
<td>1 57 302 302 57 1</td>
<td>1 26 66 26 1</td>
<td>1 11 11 1</td>
<td>1 1</td>
</tr>
<tr>
<td>5</td>
<td>6 1158 8916 8916 1158 6</td>
<td>5 400 1290 400 5</td>
<td>2 24 24 2</td>
<td>2 2</td>
</tr>
<tr>
<td>4</td>
<td>3 27 119 27 3</td>
<td>1 10 10 1</td>
<td>1 1</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>2 2 2 2 2</td>
<td>1 1</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

For \( \gamma(n, k) \) and \( \delta(n, k) \) we have a pair of graphs similar to Figs. 13 and 14 (simply replace \( \alpha \) by \( \gamma \) and \( \beta \) by \( \delta \) in Figs. 13 and 14). As before we set

\[
V_\gamma(n, k) = v_\gamma(n, k), \quad U_\gamma(n, k) = u_\gamma(n, k),
\]

\[
V_\delta(n, k) = v_\delta(n, k), \quad U_\delta(n, k) = u_\delta(n, k),
\]

and obtain the recurrences

\[
v_\gamma(n, k) = (2n - 2k + 1) v_\gamma(n - 1, k - 1) + (2k + 1) v_\gamma(n - 1, k), \quad (48)
\]

\[
u_\gamma(n, k) = (2n - 2k + 1)^3 v_\gamma(n - 1, k - 1) + (2k + 1)^3 v_\gamma(n - 1, k)
+ (2n - 2k + 1) u_\gamma(n - 1, k - 1) + (2k + 1) u_\gamma(n - 1, k), \quad (49)
\]

\[
v_\delta(n, k) = 2 n^2(n + 1)^2 v_\delta(n - 1, k - 1) + (2k + 1) v_\gamma(n - 1, k), \quad (50)
\]

\[
u_\delta(n, k) = (2k + 1)^3(n + 1) v_\gamma(n - 1, k) + (2k + 1) u_\gamma(n - 1, k)
+ 8 n^2(n + 1) u_\gamma(n - 1, k - 1) + 2 n u_\delta(n - 1, k - 1), \quad (51)
\]

we obtain the recurrences

\[
v_b(n, k) = m v_a(n - 1, k - 1) + k v_a(n - 1, k), \quad (44)
\]

\[
u_\delta(n, k) = (2k + 1)^3(n + 1) v_\gamma(n - 1, k) + (2k + 1)^3 v_\gamma(n - 1, k)
+ 8 n^2(n + 1) u_\gamma(n - 1, k - 1) + 2 n u_\delta(n - 1, k - 1), \quad (52)
\]
TABLE IV  
\(v_r(n, k), u_r(n, k), v_s(n, k),\) and \(u_s(n, k)\). The diagonals correspond to \(k = 0, 1, \ldots\).

<table>
<thead>
<tr>
<th>(n)</th>
<th>(v_r(n, k))</th>
<th>(u_r(n, k))</th>
<th>(v_s(n, k))</th>
<th>(u_s(n, k))</th>
</tr>
</thead>
<tbody>
<tr>
<td>5</td>
<td>1 237 1682 1682 237 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>1 76 230 76 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>1 23 23 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>1 6 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5 10065 124330 124330 10065 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4 2416 10520 2416 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3 477 384 384 384 377</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2 60 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>10</td>
<td>10 10065 124330 124330 10065 10</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>9</td>
<td>9 2416 10520 2416 9</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>8</td>
<td>8 477 384 384 384 477</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>7</td>
<td>7 60 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>6</td>
<td>6 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5 10065 124330 124330 10065 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4 2416 10520 2416 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3 477 384 384 384 377</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2 60 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>5 237 1920 3602 3839 3840 5</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>4 77 307 383 384 477 4</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>3 24 47 48 24 3</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>2</td>
<td>2 7 8</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>1</td>
<td>1 2</td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>0</td>
<td>1</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

The \(j\)-dimensional faces of \(a(n, k), \ldots, b(n, k)\) for any \(j\) can be found from Figs. 15 and 16.

