NOTES

Designing an Auditing Procedure, or How to Keep Bank Managers on Their Toes

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A bank inspector has N banks under his supervision, and wishes to plan his visits to these banks for many years in advance so that

(i) there is a high probability, or even certainty, that every bank will be audited at least once a year,
(ii) the visits are unexpected, and
(iii) the total number of banks he visits each year is minimized.

Furthermore the plan must be fair: every bank must be treated in the same way. We wish to design a method for selecting M banks to be audited each week so as to satisfy conditions (i)–(iii) as far as possible. We assume the banks will know (or can find out) what algorithm we intend to use.

We first analyze two of the more obvious inspection plans (Plans A and B), and show that neither meets all the requirements. Then three schemes (Plans C, D and E) are described that do satisfy (i)–(iii). Plans C and D both have drawbacks, however, and the final scheme, Plan E, appears to be the best. Some numerical values of the parameters for the case of 2000 banks are given in TABLES I–III.

We are not aware of any previous work on this problem, although [**4**], [**8**], [**9**] and the paradox of the unexpected hanging [**3**] are tangentially related.

The inspection plans

We assume that the diligent inspector spends 50 weeks each year visiting his banks (this parameter can easily be changed.) The other parameters in the analysis are:

N = total number of banks (assumed for simplicity to be a multiple of 50),
P = probability that every bank is visited at least once during the year,
Q = probability that the ith bank is visited more than once during the year (this will be independent of i),
$H_{g,t}$ = probability that the ith bank is visited in week t, conditional on that bank having been last visited g weeks previously (i.e., in week $t-g$), for $t=1,2,\ldots,50$ and $g=1,2,\ldots,t-1$,
T = average number of banks visited by the inspector during each year, and
W = average number of weeks between visits to the ith bank.

In practice, a team of inspectors would be used to visit such a large number of banks each week. But for simplicity we speak as if only a single inspector were involved. Also, unlike the notorious traveling salesman problem, we are not concerned with the distance the inspector has to travel each week. (Perhaps the inspections can be carried out by telephone.)

Our first three sampling schemes work on a calendar year basis; everything starts afresh at the beginning of the year.

Plan A: Weekly selection without replacements. At the beginning of each year N balls bearing the names of the banks are placed in an urn and thoroughly shuffled. Each week $M = N/50$ balls are drawn without replacement and these banks are inspected.

This plan certainly satisfies conditions (i) and (iii), since by the end of the year all the banks will have been visited exactly once (in fact $P = 1$, $Q = 0$ and $T = N$). Condition (ii) is partially satisfied, since a bank manager does not know in advance when he will be audited. On the other hand once he has been visited he knows he is safe from further inspection for the rest of the year, so that $H_{g,t} = 0$ for all $g, t \geqslant 1$. With this scheme the average time W between the inspector's visits is one year. Since one of the goals implied by condition (ii) is that the inspections may occur at any time, this plan is not completely satisfactory. Also there is a good chance that the time between visits will be substantially longer than one year, if a visit is made early in one year and the next visit is late in the following year.

Plan B: Weekly selection with replacement between weeks. Each week the names of *all* N banks are mixed in an urn and a random subset of M names is selected (without replacement) for inspection.

This plan certainly satisfies condition (ii), for whether or not a bank is inspected one week is independent of whether it was inspected the previous week. In fact $H_{g,t} = M/N$, independently of g and t. Furthermore it is clear that, if M is sufficiently large, condition (i) is also satisfied. How large must M be? To answer this, we note that this is a version of the classical occupancy problem [**1**], [**2**], [**5**].

