Anti-Hadamard Matrices

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ABSTRACT

An anti-Hadamard matrix may be loosely defined as a real $(0,1)$ matrix which is invertible, but only just. Let A be an invertible $(0,1)$ matrix with eigenvalues λ_i, singular values σ_i, and inverse $B=(b_{ij})$. We are interested in the four closely related problems of finding $\lambda(n)=\min_{A,i}|\lambda_i|$, $\sigma(n)=\min_{A,i}\sigma_i$, $\chi(n)=\max_{A,i,j}|b_{ij}|$, and $\mu(n)=\max_A\Sigma_{ij}b_{ij}^2$. Then A is an anti-Hadamard matrix if it attains $\mu(n)$. We show that $\lambda(n),\sigma(n)$ are between $(2n)^{-1}(n/4)^{-n/2}$ and $c\sqrt{n}\,(2.274)^{-n}$, where c is a constant, $c(2.274)^n\leqslant\chi(n)\leqslant 2(n/4)^{n/2}$, and $c(5.172)^n\leqslant\mu(n)\leqslant 4n^2(n/4)^n$. We also consider these problems when A is restricted to be a Toeplitz, triangular, circulant, or $(+1,-1)$ matrix. Besides the obvious application—to finding the most ill-conditioned $(0,1)$ matrices—there are connections with weighing designs, number theory, and geometry.

I. INTRODUCTION

If A is any real invertible matrix, with inverse $B=(b_{ij})$, we let $\mu(A)=\Sigma_{ij}b_{ij}^2$ (this is the square of the Euclidean norm of A^{-1} [23]). A *Hadamard matrix* H is an $n\times n$ matrix with entries $(+1,-1)$ satisfying $HH^{\mathrm{tr}}=nI_n$, where tr denotes transpose [11, 15, 16, 17, 35]. In 1944 Hotelling proved that if A is a $(+1,-1)$ matrix, then $\mu(A)\geqslant 1$, and $\mu(A)=1$ if and only if A is a Hadamard matrix ([18]; see also [1], [16]). A similar result appears to hold for $(0,1)$ matrices. A binary Hadamard matrix, or *S-matrix*, is an $n\times n$ $(0,1)$ matrix formed by taking an $(n+1)\times(n+1)$ Hadamard matrix in which the entries in the first row and column are $+1$, changing $+1$'s to 0's and -1's to 1's, and deleting the first row and column [16]. An S-matrix satisfies $SS^{\mathrm{tr}}=\frac{1}{4}(n+1)(I_n+J_n)$ and $SJ_n=J_nS=\frac{1}{2}(n+1)J_n$, where J_n is an all-ones matrix.

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It is conjectured that if A is a (0,1) matrix, then

$$\mu(A) \geqslant \frac{4n^2}{(n+1)^2}, \tag{1}$$

with equality if and only if A is an S-matrix [16, 27, 29].

At the opposite extreme we call a $(+1, -1)$ or (0,1) matrix that *maximizes* $\mu(A)$ an anti-Hadamard[1] matrix. More precisely, if $A = (a_{ij})$ is an $n \times n$ invertible (0,1) matrix with eigenvalues λ_i, singular values σ_i, and inverse $B = (b_{ij})$, we define

$$\lambda(A) = \min_i |\lambda_i|, \qquad \sigma(A) = \min_i \sigma_i,$$

$$\chi(A) = \max_{i,j} |b_{ij}|, \qquad \mu(A) = \sum_{i,j=1}^{n} b_{ij}^2,$$

and

$$\lambda(n) = \min_A \lambda(A), \qquad \sigma(n) = \min_A \sigma(A),$$

$$\chi(n) = \max_A \chi(A), \qquad \mu(n) = \max_A \mu(A).$$

If A is restricted to be a symmetric, Toeplitz [26], triangular, or circulant [8] (0,1) matrix, this is indicated by a subscript s, T, t, or c respectively, or if A is a $(+1, -1)$ matrix we use a subscript $\pm$.

Then the formal definition is that an anti-Hadamard matrix A is an $n \times n$ (0,1) matrix for which $\mu(A) = \mu(n)$, or a $(+1, -1)$ matrix for which $\mu(A) = \mu_{\pm}(n)$. Some examples may be seen in Figure 1. However, the four problems of finding $\lambda(n)$, $\sigma(n)$, $\chi(n)$, and $\mu(n)$ are closely related. For $\sigma(A) \leqslant \lambda(A)$ by Browne's theorem [23, p. 144], so

$$\sigma(n) \leqslant \lambda(n). \tag{2}$$

Also the definitions imply

$$\mu(A) = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \tag{3}$$

[1] Or, as C. L. Mallows has suggested, a Dramadah matrix.

$$[1] \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

(1.1) (2.1) (3.1)

$$\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

(4.1) (5.1)

FIG. 1. Examples of (0,1) anti-Hadamard matrices [with the largest possible value of $\mu(A)$].

and, if A is symmetric,

$$\mu(A) = \sum_{i=1}^{n} \frac{1}{\lambda_i^2}. \tag{4}$$

Therefore

$$\frac{1}{\lambda(n)^2} \leqslant \frac{1}{\sigma(n)^2} \leqslant \mu(n) \leqslant \frac{n}{\sigma(n)^2}. \tag{5}$$

Finally

$$\chi(A)^2 \leqslant \mu(A) \leqslant n^2 \chi(A)^2, \tag{6}$$

implying

$$\chi(n)^2 \leqslant \mu(n) \leqslant n^2 \chi(n)^2. \tag{7}$$

Therefore matrices for which λ and σ are small usually have large values of χ and μ, and conversely. We may say that whereas Hadamard matrices (since they are orthogonal) are a long way from being singular, anti-Hadamard matrices on the other hand are only just nonsingular.

Statement of Results

We shall prove the following results.

