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A Monster Lie Algebra?

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This paper defines a remarkable Lie algebra of infinite dimension and rank, and conjectures that it may be related to the Fischer–Griess Monster group.

The idea was discussed in [3] that there might be an infinite-dimensional Lie algebra (or superalgebra) L that in some sense "explains" the Fischer–Griess "Monster" group M. The present note produces some candidates for L based on recent discoveries about the Leech lattice. These candidates are described in terms of a particular Lie algebra L_∞ of infinite rank.

We first review some of our present knowledge about these matters. It was proved by character calculations in [3, p. 317] that the centralizer C of an involution of class 2A in M has a natural sequence of modules affording the head characters (restricted to C). In [12], Kac has explicitly constructed these as C-modules. Now that Atkin, Fong, and Smith [1, 9] have verified the relevant numerical conjectures of [3] for M, we know that these modules can be given the structure of M-modules, but we have no idea how to do this explicitly.

Some of the conjectures of [3] have analogues in which M is replaced by a compact simple Lie group, and in particular by the Lie group E_8. Most of the resulting statements have now been established by Kac. However, it seems that this analogy with Lie groups may not be as close as one would wish, since two of the four conjugacy classes of elements of order 3 in E_8 were shown in [15] (see [16]) to yield modular functions that are *not* Hauptmoduls for any modular group. This disproves the conjecture made on p. 267 of [11], and is particularly distressing since it was the Hauptmodul property that prompted the discovery of the conjectures in [3], and it is this property that gives those conjectures almost all their predictive power.

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There have also been a number of recent discoveries about the Leech lattice [2, 4–7], stemming mostly from the facts about "deep holes" in that lattice reported in [4]. The one that most concerns us here is the identification in [7] of the automorphism group of the even unimodular lattice $\mathrm{II}_{25,1}$. We can define $\mathrm{II}_{25,1}$ as the lattice of all points

$$(x_0, x_1, \ldots, x_{24} \mid x_{70})$$

for which the coordinates are all in $\mathbb{Z}$ or all in $\mathbb{Z} + \frac{1}{2}$ and which have integer inner product with

$$(\tfrac{1}{2}, \tfrac{1}{2}, \ldots, \tfrac{1}{2} \mid \tfrac{1}{2}),$$

all norms and inner products being evaluated in the Lorentzian metric

$$x_0^2 + x_1^2 + \cdots + x_{24}^2 - x_{70}^2.$$

Let $w = (0, 1, 2, 3, \ldots, 24 \mid 70)$. Then the main result of [6] is that the subset of vectors r in $\mathrm{II}_{25,1}$ for which $r \cdot r = 2$, $r \cdot w = -1$ is isometric to the Leech lattice, under the metric defined by $d(r, s)^2 = \mathrm{norm}(r - s)$. The main result of [2] is that $\mathrm{Aut}(\mathrm{II}_{25,1})$ is obtained by extending the Coxeter subgroup generated by the reflections in these "Leech roots" by its group of graph automorphisms together with the central inversion -1. It is remarkable that the walls of the fundamental region for this Coxeter group (which correspond one-for-one with the Leech roots) are transitively permuted by the graph automorphisms, which form an infinite group abstractly isomorphic to the group of all automorphisms of the Leech lattice, including translations.

Vinberg [17] shows that for the earlier analogues $\mathrm{II}_{9,1}$ and $\mathrm{II}_{17,1}$ of $\mathrm{II}_{25,1}$ the fundamental regions for the reflection subgroups have respectively 10 and 19 walls, and the graph automorphism groups have orders 1 and 2. For the later analogues $\mathrm{II}_{33,1}, \ldots$, there is no "Weyl vector" like w, and it again seems unlikely that the graph automorphisms can act transitively on the walls. So it appears that $\mathrm{II}_{25,1}$ is very much a unique object.

We can use the vector w to define a root system in $\mathrm{II}_{25,1}$. If $v \in \mathrm{II}_{25,1}$ then we define the height of v by $-v \cdot w$, and we say that v is positive or negative according as its height is positive or negative. We now define a Kac–Moody Lie algebra L_∞, of infinite dimension and rank, as follows: L_∞ has three generators $e(r)$, $f(r)$, $h(r)$ for each Leech root r, and is presented by the following relations:

$$[e(r), h(s)] = r \cdot s\ e(r),$$

$$[f(r), h(s)] = -r \cdot s\ f(r),$$

$$[e(r), f(r)] = h(r),$$

$$[e(r), f(s)] = 0,$$

$$[h(r), h(r)] = 0 = [h(r), h(s)],$$
$$e(r)\{ad\, e(s)\}^{1-r\cdot s} = 0 = f(r)\{ad f(s)\}^{1-r\cdot s},$$

where r and s are distinct Leech roots. (We have quoted these relations from Moody's excellent survey article [14]. Moody supposes that the number of fundamental roots is finite, but since no argument ever refers to infinitely many fundamental roots at once, this clearly does not matter.)