In one year there are a total of $\binom{N}{M}^{50}$ possible ways of selecting the banks to be inspected, each having probability $\binom{N}{M}^{-50}$. The number of these selections having the property that a particular set of k banks all fail to be visited during the year is $\binom{N-k}{M}^{50}$. The principle of inclusion and exclusion [**1**, p. 242], [**2**, Section IV.2] shows that the probability that there are exactly k banks not visited during the year is

$$\binom{N}{k}\sum_{i=0}^{N-k-M}(-1)^i\binom{N-k}{i}\binom{N-k-i}{M}^{50}\Big/\binom{N}{M}^{50}. \tag{1}$$

In particular, the probability that every bank is visited in the year is

$$P = \sum_{i=0}^{N-M}(-1)^i\binom{N}{i}\binom{N-i}{M}^{50}\Big/\binom{N}{M}^{50}. \tag{2}$$

This expression can be evaluated exactly only for small values of N and M. For larger values the following Poisson approximation is appropriate (compare [**1**, Chapter 14], [**2**, p. 94]). The probability that the ith bank is not visited during the year is

$$\frac{\binom{N-1}{M}^{50}}{\binom{N}{M}^{50}} = \left(1 - \frac{M}{N}\right)^{50},$$

and therefore the average number that are not visited is

$$\lambda = N\left(1 - \frac{M}{N}\right)^{50}. \tag{3}$$

It is straightforward to show from (1) that if N and M increase while λ remains fixed, then the probability that exactly k banks remain unvisited is approximately

$$e^{-\lambda}\frac{\lambda^k}{k!}.$$

In particular, the probability that every bank is visited is

$$P \approx e^{-\lambda}. \tag{4}$$

Thus using Plan B with N banks and a desired probability P, we choose a value of λ from (4), and take

$$M = N\left\{1 - \left(\frac{\lambda}{N}\right)^{1/50}\right\}. \tag{5}$$

The total number of banks visited during the year is then

$$T = 50M \approx 50N\left\{1 - \left(\frac{\ln P^{-1}}{N}\right)^{1/50}\right\}. \tag{6}$$

The probability that the ith bank is visited more than once is

$$\begin{aligned} Q &= 1 - \left(1 - \frac{M}{N}\right)^{50} - \frac{50M}{N}\left(1 - \frac{M}{N}\right)^{49} \\ &\approx 1 - \frac{\lambda}{N} \cdot \frac{N + 49M}{N - M}, \end{aligned} \tag{7}$$

using (3), and the average time between visits is approximately

$$W = \frac{50}{P} = 50e^{\lambda} \text{ weeks}. \tag{8}$$

The difference $N - M$ is $\lambda^{.02}N^{.98}$ (from (5)), which approaches 0 as N increases (for λ fixed). TABLE I gives some typical values of P, λ, M and T for $N = 2000$. It is clear that although conditions (i) and (ii) are satisfied, (iii) is not, as T is unacceptably large.

It is worth remarking that, in the case $M = 1$, (4) also leads to an approximation for the Stirling numbers of the second kind. Although this approximation seems very natural in the present context, it does not appear to have been pointed out before. (It is not included for example among the half-dozen similar approximations analyzed in Chapter 16 of [1].)

Our next scheme is a generalization of Plans A and B, and appears to satisfy all three conditions.

TABLE I. **Typical values of the parameters for Plan B when visiting $N = 2000$ banks. The columns show P = probability that every bank is visited in the year, λ = average number not visited, M = number visited each week, $T = 50M$ = total number of visits during a year.**

P	λ	M	T
.75	.288	324	16220
.80	.223	333	16640
.85	.163	343	17170
.90	.105	358	17880
.95	.051	381	19060

Plan C: A and B combined. As in Plan A, $M = N/50$ names are drawn without replacement for each of the 50 weeks. In addition, for each week we place the $N - M$ names not drawn for that week in an urn, and choose from it a random subset of K additional names. All $M + K$ banks are visited that week.