THEOREM 1 [The bounds for (0,1) matrices]. *For all n,*

$$\lambda(n), \sigma(n) \geqslant \frac{1}{2n}\left(\frac{4}{n}\right)^{n/2}, \tag{8}$$

$$\chi(n) \leqslant 2\left(\frac{n}{4}\right)^{n/2}, \qquad \mu(n) \leqslant 4n^2\left(\frac{n}{4}\right)^{n}, \tag{9}$$

and there are infinitely many values of n for which

$$\lambda(n), \sigma(n) \leqslant \frac{c\sqrt{n}}{2.274^n}, \tag{10}$$

$$\chi(n) \geqslant c(2.274)^n, \qquad \mu(n) \geqslant c(5.172)^n, \tag{11}$$

where c is a constant (and different occurrences of c in general represent different constants).

THEOREM 2. *For (+1, −1) matrices we have*

$$\lambda_{\pm}(n), \sigma_{\pm}(n) \geqslant \frac{1}{n}\left(\frac{4}{n-1}\right)^{(n-1)/2}, \tag{12}$$

$$\chi_{\pm}(n) \leqslant \left(\frac{n-1}{4}\right)^{(n-1)/2}, \qquad \mu_{\pm}(n) \leqslant n^2\left(\frac{n-1}{4}\right)^{n-1}, \tag{13}$$

while in the other direction (10) *and* (11) *still hold (although with different values of c).*

THEOREM 3. *For* (0,1) *symmetric and Toeplitz matrices the bounds of Theorem* 1 *still hold, except that for Toeplitz matrices the constants* 2.274 *and* 5.172 *in* (10), (11) *should be replaced by* 1.754 *and* 3.079 *respectively.*

THEOREM 4. *For triangular* (0,1) *matrices we have* $\lambda_t(n) = 1$,

$$\chi_t(n) = F_{n-2}, \tag{14}$$

$$\mu_t(n) = F_{n-1}^2 + n + \tfrac{1}{2}[(-1)^n - 1]$$
$$= \frac{1}{5}\left(\frac{3+\sqrt{5}}{2}\right)^n + O(n) = \tfrac{1}{5}(2.618\ldots)^n + O(n), \tag{15}$$

TABLE 1[a]

n	Arbitrary $\mu(n)$	Symmetric $\mu_s(n)$	Toeplitz $\mu_T(n)$	Triangular $\mu_t(n)$	$(+1,-1)$ $\mu_{\pm}(n)$
1	1* (1.1)	1* (1.1)	1* (1.1)	1* (1.1)	1* (1.1)
2	3* (2.1)	3* (2.1)	3* (2.1)	3* (2.1)	1* ϕ(1.1)
3	7* (3.1)	7* (3.1)	7* (3.1)	6* Eq. (35)	$1\frac{1}{2}$* ϕ(2.1)
4	16* (4.1)	16* (4.1)	16* (4.1)	13* Eq. (35)	$2\frac{1}{2}$* ϕ(3.1)
5	46* (5.1)	46* (5.1)	46* (5.1)	29* Eq. (35)	5* ϕ(4.1)
6	146† (6.2)	138* (6.1)	138* (6.1)	70* Eq. (35)	$12\frac{1}{2}$* ϕ(5.1)
7	624† (7.2)	624† (7.2)	601* (7.1)	175* Eq. (35)	38 ϕ(6.2)
8	2955 (8.3)	2646† (8.2)	2619* (8.1)	449* Eq. (35)	158 ϕ(7.2)
9	16,162 (9.1)	16,162 (9.1)	16,162* (9.1)	1164* Eq. (35)	$775\frac{1}{2}$ ϕ(8.3)
10	93,531 (10.1)	93,531 (10.1)	93,531* (10.1)	3035* Eq. (35)	$4291\frac{1}{2}$ ϕ(9.1)
11	654,700 (11.1)	654,700 (11.1)	654,700* (11.1)	7931* Eq. (35)	$23{,}773\frac{1}{2}$ ϕ(10.1)
12	4,442,304 (12.1)	4,442,304 (12.1)	4,442,304† (12.1)	20,748* Eq. (35)	169,250 ϕ(11.1)
13	32,609,366 (13.1)	32,609,366 (13.1)	32,609,366† (13.1)	54,301* Eq. (35)	1.252×10^6 ϕ(12.1)

[a]The first column refers to arbitrary invertible (0,1) matrices, and gives lower bounds on $\mu(n)$, and the names of matrices attaining these bounds. Entries marked with * are exact, and the corresponding matrices are anti-Hadamard matrices. Entries marked with † are believed to be exact. The matrices themselves can be found in Figures 1,2 and Table 2. The remaining columns give lower bounds for (0,1) symmetric, Toeplitz, and triangular matrices, and the last column refers to arbitrary $(+1,-1)$ matrices. The map ϕ is defined in Equation (18).

where $F_0 = F_1 = 1$, $F_2 = 2$, $F_3 = 3, \ldots$ *are the Fibonacci numbers. Matrices attaining* (15) *are given in Equation* (35) *below.*

THEOREM 5. *For circulant* (0,1) *matrices* (8) *and* (9) *still hold, but* (10) *and* (11) *must be replaced by*

$$\lambda_c(n), \sigma_c(n) \leqslant \frac{cn}{2^{n/8}}, \tag{16}$$

$$\chi_c(n) \geqslant \frac{c}{n^2} 2^{n/8}, \qquad \mu_c(n) \geqslant \frac{c}{n^2} 2^{n/4}. \tag{17}$$

For small values of n we have made extensive computer calculations of many of these quantities, and the results are shown in Tables 1–4. Table 1 gives lower bounds (which in some cases are exact) for $\mu(n)$, $\mu_s(n)$, etc., together with the names of matrices attaining the bounds. The names refer to the matrices given in Figures 1, 2 and Table 2. Our notation is that (8.1), (8.2), ..., for example, are particular matrices of order 8, and $T(a_{-(n-1)}, \ldots, a_0, \ldots, a_{n-1})$ is a Toeplitz matrix of order n with (i, j)th entry a_{j-i}. We usually write a_0, the entry on the main diagonal, in boldface for emphasis.