The special cases we are most interested in are \(b(2t, t)\) and \(d(2t + 1, t)\), the Voronoi regions for \(D_{2t}\) and \(D_{2t+1}\) respectively. For \(D_{2t}\), we have

\[
\text{vol}(V(0)) = v_b(2t, t) \frac{2^{2t}}{(2t)!} = 2^{2t-1}, \quad \text{from (47),}
\]

\[
U(V(0)) = u_b(2t, t) \frac{2^{2t}}{(2t)!} ,
\]

\[
G(D_{2t}^*) = \frac{u_b(2t, t)}{2^{2t-1/4}(2t+2)!},
\]

\[
\rho = 1,
\]

and covering radius

\[
R_c = R_b(2t, t) = \sqrt{t} - \sqrt{t}.
\]

For this version of \(D_{2t+1}^*\) (which differs by a scale factor from the definitions given in Section III-A) we have

\[
\text{vol}(V(0)) = v_b(2t + 1, t) \frac{2^{2t+1}}{(2t+1)!} = 2^{2t+1}, \quad \text{from (52),}
\]

\[
U(V(0)) = u_b(2t + 1, t) \frac{2^{2t+1}}{(2t+3)!} ,
\]

\[
G(D_{2t+1}^*) = \frac{u_b(2t + 1, t)}{(2t+1)(2t+3)!2^f(t)},
\]

where

\[
f(t) = \frac{2(2t^2 + 5t + 1)}{2t + 1},
\]

\[
\rho = \frac{\sqrt{3}}{3} \quad \text{if } t = 1, \quad \rho = 2 \quad \text{if } t > 1,
\]

\[
R_c = R_b(2t + 1, t) = \frac{\sqrt{4t + 1}}{\frac{\sqrt{t}}{\sqrt{3}}} \quad \text{if } t = 1, \quad \rho \sqrt{t + \frac{1}{4}} \quad \text{if } t > 1.
\]

For example \(D_{2}^* = A_{2}^*\) is the body-centered cubic lattice, the Voronoi region is a truncated octahedron (see [22, page 294 and anaglyph XI], [24, fig. 4], and [33, p. 129]), and \(G(D_{2}^*) = 19/192\sqrt{2}\) (see (7) and Section II-G). Also

\[
G(D_{3}^*) = 13/120\sqrt{2} = G(D_{4}),
\]

\[
G(D_{5}^*) = \frac{2641}{23040 \cdot 2^{3/5}} = 0.0756254 \cdots , \quad (55)
\]

\[
G(D_{6}^*) = \frac{601 \cdot 2^{1/5}}{10080} = 0.0751203 \cdots . \quad (56)
\]

The values of \(G(D_{n}^*)\) are plotted in Fig. 20. The minimum value is 0.0746931 \cdots at \(n = 9\).

**G. Voronoi Region for \(A_{n}^*\)**

**Theorem 7:** The Voronoi region for the lattice \(A_{n}^*\) is the polytope \(P_{n}\) defined in Fig. 17. If we rescale \(A_{n}^*\) by multiplying it by \(n + 1\), the vertices of the Voronoi region may
be taken to be the images of the point
\[
\sigma = \left(-\frac{n}{2}, -\frac{n-2}{2}, -\frac{n-4}{2}, \ldots, -\frac{n-2}{2}, \frac{n}{2}\right)
\]
under the action of the Weyl group \(W(A_n)\).

Since \(\sigma\) is sometimes referred to as the Weyl vector for \(A_n\) (see [10]), it is appropriate to call \(P_n\) the Weyl polytope of \(A_n\), The case \(n = 4\) of this theorem may be found in [15, pp 72-73].

**Sketch of Proof:** It is easy to check that \(\sigma\) is equidistant from the walls of the fundamental simplex \(S\) for \(A_n\); i.e., that \(\sigma\) is the incenter of \(S\). Let \(P\) be the convex hull of the images of \(\sigma\) under \(W(A_n)\). Since the walls of \(S\) are reflecting hyperplanes for \(W(A_n)\), \(P\) and its images under \(W(A_n)\) tile \(\mathbb{R}^n\). Thus \(P\) is the Voronoi region for some lattice \(\Lambda \subseteq A_n^*\). But \(\Lambda\) must contain all the points (28), since these are the images of \(0\) in the walls of \(P\). Since these points span \(A_n^*\), \(\Lambda = A_n^*\).