This plan combines some of the best features of Plans A and B, since each bank will be drawn exactly once during the year in one of the sets of size M, and in addition may also be drawn in any other week in one of the sets of size K. If $K = 0$, it reduces to Plan A, while if we change M from $N/50$ to zero, it reduces to Plan B. For this scheme $P = 1$, $T = 50(M + K)$ and

$$Q = 1 - \left(1 - \frac{K}{49M}\right)^{49}. \tag{9}$$

The parameters $H_{g,t}$ and W are more difficult to calculate, and a formula for W is derived at the end of this paper. We can see, however, that $H_{g,t}$ depends strongly on g and t. Suppose for example that $g \geqslant t$, so that in the current year the ith bank has not yet been visited. Then

$$\begin{aligned} H_{g,t} &= \frac{M}{N - M(t-1)} + \frac{N - Mt}{N - M(t-1)} \cdot \frac{K}{N - M} \\ &= \frac{1}{51 - t}\left(1 + (50 - t)\frac{K}{49M}\right), \end{aligned}$$

which increases from $(M + K)/N$ to 1 as t goes from 1 to 50. If $g < t$, the situation is improved; for these cases we have $K/N \leqslant H_{g,t} \leqslant (K + M)/N$. (We are assuming that when it is visited, a bank does not learn whether the visit is an M-type or a K-type.)

Some typical values of the parameters when $N = 2000$ are shown in TABLE II. Taking K in the range 10 to 40 gives an inspection scheme which meets requirements (i)–(iii), and were it not for the strong dependence of $H_{g,t}$ on g and t in the range $g \geqslant t$, Plan C would be a quite satisfactory solution.

The last two plans to be considered are *stationary*, in the technical sense that no part of the calendar year has any special role. They do not have the drawbacks of Plans A, B and C that the probability that a particular bank is visited in a particular week can vary over a wide range as a function of the calendar date. As a side benefit, this approach will result in our satisfying condition (i) in a stronger sense than before; we shall be able to guarantee that no bank is ever left unvisited for more than twelve months.

TABLE II. **Typical values of the parameters for Plan C when $N = 2000$ and $M = 40$. The columns show K = number of banks chosen randomly (in addition to the M that are chosen without replacement), $1 - Q$ = probability that any given bank is visited more than once during the year, $T = 50(M + K)$ = total number of visits, W = average time between visits.**

K	$1 - Q$	T	W
0	1	2000	50
10	.778	2500	40
20	.605	3000	33.3
30	.470	3500	28.6
40	.364	4000	25
50	.282	4500	22.2
60	.218	5000	20
70	.168	5500	18.2
80	.130	6000	16.7

Plan D: Independent renewals. To implement this scheme, we first specify a probability distribution $\pi = \{\pi_1, \pi_2, \ldots, \pi_{50}\}$ on the integers $1, 2, \ldots, 50$. (We shall soon see what properties this distribution should have.) Then, for each bank separately, the following plan is followed. First, an initial visit is scheduled, for week t_1, say, in a way that will be described shortly. Then an integer g_1 between 1 and 50 is chosen, with $\text{Prob}\{g_1 = j\} = \pi_j$, for $1 \leqslant j \leqslant 50$, and the second visit is scheduled for week $t_1 + g_1$. A second integer g_2 is chosen (with the same probability distribution as g_1), independent of everything else so far, and the third visit is scheduled for $t_1 + g_1 + g_2$, and so on. Once the plan is under way, the average time between visits is $W = \pi_1 + 2\pi_2 + \cdots + 50\pi_{50}$, which is simply the mean of the distribution π.

We have still to specify how t_1 is to be chosen. We do this in such a way as to ensure that the probability that a chosen bank is visited in any particular week is a constant, independent of the week. The constant will turn out to be simply $1/W$. It is easy to see that for this to happen we must make

$$\text{Prob}\{t_1 = k\} = \frac{1}{W}(\pi_k + \pi_{k+1} + \cdots + \pi_{50}), \tag{10}$$

for $k = 1, \ldots, 50$. This much complication seems to be unavoidable if the sampling plan is to be completely independent of the calendar.