TABLE 2

TOEPLITZ MATRICES WITH LARGEST VALUE OF $\mu(A)$[a]

Order	Name	Matrix
1	(1.1)	T(1)
2	(2.1)	T(110)
3	(3.1)	T(11101)
4	(4.1)	T(1001101)
5	(5.1)	T(110011010)
6	(6.1)	T(01100110100)
7	(7.1)	T(1100011101100)
8	(8.1)	T(101111001101110)
9	(9.1)	T(10010011110001001)
10	(10.1)	T(0101110011110100010)
11	(11.1)	T(100110010111100110011)
12	(12.1)	T(11000100101111100010010)
13	(13.1)	T(1100011101001001110101001)

[a]See Table 1. The first five matrices are written in full in Figure 1.

TABLE 3
VALUES OF $\mu_c(n)$, $\lambda_c(n)$, $\chi_c(n)$
FOR $n \times n$ INVERTIBLE
(0, 1) CIRCULANT MATRICES[a]

n	$\mu_c(n)$	$\lambda_c(n)$	$\chi_c(n)$
1	1*	1*	1*
	I_1	I_1	I_1
2	2*	1*	1*
	I_2	I_2	I_2
3	3*	1*	1*
	I_3	I_3	I_3
4	4*	1*	1*
	I_4	I_4	I_4
5	6.25*	0.6180*	1*
	(5.2)	(5.2)	I_5
6	6*	1*	1*
	I_6	I_6	I_6
7	12.25*	0.4450*	I*
	(7.3)	(7.3)	I_7
8	15.111*	0.4142*	1*
	(8.4)	(8.4)	I_8
9	20.25*	0.3473*	1*
	(9.2)	(9.2)	I_9
10	21.111*	0.3820*	1*
	(10.2)	(10.2)	I_{10}
11	260.04*	0.08816*	2.2*
	(11.2)	(11.2)	(11.2)
12	35.04*	0.2679*	1*
	(12.2)	(12.2)	I_{12}
13	412.03*	0.07010*	2.167*
	(13.2)	(13.2)	(13.2)
14	173.04*	0.1099*	1.8*
	(14.1)	(14.1)	(14.1)
15	267.97*	0.08693*	1.641*
	(15.1)	(15.1)	(15.1)
16	302.15*	0.08239*	1.867*
	(16.1)	(16.1)	(16.1)

TABLE 3 (Continued)

n	$\mu_c(n)$	$\lambda_c(n)$	$\chi_c(n)$
17	4660.04*	0.02073*	5.6*
	(17.1)	(17.1)	(17.1)
18	160.31*	0.1206*	1.4*
	(18.2)	(18.1)	(18.1)
19	7878.02*	0.01594*	6.583*
	(19.1)	(19.1)	(19.2)
23	4.51×10^5	0.002106*	41.125
	(23.1)	(23.1)	(23.1)
29	5.77×10^6	0.0005888*	117.309
	(29.1)	(29.1)	(29.1)

[a]Conventions as in Table 1. The matrices themselves are given in Table 4.

TABLE 4

CIRCULANT MATRICES WITH LARGEST VALUES OF $\mu(A)$, $\chi(A)$ OR SMALLEST VALUE OF $\lambda(A)$[a]

Order	Name	Matrix	|det|
5	(5.2)	C(11000)	2
7	(7.3)	C(1100000)	2
8	(8.4)	C(11100000)	3
9	(9.2)	C(110000000)	2
10	(10.2)	C(1110000000)	3
11	(11.2)	C(10111010000)	5
12	(12.2)	C(111110000000)	5
13	(13.2)	C(1011110100000)	6
14	(14.1)	C(11001001100000)	5
15	(15.1)	C(110010010101000)	198
16	(16.1)	C(1011101000000000)	15
17	(17.1)	C(11010110000000000)	5
18	(18.1)	C(101000010010010000)	80
18	(18.2)	C(111000000010000000)	8
19	(19.1)	C(1101110110000000000)	7
19	(19.2)	C(0010001001111111111)	12
23	(23.1)	C(00100110000110000110010)	8
29	(29.1)	C(10011100001010110101000011100)	13

[a]See Table 3.

$$\begin{bmatrix} 1&1&0&0&0&0\\ 1&0&1&1&0&0\\ 0&1&1&0&1&0\\ 0&1&0&0&1&1\\ 0&1&1&1&0&1\\ 1&0&0&1&1&1 \end{bmatrix} \qquad \begin{bmatrix} 1&1&1&1&0&0&0\\ 1&1&0&1&0&1&1\\ 1&0&0&0&1&0&1\\ 1&1&0&0&1&0&0\\ 0&0&1&1&1&1&0\\ 0&1&0&0&1&1&0\\ 0&1&1&0&0&0&1 \end{bmatrix}$$

(6.2) (7.2)

$$\begin{bmatrix} 0&1&1&1&1&0&0&0\\ 1&0&1&1&0&1&1&0\\ 1&1&1&0&0&1&0&1\\ 1&1&0&1&1&0&0&1\\ 1&0&0&1&1&1&1&1\\ 0&1&1&0&1&0&1&1\\ 0&1&0&0&1&1&0&0\\ 0&0&1&1&1&1&0&1 \end{bmatrix} \qquad \begin{bmatrix} 0&0&0&1&1&0&0&1\\ 0&1&1&1&0&0&1&0\\ 0&0&1&0&1&1&1&0\\ 1&1&0&0&1&0&1&0\\ 0&1&0&0&0&1&1&1\\ 1&0&1&1&0&1&0&1\\ 1&0&0&1&0&1&1&1\\ 1&1&1&0&1&1&0&1 \end{bmatrix}$$

(8.2) (8.3)

FIG. 2. Some record-holders (see Table 1). (7.2) and (8.2) are symmetric, but the other two are not.