Then we conjecture that L_∞ provides a natural setting for the Monster, and more specifically that the Monster can be regarded as a subquotient of the automorphism group of some naturally determined subquotient algebra of L_∞.

The main problem is to "cut L_∞ down to size." Here are some suggestions. A rather trivial remark is that we can replace the Cartan subalgebra H of L_∞ by the homomorphic image obtained by adding the relations

$$c_1 h(r_1) + c_2 h(r_2) + \ldots, = 0$$

for Leech roots $r_1, r_2, \ldots,$ whenever $c_1, c_2, \ldots,$ are integers for which

$$c_1 r_1 + c_2 r_2 + \ldots, = 0.$$

A more significant idea is to replace L_∞ by some kind of completion allowing us to form infinite linear combinations of the generators, and then restrict to the subalgebra fixed by all the graph automorphisms. The resulting algebra, supposing it can be defined, would almost certainly not have any notion of root system. Other subalgebras of L_∞ are associated with the holes in the Leech lattice, which are either "deep" holes or "small" holes (see [4]).

(i) By [4], any deep hole corresponds to a Niemeier lattice N, which has a Witt part which is a direct sum of root lattices chosen from the population A_n $(n = 1, 2, \ldots)$, D_n $(n = 4, 5, \ldots)$, E_6, E_7, and E_8. Only 23 particular combinations arise, and we shall take $A_{11}D_7E_6$ as our standard example. The graph of Leech roots contains a finite subgraph which is the disjoint union of extended Dynkin diagrams corresponding to these Witt components W of N, and so our algebra L_∞ has a subalgebra $L[N]$ which is a direct sum of the Euclidean Lie algebras $E(W)$ corresponding to those components (see [10, 13]). For example, L_∞ has a subalgebra

$$E(A_{11}) + E(D_7) + E(E_6).$$

Each such subalgebra of L_∞ can be extended to a larger subalgebra $L^*(N)$ having one more fundamental root, corresponding to a "glue vector" of the appropriate hole (see [5]). In the corresponding graph, the new node is joined to a single special node in each component. The graph for

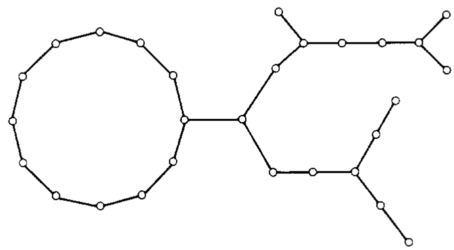

FIG. 1. The fundamental root diagram for $L^*[A_{11}D_7E_6]$.

$L^*[A_{11}D_7E_6]$ is shown in Fig. 1. (A special node of a connected extended Dynkin diagram is one whose deletion would result in the corresponding ordinary diagram.) These hyperbolic algebras $L^*[N]$, having finite rank, are certainly more manageable than L_∞ itself. Since the 23 Niemeier lattices yield 23 constructions for the Leech lattice [5], it is natural to ask if we can obtain 23 different constructions for the Monster using the Lie algebras $L[N]$.

(ii) Each small hole in the Leech lattice (these have been enumerated by Borcherds, Conway, and Queen) corresponds to a maximal subalgebra of L_∞ of finite rank.

We are making various calculations concerning L_∞ (finding the multiplicities of certain roots via the Weyl–MacDonald–Kac formula, etc.). It is worth noting that these calculations are facilitated by the remarkable recent discovery (see [8]) that the Mathieu group M_{12} is generated by the two permutations

$$t \to |2t|, \qquad t \to 11 - t \pmod{23},$$

of the set $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11\}$, where $|x|$ denotes the unique y in this set for which $y \equiv \pm x \pmod{23}$. (This discovery arose from the study of properties of various standard playing-card shuffles. We have noticed that there are many other elements of M_{12} which have simple formulae in this "card numbering," for example $t \to |t^3|$.) The simplest transformation between the usual Euclidean coordinates for the Leech lattice and its Lorentzian coordinates uses this description of M_{12}.

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