Since under Plan D the banks are treated independently, it is straightforward to derive the following formulae for P, Q (and more generally the probability that a particular bank is visited more than once during *any* period of 50 consecutive weeks), $H_{g,t}$, and T (and more generally the average number of banks visited during any period of 50 consecutive weeks). We denote the right-hand side of (10) by p_k. Then

$$P = 1,$$

$$Q = \sum_{k=1}^{49} p_k(\pi_1 + \pi_2 + \cdots + \pi_{50-k})$$

$$= 1 - W \sum_{k=1}^{50} p_k p_{51-k}, \tag{11}$$

$$H_{g,t} = \frac{\pi_g}{\pi_g + \pi_{g+1} + \cdots + \pi_{50}} = \frac{\pi_g}{W p_g}, \tag{12}$$

$$T = 50N/W. \tag{13}$$

Thus $H_{g,t}$ is independent of t, as desired, and can be written simply as H_g. Note that $H_{50} = 1$. Also M, the number of banks that are visited in any given week, has an average value of N/W and a variance of $N(W-1)/W^2$. These quantities do not depend on the distribution π, except through its mean W, and we choose this so that the resulting value of M has a high probability of being acceptable.

For example, when $N = 2000$, if we wish to make no more than 70 inspections per week, we may take $W = 40$, so that M has a mean of 50 and a standard deviation of 6.98, and so has only a small chance of exceeding 70. (This would happen with probability 0.003, i.e., about once every six years.) The total number T of inspections per 50-week period is 2500 ± 49.4.

Now we consider how the probability distribution π should be chosen, supposing that its mean W is given in advance. Clearly it is undesirable to have $\pi_g = 0$ for any g, since this makes $H_g = 0$ and the bank is certain not to be visited that week. Also H_{50} is constrained to be 1. A reasonable choice for π is to make $H_1 = H_2 = \cdots = H_{49} = H$, say, which we can do by setting

$$\pi_k = H(1-H)^{k-1}, \quad k = 1, \ldots, 49, \tag{14}$$

$$\pi_{50} = (1-H)^{49}, \tag{15}$$

TABLE III. **Typical values of the parameters for Plans D and E when $N = 2000$. For Plan D the columns show W = average time between visits to any bank, T = average number of banks visited in any 50-week period, H = probability that a particular bank is visited in any week, conditioned on the previous visit having been less than 50 weeks earlier, π_{50} = probability that a bank goes 49 weeks without a visit, and M = average number of banks visited per week. These values also apply (approximately) to Plan E, in which case T and M are constants.**

W	T	H	π_{50}	M
44.34	2255	.005	.782	45
39.50	2532	.01	.611	51
35.35	2829	.015	.477	57
31.79	3146	.02	.371	63
28.72	3482	.025	.289	70
26.06	3837	.03	.224	77
23.76	4209	.035	.174	84
21.75	4598	.04	.135	92
20.00	5000	.045	.105	100
18.46	5417	.05	.081	108

where H is determined by the equation

$$\text{mean}\{\pi\} = \frac{1}{H}\left(1-(1-H)^{50}\right) = W. \tag{16}$$

Also

$$p_k = (1-H)^{k-1}/W \text{ for } k = 1,\ldots,50. \tag{17}$$

Some numerical values are given in TABLE III.

The probability distribution π we have arrived at, given by equations (14) and (15), is very nearly the distribution of waiting-time until the appearance of the first head in a sequence of independent coin-tosses with Prob{head} = H at each trial. The only difference is that at trial 50 the outcome "head" is forced. Thus Plan D is only a minor modification of Plan B! This suggests two things: Plan D should be easy to implement, and can be easily modified to make the number of inspections each week a constant.

To implement Plan D, once the start-up phase is over and every bank has been visited at least once, each week the inspector must visit

(a) all banks that have gone 49 weeks without a visit, and
(b) a random selection of the remaining banks, with each bank having independently a chance H of being selected.

In the start-up phase, we must implement the probability distribution $(p_1,\ldots,p_{50})$ given in (10), (17). For each bank, we simply toss a coin having Prob{head} = H until the first head appears, and schedule the first visit to occur in the corresponding week, except that if no head appears in the first 50 tosses, we start again at week 1.