The final column of Table 1 is concerned with $(+1, -1)$ matrices. There is a standard mapping from $(n-1)\times(n-1)$ $(0,1)$ matrices A to $n\times n$ $(+1, -1)$ matrices $\phi(A)$ with first row and column consisting of $+1$'s, given by

$$\phi(A)=\begin{bmatrix} 1 & \mathbf{1} \\ \mathbf{1}^{\text{tr}} & J_{n-1}-2A \end{bmatrix}, \tag{18}$$

where $\mathbf{1}=(1,1,\ldots,1)$ [6]. S-matrices and Hadamard matrices are related in precisely this manner, as we mentioned at the beginning of this section. The mapping is invertible,

$$|\det\phi(A)|=2^{n-1}|\det A|, \tag{19}$$

and, if A^{-1} exists,

$$\phi(A)^{-1}=\begin{bmatrix} 1-\frac{1}{2}\mathbf{1}A^{-1}\mathbf{1}^{\text{tr}} & \frac{1}{2}\mathbf{1}A^{-1} \\ \frac{1}{2}A^{-1}\mathbf{1}^{\text{tr}} & -\frac{1}{2}A^{-1} \end{bmatrix}. \tag{20}$$

It seems likely (although we have not been able to give a proof) that

$$A \text{ is a } (0,1) \text{ anti-Hadamard matrix} \quad \text{if and only if}$$
$$\phi(A) \text{ is a } (+1,\,-1) \text{ anti-Hadamard matrix.} \qquad (21)$$

Certainly the best $(+1,\,-1)$ matrices we have found are obtained by applying ϕ to the best (0,1) matrices, as shown in the final column of Table 1. The table also suggests that

$$\mu_{\pm}(n) \approx \tfrac{1}{4}\mu(n-1),$$

an approximation that is explained by Equation (20). [For Eq. (20) implies that the sum of the squares of the entries of $\phi(A)^{-1}$ is equal to one quarter of the sum of the squares of the entries of A^{-1}, plus a presumably smaller contribution from the first row and column of $\phi(A)^{-1}$.]

The results for circulant matrices are given in Tables 3 and 4, using the notation that a circulant matrix with first row $(a_0, a_1, \ldots, a_{n-1})$, and (i, j)th entry a_{j-1}, the subscript being read modulo n, is denoted by $C(a_0, a_1, \ldots, a_{n-1})$.

Remarks

(1) The bounds given in Theorem 1 unfortunately do not determine the rate of growth of $\mu(n)$, although the data in Table 1 suggest that even for Toeplitz matrices $\mu(n)$ grows faster than c^n.

(2) Anti-Hadamard matrices of order $n \leqslant 5$ can be put into Toeplitz form (Figure 1 and Table 1), but not for $n = 6, 7$ or (probably) any larger value of n. Indeed, the presumed anti-Hadamard matrices (6.2) and (7.2) have no apparent structure. At order 8 we have

$$\mu(n) > \mu_s(n) > \mu_T(n) > \mu_t(n) > \mu_c(n) \qquad (n = 8).$$

These strict inequalities almost certainly hold for all larger n, although this cannot be seen from Table 1, since for $n \geqslant 9$ we restricted our computer search to Toeplitz matrices. Toeplitz matrices have the advantage that $\mu_T(n)$ appears to grow very rapidly, and besides have been extensively studied [3, 4, 13, 14, 19, 20, 26, 36]. Unfortunately the best *infinite* sequence of Toeplitz matrices we have been able to construct [see Theorem 3 and Equation (33)] only has $\mu(A) \approx c(3.0796)^n$. The reader is invited to try and continue the sequence of matrices begun in Table 2.

(3) To make $\mu(A)$ large, we must make $|\det A|$ small and the cofactors A_{ij} large. In fact the matrices with the largest values of $\mu(A)$, $\mu_s(A)$, and $\mu_T(A)$ always seem to have determinant ± 1, although again we are unable to prove this. Any process that generates random-looking matrices with determinant ± 1 should make $\mu(A)$ large.

(4) Finally, we note the following useful identities: $\mu(A) = \mu(A^{\text{tr}}) = \text{Trace}(AA^T)^{-1} = \mu(AU) = \mu(UA)$, where U is any orthogonal matrix.

In the following section we describe a number of applications, and then in Sections III–VII give the proofs of Theorems 1–5. The paper concludes with a list of open problems.

II. APPLICATIONS

Ill-Conditioned Matrices

Our results can be applied directly to discover how ill conditioned a (0, 1) matrix of order n can be. The M- and N-condition numbers of a (0, 1) matrix A are (in the notation of Section I)

$$M(A) = n \max|a_{ij}| \max|b_{ij}| = n\chi(A)$$

and

$$N(A) = \frac{1}{n}\left(\sum a_{ij}^2\right)^{1/2}\left(\sum b_{ij}^2\right)^{1/2} = \frac{1}{n}\left(\sum a_{ij}^2\right)^{1/2}\mu(A)^{1/2}$$

(see [22], [31], [32]). Then Theorem 1 implies that the largest M-condition number of a (0, 1) matrix lies in the range

$$cn(2.274)^n \leqslant M(A) \leqslant 2n\left(\frac{n}{4}\right)^{n/2}, \tag{22}$$

with a similar result for the N-condition number. Although ill-conditioned matrices have been studied by many authors [5, 12, 37], these results appear to be new.