The second moment of \(P_n\) may be found as follows. First, the covering radius of \(A_n^*\), \(R_c(n)\) say, is the circumradius of \(P_n\), which is
\[
R_c(n) = \sqrt{\rho^2} = \left(\frac{1}{2} \left\{ n + 2 \right\} \right)^{1/2} = \rho \sqrt{n + 2},
\]
(57)
since now \(\rho = (n+1)/4\), and the volume of \(P_n\) is
\[
V_n = (n+1)^{n-1/2} \text{ from (27).}
\]
(58)

Let \(I_n = I(P_n)\) be the normalized second moment. A typical cell of \(P_n\) is obtained by deleting the \(r\)th node from the left in Fig. 17, and is a prism \(P_r \times P_s\) with \(r + s = n - 1\) (see Fig. 18). The number of such faces is
\[
\frac{|W(A_n)|}{|W(A_r)| |W(A_s)|} = \left(\frac{n+1}{r+1}\right).
\]
(59)

Furthermore \(I(P_r \times P_s) = I_r + I_s\).

Let \(h_{rs}\) be the height of the perpendicular from the center of \(P_n\) to a typical face \(P_r \times P_s\). Then (see Fig. 19)
\[
h_{rs}^2 = R_c(r+s+1)^2 - R_c(r)^2 - R_c(s)^2 = \frac{(r+1)(s+1)(n+1)}{4},\text{ using (57).}
\]

We may now apply Theorem 3, to obtain
\[
I_nV_n = \frac{1}{n+2} \sum_{r=0}^{n-1} \left(\frac{n+1}{r+1}\right)h_{rs}V_s\left(h_{rs}^2 + I_r + I_s\right)
\]
(with \(r + s = n - 1\)), which, if we write \(J_n = I_{n-1}/n\), becomes
\[
J_n = \frac{1}{2(n+1)n^{n-1}} \sum_{r=1}^{n-1} \left(\frac{n}{r}\right)^r \frac{n}{4} \frac{J_r}{s} \frac{J_s}{r}.
\]

Finally, the dimensionless second moment of the Voronoi region of \(A_n^*\) is
\[
G(A_n^*) = \frac{J_{n+1}}{n(n+1)^{1-(1/n)}}.
\]
(61)

The values for \(n \leq 9\) are plotted in Fig. 20. The curve is extremely flat, the minimum value of 0.0754913 occurring at \(n = 16\).

**H. When is the Voronoi Region Determined by the First Layer of the Lattice?**

We have seen in the Corollary to Theorem 5 that for a root lattice the walls of the Voronoi region are determined solely by the minimum vectors of the lattice. To give a precise statement of this property for an arbitrary lattice \(\Lambda\), let us write \(\Lambda = \Lambda_0 \cup \Lambda_1 \cup \Lambda_2 \cup \cdots\), where \(\Lambda_0\), the 0th layer from the origin, is chosen so that \(u \cdot u\) is a constant \(\lambda_0\) (say) for all \(u \in \Lambda_0\), and \(0 = \lambda_n < \lambda_1 < \lambda_2 < \cdots\). We say that the Voronoi region is determined by the first layer of the lattice if the walls of the Voronoi region...
around the origin, \( V(0) \), are bounded by the hyperplanes
\[
x \cdot u = \frac{1}{2} \lambda_1, \quad \text{for } u \in \Lambda_1,
\]
(62)

If this property holds then there is a simple description of the Voronoi region.

**Theorem 8.** If the Voronoi region \( V(0) \) is determined by the first layer of the lattice, then \( V(0) \) is the reciprocal of the vertex figure of \( \Lambda \) at the origin. Equivalently, \( V(0) \) is (on a suitable scale) the reciprocal of the polytope with vertices \( \Lambda_1 \).

**Proof:** This follows immediately from the definitions of vertex figure and reciprocal polytope—see [16].

The final two theorems give sufficient conditions for this property to hold.

**Theorem 9.** Suppose that (i) \( \Lambda_r \subseteq \Lambda_1 \mid \Lambda_1 \mid \cdots \mid \Lambda_1 \) (\( r \) times) and (ii) \( r \lambda_1 \leq \lambda_r \), for all \( r = 1, 2, \ldots \). Then the Voronoi region is determined by the first layer.

Condition (i) states that \( \Lambda_1 \) spans \( \Lambda \), and moreover does it economically in the sense that any vector in \( \Lambda_r \) is the sum of not more than \( r \) vectors of \( \Lambda_1 \). In practice this condition is very easily checked by induction. An important class of lattices satisfying (i) are those obtained by applying Construction A of [32] or [45] to a linear binary code with minimum distance \( \leq 4 \), which is spanned by the codewords of minimum weight, and which if \( d < 4 \) has the additional property that no coordinate of the code is always 0.