The modification of Plan D in which the number of inspections per week is constant is given by our final scheme, Plan E, which combines most of the attractive features of Plans B and D.

Plan E: Truncated geometric renewals, constrained to have fixed sample size. In this scheme, after a start-up phase that is described below, each week we determine which banks have gone 49 weeks without a visit. Suppose there are S of them. Then that week the inspector visits

(a) these S banks, and
(b) a random subset of size $M - S$ of the remaining banks.

Thus every week exactly M banks are visited, and no bank goes for more than 50 weeks between visits; also for each bank there is a constant probability each week that it will be visited (except if it is 50 weeks since the last visit, in which case another visit is sure). It is easy to see that we shall never find $S > M$, since the banks in the S group are a subset of the M banks that were inspected exactly 50 weeks ago.

The number of different ways a schedule for 50 consecutive weeks can be written down (with M visits each week, and so that every bank is visited at least once) is simply $Z = \binom{N}{M}^{50} P$ where P is given in (2) above. It turns out that when Plan E is used, then for any set of 50 consecutive weeks, each of these Z ways is equally likely. A proof of this result is given below.

Now we must consider how to start up Plan E. If we simply use Plan B (weekly random selections with replacement) for the first 49 weeks, we run the risk that more than M banks will escape visitation, so that in the 50th week we have an impossible task. Better would be Plan C, which does force every bank to be visited at least once in the first year. However, this start-up rule does not exactly achieve the stationarity condition, though the approximation appears to be quite good in the cases (small N and a "year" with only 2 or 3 weeks) that we have been able to work out in detail. To achieve a better approximation, it suffices to generate several years' worth of Plan E (with Plan C start-up) before coming to a part that will actually be used.

Although Plan E is simple to state and to implement, we have not been able to obtain tractable expressions for all of its parameters. As an approximation, however, we can use the values given in TABLE III (for $N=2000$ banks), interpreting the columns as follows:

M = number of banks visited each week,
W = average time between visits to any bank,
T = $50M$ = total number of visits per 50-week period,
H = approximate probability that a particular bank is visited in any week, except when the previous visit was 49 weeks earlier, and
π_{50} = approximate probability that a bank goes 49 weeks without a visit.

Calculation of the expected time between visits in Plan C

We consider the ith bank, and say that it is in state $(j,0)$, where $1 \leqslant j \leqslant 50$, if at the beginning of the jth week it has not yet been drawn in one of the M-sets, and in state $(j,1)$ if it has. Let $P(j,0)$ and $P(j,1)$ denote the probabilities of being in these states, and let $\pi(j,0)$ [resp. $\pi(j,1)$] be the probability of going from state $(j,0)$ to state $(j+1,0)$ [resp. $(j+1,1)$]. These states and probabilities are related by the Markov chain shown in FIGURE 1. For $1 \leqslant j \leqslant 49$ we have (using $N=50M$)

$$\pi(j,0)=\frac{\binom{N-jM-1}{M}}{\binom{N-jM}{M}}=\frac{50-j}{51-j},$$

$$\pi(j,1)=\frac{1}{51-j},$$

and therefore $P(j,0)=(51-j)/2500$, $P(j,1)=(j-1)/2500$, for $1 \leqslant j \leqslant 50$. The probability of this bank being drawn during the jth week in the K-set, given that it was not drawn in the M-set,