Weighing Designs and Spectroscopy

If an invertible (0, 1) matrix A is used as a weighing design (for weighing small objects, or in spectroscopy), then under suitable conditions the mean squared error in the measurements is reduced by a factor of $n/\mu(A)$ [1, 16, 25, 27–29]. If A is chosen to be an S-matrix, this is a reduction by about $n/4$ [using (1)], a substantial improvement. On the other hand, Theorem 1 shows that ill-chosen weighing designs can greatly *increase* the errors, and Theorem 4 shows that circulant matrices (which are the ones used in practice) can also be bad.

The Hyperplane Problem

The problem of maximizing $\chi(A)$ or $\mu(A)$ is related to the following simple geometrical question. Consider the cube in n-dimensional Euclidean space whose vertices are all 2^n vectors of 0's and 1's. Take any n vertices of this cube, and consider the hyperplane H passing through them. The problem is to determine how close H can be to the origin (with the optimal choice of the original n vertices), without actually passing through the origin.

Let the answer be $\delta(n)$, and let the n vertices be $(a_{i1},\ldots,a_{in})$, $1 \leqslant i \leqslant n$. The equation to the hyperplane is

$$\det\begin{bmatrix} a_{11} & \cdots & a_{1n} & 1 \\ a_{21} & \cdots & a_{2n} & 1 \\ & \cdots & & \\ a_{n1} & \cdots & a_{nn} & 1 \\ x_1 & \cdots & x_n & 1 \end{bmatrix} = 0$$

(since this vanishes at the n vertices). When expanded this reads

$$c_1x_1 + \cdots + c_nx_n + x_0 = 0,$$

where

$$c_0 = \det A,$$

$$c_j = \sum_{i=1}^{n} A_{ij} \qquad (1 \leqslant j \leqslant n),$$

where $A = (a_{ij})$ and A_{ij} is the (i, j)th cofactor of A. The distance from this hyperplane to the origin is [2, p. 36]

$$\frac{|\det A|}{\sqrt{c_1^2 + \cdots + c_n^2}} = \left\{\left(\sum_{i=1}^{n} b_{1i}\right)^2 + \cdots + \left(\sum_{i=1}^{n} b_{ni}\right)^2\right\}^{-1/2}, \qquad (23)$$

where $B=(b_{ij})=A^{-1}$. Therefore H does not pass through $\mathbf{0}$ provided that $\det A \neq 0$. To find $\delta(n)$ we must choose an invertible (0,1) matrix to maximize (23). From Theorem 1 we have

$$\delta(n) \geqslant \frac{1}{n^{3/2}\chi(n)} \geqslant \frac{1}{2n^{3/2}}\left(\frac{4}{n}\right)^{n/2},$$

while the triangular matrices t_n given in Section VI imply $\delta(n) \leqslant c\,1.618^{-n}$.

Sums of Roots of Unity

Since the eigenvalues of a circulant matrix with first row $(a_0, a_1, \ldots, a_{n-1})$ are the numbers $\sum_{j=0}^{n-1} a_j \omega^{jk}$, $k=0,1,\ldots,n-1$, where $\omega = e^{2\pi i/n}$ [8], the problem of finding $\chi_c(n)$ can be restated as follows: What is the smallest magnitude of any nonvanishing sum of distinct nth roots of unity? Theorem 5 gives the best bounds we have been able to obtain.

Other Applications

Finally, $\chi_T(n)$ and $\mu_T(n)$ (referring to Toeplitz matrices) are relevant for studying Padé approximations and the Euclidean algorithm, via the connections between these problems and the solution of Toeplitz equations [4, 20].

III. UPPER BOUNDS

Let A be any $n \times n$ invertible (0,1) matrix, with inverse $B=(b_{ij})$. Then $|\det A| \geqslant 1$, and $|b_{ij}| \leqslant f(n-1)$, where $f(n)$ is the greatest determinant of any $n \times n$ (0,1) matrix. From Hadamard's inequality [6, 22, 23], $f(n) \leqslant 2^{-n}(n+1)^{(n+1)/2}$, which implies $\chi(n) = \max|b_{ij}| \leqslant 2(n/4)^{n/2}$. The rest of (8), (9) now follow from (5), (7). Of course the same bounds also apply to symmetric, Toeplitz, and circulant matrices.

If A is an $n \times n$ $(+1,-1)$ matrix, then $|\det A| \geqslant 2^{n-1}$ by (19), and $|b_{ij}| \leqslant 2^{-(n-1)} g(n-1)$, where $g(n)$ is the greatest determinant of any $n \times n$ $(+1,-1)$ matrix. From Hadamard's inequality, $g(n) \leqslant n^{n/2}$, which implies $\chi_{\pm}(n) = \max|b_{ij}| \leqslant [(n-1)/4]^{(n-1)/2}$. The rest of (10), (11) follow from (5), (7).

IV. AN ITERATIVE CONSTRUCTION FOR SYMMETRIC MATRICES

We will prove the second half of Theorem 1 by constructing an infinite sequence $A_0, A_1, A_2, \ldots$ of symmetric (0,1) matrices whose inverses contain large entries. We must first introduce the notion of a well-signed matrix X and its associated (0,1) matrix $P(X)$. A real $n \times n$ matrix $X = (x_{ij})$ is said to be *well signed* if the componentwise product of any two rows does not contain both positive and negative entries. Stated informally, each row either has the same signs as the first row, or the opposite signs. For examples see (29), (34). If the entries of X are nonzero, we define a (0,1) matrix $P(X)$ by $P(X)_{ij} = 1$ if $x_{ij} > 0$, $= 0$ if $x_{ij} < 0$.