**Proof of Theorem 9:** Suppose the contrary, so that there is a point \( u \in \Lambda_x \), with \( r > 1 \) and a point \( x \in \mathbb{R}^n \) such that
\[
x \cdot u > \frac{1}{2} \lambda_r,
\]
but
\[
x \cdot v \leq \frac{1}{2} \lambda_1, \quad \text{for all } v \in \Lambda_1.
\]
From (i), \( u = \sum_i v_i \) with \( v_i \in \Lambda_1 \), \( n_i > 0 \) and \( \sum n_i \leq r \). Then
\[
x \cdot u = \sum \lambda_i (x \cdot v_i) \leq \frac{1}{2} \lambda_1 \sum n_i
\]
\[
\leq \frac{1}{2} r \lambda_1 \leq \frac{1}{2} \lambda_r,
\]
a contradiction.

Theorem 9 can be used to give an alternative proof that any root lattice has the property. On the other hand the dual lattices \( A_n^* \) for \( n \geq 3 \) and \( D_n^* \) for \( n \geq 5 \) do not, as the previous section demonstrated, nor does the Leech lattice in 24 dimensions [9], [32]. In fact one can show that the Voronoi region of the Leech lattice is determined just by the first two layers: in other words the Voronoi region has 196560 + 16773120 = 16969680 cells.

A second test is the following.

**Theorem 10 (A. Gersho, private communication):** Let \( V \) be the intersection of the half-planes defined by (62). If \( x \cdot x \leq \frac{1}{2} \lambda_2 \) holds for all \( x \in V \), then \( V \) is the Voronoi region.

**Note added in proof:** It has recently been shown that the body-centered cubic lattice \( A^*_3 \) is the optimal three-dimensional lattice quantizer for uniformly distributed data: see E. S. Barnes and N. J. A. Sloane, "The optimal lattice quantizer..."

## COMPARISON OF QUANTIZERS

Comparing the different lattices analyzed in this section we see that the best quantizers found so far in dimensions 1–10 are the following:

<table>
<thead>
<tr>
<th>Dimension</th>
<th>Lattice</th>
<th>Dimension</th>
<th>Lattice</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>( A_1(=A^*_1) )</td>
<td>6</td>
<td>( E_6 )</td>
</tr>
<tr>
<td>2</td>
<td>( A_2(=A^*_2) )</td>
<td>7</td>
<td>( E_7 )</td>
</tr>
<tr>
<td>3</td>
<td>( A_3(=A^*_3) )</td>
<td>8</td>
<td>( E_8(-E_8) )</td>
</tr>
<tr>
<td>4</td>
<td>( D_4(=D^*_4) )</td>
<td>9</td>
<td>( D^*_5 )</td>
</tr>
<tr>
<td>5</td>
<td>( D^*_5 )</td>
<td>10</td>
<td>( D^*_5 )</td>
</tr>
</tbody>
</table>

The values of the dimensionless second moment \( G(\Lambda) \), which is our measure of the mean-squared quantization error per symbol, are shown in Table V and Fig. 20, together with Zador's bounds (3). It is known that \( A_1 \) and \( A_2 \) are optimal, and it is tempting to make the following conjecture.

**Conjecture**

The best lattice quantizer in \( \mathbb{R}^n \)—that with the lowest \( G(\Lambda) \)—is the dual of the densest lattice packing.

**Certainly** \( E_8^* \), \( E_7^* \), and the Leech lattice should be investigated.

It is worth drawing attention to the remarkably low value of the mean-squared error for \( E_8 \) (see Fig. 20). Furthermore there is a fast algorithm [12] available for performing the quantization with this lattice (and in fact for any of the lattices described here).

ACKNOWLEDGMENT

During the early stages of this work we were greatly helped by several discussions with Allen Gersho. Some of the calculations were performed on the MACSYMA system [35]. We should also like to thank E. S. Barnes and H. S. M. Coxeter for their comments on the manuscript.

REFERENCES


Fig. 20. Comparison of mean-squared quantization error per symbol, \( G(A) \), for different lattices \( A \) in dimensions 1–9.