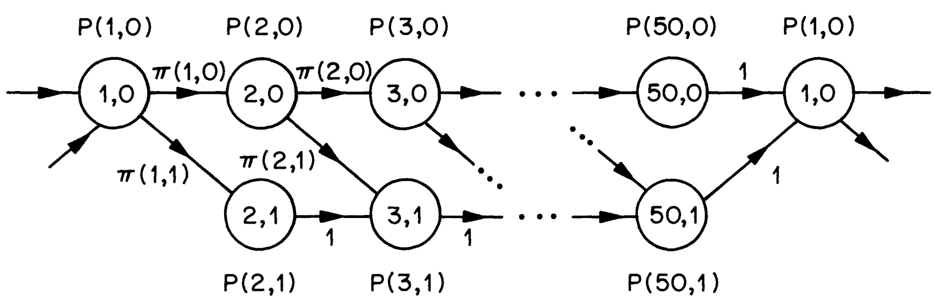

FIGURE 1. **Markov chain describing Plan C.**

is

$$\sigma = 1 - \frac{\binom{N-M-1}{K}}{\binom{N-M}{K}} = \frac{K}{49M}.$$

We wish to determine W, the average number of weeks to the next time this bank is drawn, given that it has just been drawn. There are three possibilities: (a) the bank was in state $(j,0)$ and in the jth week was drawn in the M-set, (b) the bank was in state $(j,1)$ and in the jth week was drawn in the K-set, and (c) the bank was in state $(j,0)$ and in the jth week was not drawn in the M-set but was drawn in the K-set. The expression for W contains terms corresponding to these three possibilities; we omit the details and simply state the result, which is

$$W = \frac{W_a + W_b + W_c}{D_a + D_b + D_c}, \tag{18}$$

where

$$W_a = \sum_{j=1}^{50} P(j,0)\pi(j,1)W_a',$$

$$W_a' = \sum_{r=1}^{50-j} r(1-\sigma)^{r-1}\sigma + \sum_{r=51-j}^{100-j} r(1-\sigma)^{r-1}\left\{\prod_{i=1}^{r+j-51}\pi(i,0)\right\}\{\pi(r+j-50,1)+\pi(r+j-50,0)\sigma\},$$

$$W_b = \sum_{j=1}^{50} P(j,1)\sigma W_a',$$

$$W_c = \sum_{j=1}^{50} P(j,0)\pi(j,0)\sigma W_c',$$

$$W_c' = \sum_{r=1}^{50-j} r\left\{\prod_{i=j+1}^{r+j-1}\pi(i,0)\right\}(1-\sigma)^{r-1}\{\pi(r+j,1)+\pi(r+j,0)\sigma\},$$

and

$$D_a = \sum_{j=1}^{50} P(j,0)\pi(j,1),$$

$$D_b = \sum_{j=1}^{50} P(j,1)\sigma,$$

$$D_c = \sum_{j=1}^{50} P(j,0)\pi(j,0)\sigma.$$

(An empty product is equal to 1 by convention.) Therefore

$$W_a = \sum_{j=1}^{50} \frac{49M+(j-1)K}{2500\times 49M}\cdot\left[\sum_{r=1}^{50-j} r(1-\sigma)^{r-1}\sigma + \sum_{r=51-j}^{100-j}\frac{r(1-\sigma)^{r-1}}{50}\{1+(100-r-j)\sigma\}\right], \tag{19}$$

$$W_c = \sum_{j=1}^{50}\frac{\sigma}{2500}\sum_{r=1}^{50-j} r(1-\sigma)^{r-1}\{1+(50-r-j)\sigma\}, \tag{20}$$

and the denominator is

$$D_a + D_b + D_c = \frac{M+K}{50M}. \tag{21}$$

Equations (18)–(21) were used to calculate the values shown in TABLE II.

Proof of the equidistribution property of Plan E

We first establish some notation. Let $X_1, X_2, \ldots$ be the subsets of $\{1, 2, \ldots, N\}$ that indicate which banks are to be visited in weeks $1, 2, \ldots$. The number of visits each week is $|X_t| = M$, for all t. We have to show that under Plan E, if

$$\left| \bigcup_{j=t-49}^{t} X_j \right| = N \tag{22}$$

so that all the banks are visited in the 50-week period $(t-49, \ldots, t)$, then

$$\operatorname{Prob}\{X_{t-49}, \ldots, X_{t-1}, X_t\} = \frac{1}{Z}. \tag{23}$$