However, if X contains zero entries, the definition of $P(X)$ is more subtle. Let $X^* = (\mathrm{sgn}(x_{ij}))$, where

$$\mathrm{sgn}(x) = \begin{cases} 1 & \text{if} \quad x > 0, \\ 0 & \text{if} \quad x = 0, \\ -1 & \text{if} \quad x < 0. \end{cases}$$

It is not difficult to see that if X is well-signed and symmetric, it is possible to find (in perhaps more than one way) a vector $\mathbf{t} = (t_1, t_2, \ldots, t_n)$ with each $t_i \in \{+1, -1\}$ and a number $a \in \{+1, -1\}$ such that X^* can be obtained by forming the $(+1, -1)$ matrix $a\mathbf{t}^{\mathrm{tr}}\mathbf{t}$ and replacing some entries by zero. To get $P(X)$ we replace every -1 in $a\mathbf{t}^{\mathrm{tr}}\mathbf{t}$ by 0. [For an example see (29), (30).]

LEMMA 6. *If X is symmetric and well signed, then so is $Y = XP(X)X$, and $P(Y) = P(X)$. In fact*

$$Y_{ij} = \begin{cases} \pi_i\pi_j + \nu_i\nu_j & \text{if} \quad P(X)_{ij} = 1, \\ \pi_i\nu_j + \nu_i\pi_j & \text{if} \quad P(X)_{ij} = 0, \end{cases} \tag{24}$$

where π_i is the sum of the positive entries in the ith row of X, and ν_i is the sum of the negative entries.

The straightforward proof is omitted. We also record the fact that $XP(X)$ has rank $\leqslant 2$, and trace equal to $\pi_1 + \cdots + \pi_n$, the sum of all the positive entries in X. Therefore the eigenvalues of $XP(X)$ are $\lambda_1, \lambda_2, 0, \ldots, 0$, with $\lambda_1 \geqslant |\lambda_2| \geqslant 0$.

We can now state the construction. Let A_0 be an $n_0 \times n_0$ symmetric (0,1) matrix with determinant ± 1 and possessing a well-signed inverse. Then define

$$A_k = \begin{bmatrix} 0 & A_{k-1} \\ A_{k-1} & P(A_{k-1}^{-1}) \end{bmatrix}, \qquad k = 1, 2, \ldots. \tag{25}$$

Clearly A_k has order $2^k n_0$, determinant ± 1, and inverse

$$A_k^{-1} = \begin{bmatrix} -A_{k-1}^{-1}P(A_{k-1}^{-1})A_{k-1}^{-1} & A_{k-1}^{-1} \\ A_{k-1}^{-1} & 0 \end{bmatrix}. \tag{26}$$

The reason the construction works is that in the product $-A_{k-1}^{-1}P(A_{k-1}^{-1})A_{k-1}^{-1}$, the middle term $P(A_{k-1}^{-1})$ matches up the positive and negative entries of A_{k-1}^{-1} [using (24)] so as to maximize the entries of A_k^{-1}. The following theorem gives bounds on the entries of A_k^{-1}.

THEOREM 7. *Assume $A_0^{-1}P(A_0^{-1})$ has two distinct nonzero eigenvalues λ_1, λ_2 with $\lambda_1 > |\lambda_2|$. Then $\chi(A_k)$ (the greatest entry of A_k^{-1}) lies in the range*

$$c\rho_1^n \leqslant \chi(A_k) \leqslant c\rho_2^n, \tag{27}$$

where $n = 2^k n_0$ is the order of A_k, $\rho_1 = \lambda_1^{1/n_0}$, and $\rho_2 = [4n_0^2\chi(A_0)]^{1/n_0}$.

Proof. For the lower bound we define matrices $D_k = (d_{ij}^{(k)})$ of order n_0 by

$$D_0 = A_0^{-1}, \qquad D_k = D_{k-1}PD_{k-1}, \tag{28}$$

where $P = P(A_0^{-1})$. By Lemma 6 all the D_k are symmetric and well signed, and $P(D_k) = P$. The entries of D_k are built up from the entries of D_{k-1} in the same way that A_k^{-1} is formed from A_{k-1}^{-1}, except that there is less buildup in D_k. In particular, for every k there is an $n_0 \times n_0$ submatrix (s_{ij}) of A_k^{-1} with $|s_{ij}| \geqslant |d_{ij}^{(k)}|$ for all i, j. The solution of (28) is

$$D_k = (D_0P)^{2^k-1}D_0,$$

and, writing $D_0P = U\Delta U^{-1}$ where $\Delta = \operatorname{diag}(\lambda_1, \lambda_2, 0, \ldots, 0)$, we obtain

$$d_{11}^{(k)} = c_1\lambda_1^{2^k-1} + c_2\lambda_2^{2^k-1},$$

which implies the left-hand side of (27).

For an upper bound, by definition the entries of A_0^{-1} do not exceed $m_0 := \chi(A_0)$. Replacing A_0^{-1} by m_0J in (26), we find that the entries of A_1^{-1} do not exceed $m_1 = n_0^2m_0^2$, and, continuing in this way, that the entries of A_k^{-1}

do not exceed $m_k = (2^{k-1}n_0)^2 m_{k-1}^2$. Solving for m_k, we find

$$m_k = \frac{\left(4n_0^2 m_0\right)^{2^k}}{4^{k+1}n_0^2},$$

which establishes the right-hand side of (27) and completes the proof of Theorem 7. ∎

EXAMPLES. By reflecting a Toeplitz matrix in a horizontal mirror we obtain a symmetric matrix, known as a Hankel or orthosymmetric [26] matrix. For the first example we take as A_0 the Hankel matrix corresponding to (4.1):

$$A_0 = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}, \quad A_0^{-1} = \begin{bmatrix} 2 & -1 & 1 & -1 \\ -1 & 0 & 0 & 1 \\ 1 & 0 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}. \tag{29}$$

A_0^{-1} is well signed and symmetric, and we may take $\mathbf{t} = (1, -1, 1, -1)$, $a = 1$, so that

$$a\mathbf{t}^{\text{tr}}\mathbf{t} = \begin{bmatrix} 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ -1 & 1 & -1 & 1 \end{bmatrix}, \quad P\left(A_0^{-1}\right) = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{30}$$

Then

$$A_0^{-1}P\left(A_0^{-1}\right) = \begin{bmatrix} 3 & -2 & 3 & -2 \\ -1 & 1 & -1 & 1 \\ 2 & -1 & 2 & -1 \\ -2 & 2 & -2 & 2 \end{bmatrix},$$

a rank-2 matrix with nonzero eigenvalues $\lambda = 7.162\ldots$, $\lambda_2 = 0.838\ldots$, and the construction produces a sequence of matrices A_k with

$$c(1.63)^n \leqslant \chi(A_k) \leqslant c(3.37)^n.$$

Although in general (0,1) Toeplitz or Hankel matrices do not have well-signed inverses, Toeplitz matrices for which $\mu(A)$ is large (those in Table 2 for example) do seem to have this property. If the corresponding Hankel matrix is used as A_0 in the construction (25), we find that $\lambda_1 \gg \lambda_2$, and therefore λ_1 is approximately equal to the sum of the positive entries of A_0^{-1}.

A computer search based on this observation produced the following Toeplitz matrix of order 17:

$$T(111111010011000111111010010000000). \qquad (31)$$

We take A_0 to be the Hankel matrix corresponding to (31), and find $\lambda_1 = 1.165\ldots \times 10^6$, $\lambda_2 = -2.90\ldots$. Then A_k has order $n = 17 \times 2^k$, and

$$c(2.274)^n \leqslant \chi(A_k) \leqslant c(2.890)^n, \qquad (32)$$

which establishes the first assertion of (11). The rest of (10) and (11) follow from (2), (4), (7), completing the proof of Theorem 1. To get the lower bounds in Theorem 2 we use the symmetric $(+1, -1)$ matrices $\phi(A_k)$.

V. TOEPLITZ MATRICES

In this section we exhibit a sequence $\{T_n\}$ of Toeplitz matrices whose inverses contain large entries. T_n is the following Toeplitz matrix of order n:

$$T_n = T(\ldots 11001100\mathbf{1}10100000\ldots), \qquad (33)$$

where the bold 1 indicates the entry on the main diagonal. For example

$$T_8 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \end{bmatrix},$$

$$T_8^{-1} = \begin{bmatrix} -1 & -2 & -3 & -5 & -9 & 12 & -7 & 16 \\ 1 & 1 & 1 & 2 & 4 & -5 & 3 & -7 \\ -1 & -1 & -2 & -4 & -7 & 9 & -5 & 12 \\ 1 & 1 & 2 & 3 & 5 & -7 & 4 & -9 \\ 0 & 1 & 1 & 2 & 3 & -4 & 2 & -5 \\ 0 & 0 & 1 & 1 & 2 & -2 & 1 & -3 \\ 0 & 0 & 0 & 1 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & -1 \end{bmatrix}. \qquad (34)$$

In order to describe T_n^{-1} we define two sequences of integers $\{p_n\}, \{q_n\}$ by

n	0	1	2	3	4	5	6	7	8	9	10	...
p_n	1	1	2	3	5	9	16	28	49	86	151	...
q_n	0	1	1	2	4	7	12	21	37	65	114	...

and

$$p_n = p_{n-1} + q_{n-1} \qquad (n \geqslant 1),$$

$$q_n = q_{n-1} + p_{n-2} \qquad (n \geqslant 3).$$

Therefore

$$p_n = 2p_{n-1} - p_{n-2} + p_{n-3} \qquad (n \geqslant 4),$$

$$q_n = 2q_{n-1} - q_{n-2} + q_{n-3} \qquad (n \geqslant 4),$$

and so

$$p_n = c\rho_3^n + o(1), \qquad q_n = c'\rho_3^n + o(1),$$

where $\rho_3 = 1.75488\ldots$ is the largest zero of $x^3 - 2x^2 + x - 1$.

THEOREM 8. *$\det T_n = \pm 1$, T_n^{-1} is as shown in Figure 3, $\chi(T_n) = p_{n-2}$, and $\mu(T_n) > c\rho_3^{2n}$.*

Proof. The determinant and inverse can be calculated by Trench's algorithm [33, 34] for inverting a Toeplitz matrix. Using Zohar's description

$$\left[\begin{array}{cccccc|ccc}
-1 & -p_2 & -p_3 & \cdots & -p_{n-4} & -p_{n-3} & q_{n-2} & -q_{n-1} & p_{n-2} \\
1 & q_1 & q_2 & \cdots & q_{n-5} & q_{n-4} & -p_{n-4} & p_{n-5} & -q_{n-1} \\
-1 & -q_2 & -q_3 & \cdots & -q_{n-4} & -q_{n-3} & p_{n-3} & -p_{n-4} & q_{n-2} \\
\hline
1 & p_1 & p_2 & \cdots & p_{n-3} & p_{n-4} & -q_{n-3} & q_{n-4} & -p_{n-3} \\
0 & 1 & p_1 & \cdots & p_{n-6} & \cdots & & & \\
0 & 0 & 1 & \cdots & & & & & \\
\cdots & & & & & & & &
\end{array}\right]$$

FIG. 3. The matrix T_n^{-1}. This is persymmetric, i.e. is symmetric about the diagonal extending from the top right-hand corner to the bottom left-hand corner.