The defining property of Plan E is that the conditional probability

$$\operatorname{Prob}\{X_t | X_{t-49}, \ldots, X_{t-1}\} \tag{24}$$

is constant over all sets X_t such that (22) holds. Let the *coverage* of $\{X_{t-49}, \ldots, X_{t-1}\}$ be

$$C_t = \left| \bigcup_{j=t-49}^{t-1} X_j \right|.$$

Then, at week t, $S = N - C_t$ banks are forced into X_t, leaving $M - (N - C_t)$ to be chosen randomly from the remaining $N - (N - C_t) = C_t$ banks. Thus the probability in (24) equals

$$\frac{1}{\binom{C_t}{M-N+C_t}} = \frac{1}{\binom{C_t}{N-M}}.$$

To prove that (23) is correct, it is sufficient to show that this specification satisfies the recurrence condition

$$\operatorname{Prob}\{X_{t-49}, \ldots, X_t\} = \sum_{X_{t-50}} \operatorname{Prob}\{X_{t-50}, \ldots, X_{t-1}\} \operatorname{Prob}\{X_t | X_{t-49}, \ldots, X_{t-1}\}, \tag{25}$$

where the sum is over all sets X_{t-50} such that

$$\left| \bigcup_{j=t-50}^{t-1} X_j \right| = N.$$

By the same argument as before, the number of such sets is just $\binom{C_t}{N-M}$, so the sum in (25) has this many terms, each equal to $1/Z\binom{C_t}{N-M}$; the sum is thus $1/Z$, and the result is established.

Finally, we note that there is no difficulty in implementing any of these schemes. In particular efficient algorithms are readily available for choosing random subsets from a larger set (see [**6**, §3.4.2], [**7**]).

Acknowledgements. The question studied in this paper was raised by Mr. Chungming An of Bell Laboratories, Indian Hill, Illinois, in connection with the auditing of telephone offices. We are grateful to him for telling us about the problem, and to J. Inglis, H. O. Pollak, and L. A. Shepp for helpful discussions.

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On the Existence of Group Automorphisms Whose Inverse Is Their Reciprocal

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Students of elementary mathematics frequently confuse $f^{-1}(x)$ with $1/f(x)$ when working with functions of a real variable, but students in junior and senior level mathematics courses have matured sufficiently to understand the difference between the two expressions. Since such students normally take a course in abstract algebra, we propose finding an algebraic setting in which $f^{-1}(x)$ and $(f(x))^{-1}$ actually coincide. We use elementary number theory to determine conditions on a finite group which guarantee the existence of an automorphism f such that $f^{-1}(x)=(f(x))^{-1}$. Our methods also yield the answer to the analogous problem for a finite dimensional vector space over a field. Since the existence of an automorphism of the desired type will require the group to be abelian, we shall adopt additive notation for groups. Our formula then becomes $f^{-1}(x)=-f(x)$.

Let $(G,+)$ be a finite group. A function $f\colon G\to G$ is called **naturally invertible** provided f is one-to-one (hence onto) and $f^{-1}(x)=-f(x)$ for each $x\in G$. We use the terminology **ni-function** to mean a naturally invertible function. Note that if $f\colon G\to G$ is a ni-function, then for each $x\in G$,

$$f^{-1}(f(x))=x=-f(f(x))=-f^2(x), \tag{1}$$

hence (replacing x by $-x$ in (1))

$$x=f^2(-x). \tag{2}$$

Immediate consequences of (1) and (2) are the properties

(i) $f^2(x)=-x$, for each $x\in G$,
(ii) $f^{-1}(x)=f(-x)$, for each $x\in G$ and
(iii) $f^4=i_G$, the identity function on G.

If f is an automorphism of G, it is one-to-one, and $f(-x)=-f(x)$ for all $x\in G$, so f is a ni-function if and only if (i) holds. In this case, we call f a **ni-automorphism**. From (i) and the fact that f^2 is an automorphism, we conclude that a group having a ni-automorphism must be abelian.

Our first theorem characterizes cyclic groups which have a ni-automorphism.