[38] of this algorithm one finds (in his notation)

$$\eta_i = p_i \text{ and } \gamma_i = 0 \qquad (i \geq 3),$$

$$\lambda_1 = \lambda_2 = 1, \qquad \lambda_i = -1 \quad (i \geq 3),$$

so $\det T_n = \prod \lambda_i = \pm 1$, and, for $i \geq 4$,

$$e_i = [p_2, p_3, \ldots, p_{i-2}, -q_{i-1}, q_{i-2}, -p_{i-1}]^{\text{tr}},$$

$$g_i = [-1, 1, 1, 0, \ldots, 0]^{\text{tr}}.$$

The inverse matrix can now be easily obtained. The largest entry is in the top right-hand corner, and $\mu(T_n) = p_{n-2}^2 + 2q_{n-2}^2 + \cdots > p_{n-2}^2 = c''\rho_3^{2n}$. This completes the proof of Theorems 3 and 8. ■

REMARK. One can show that any infinite sequence of Toeplitz matrices in which there are only a fixed number of nonzero rows above the main diagonal [such as (33)] satisfies $\mu(A) \leq \rho^n$, for some constant ρ.

VI. TRIANGULAR MATRICES

Consider the $n \times n$ triangular matrix

$$t_n = T(0\ldots01101010\ldots), \tag{35}$$

having inverse

$$t_n^{-1} = T(0, \ldots, 0, 1, -F_0, F_1, -F_2, \ldots, \pm F_{n-2}). \tag{36}$$

For example

$$t_6 = \begin{bmatrix} 1 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

$$t_6^{-1} = \begin{bmatrix} 1 & -1 & 1 & -2 & 3 & -5 \\ 0 & 1 & -1 & 1 & -2 & 3 \\ 0 & 0 & 1 & -1 & 1 & -2 \\ 0 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

Then $\chi(t_n) = F_{n-2}$, and $\mu(t_n) = F_{n-2}^2 + 2F_{n-3}^2 + \cdots + n$, which with the help of the identity [7]

$$F_0^2 + F_1^2 + \cdots + F_n^2 = F_n F_{n+1}$$

may be simplified to

$$\mu(t_n) = F_{n-1}^2 + n + \tfrac{1}{2}[(-1)^n - 1]. \tag{37}$$

We shall show that t_n is optimal in the sense that $\chi_t(n) = \chi(t_n)$ and $\mu_t(n) = \mu(t_n)$ for all n, thereby establishing Theorem 4. At the same time we shall prove the following result about (0,1) determinants.

LEMMA 9. *Let $h(n)$ be the greatest determinant of any $n \times n$ (0,1) matrix $A = (a_{ij})$ in which all entries above one row beyond the main diagonal are zero (thus $a_{ij} = 0$ if $j - i \geqslant 2$). Then $h(1) = 1$ and $h(n) = F_{n-2}$ for $n = 2, 3, \ldots$.*

We also set $h(0) = 0$. Clearly $h(n)$ is monotone:

$$h(n) \leqslant h(n+1). \tag{38}$$

Let A be any upper triangular invertible $n \times n$ (0,1) matrix, with inverse $B = (b_{ij})$. By examining the cofactors of A we find that

$$|b_{ij}| \leqslant h(j-i), \tag{39}$$

and therefore

$$\mu(A) \leqslant h(n-1)^2 + 2h(n-2)^2 + \cdots + nh(0)^2. \tag{40}$$

Now suppose A is such that $\chi(A) = \chi_t(n)$. By (38) and (39) we may assume b_{1n} is the greatest entry in A^{-1}. (38) and (39) also imply

$$\chi_t(n) = h(n-1) \qquad \text{for all } n. \tag{41}$$

$$
\text{(a)}\quad \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 & v_1 \\ & 1 & 0 & \cdots & 0 & v_2 \\ & & 1 & \cdots & 0 & v_3 \\ & & & \cdots & & \cdots \\ 0 & & & & 1 & v_{n-1} \\ & & & & & 1 \end{bmatrix}
$$

$$
\text{(b)}\quad A^{-1} = \begin{bmatrix} 1 & * & * & \cdots & * & -v_1 \\ & 1 & * & \cdots & * & -v_2 \\ & & 1 & \cdots & * & -v_3 \\ & & & \cdots & & \cdots \\ 0 & & & & 1 & -v_{n-1} \\ & & & & & 1 \end{bmatrix}
$$

FIG. 4. (a) Using row operations we put A into this form. (b) Then A^{-1} is as shown.

b_{1n} may be found by the following procedure. We subtract certain of rows $2,\ldots,n-1$ from row 1, then certain of rows $3,\ldots,n-1$ from row 2, and so on until A has been transformed into the form shown in Figure 4(a). If the last column of this matrix is $(v_1, v_2,\ldots,v_{n-1},1)^{\text{tr}}$, then $(-v_1, -v_2,\ldots, -v_{n-1}, 1)$ is orthogonal to the rows of A, and is in fact the last column of A^{-1} [Figure 4(b)]. In particular $b_{1n} = -v_1$.

It is not difficult to keep track of how the top right-hand entry of A changes during the subtraction process. At the rth step rows $2,\ldots,r+1$ have been used to make the top row equal to

$$(1,0,\ldots,0,x_r,\ldots)$$

with r initial 0's, and one can show by induction that either $x_r \in [-F_{r-1}, F_r]$ or $x_r \in [-F_r, F_{r-1}]$. Therefore $|x_r| \leqslant F_r$, and when the procedure terminates $|x_{n-2}| = |v_1| = |b_{1n}| \leqslant F_{n-2}$. Since the matrix t_n attains this bound, we have proved (14), and [by (41)] Lemma 9. Finally (15) follows from (40), completing the proof of Theorem 4.

VII. CIRCULANT MATRICES

We have been less successful in constructing circulant matrices and are only able to establish the lower bounds of Theorem 5 by an existence argument.

We shall show that if $n = 2p$, p prime, there is a circulant of order n with an eigenvalue of magnitude less than $cn2^{-n/8}$. Thus $\lambda_c(n) \leqslant cn2^{-n/8}$, and then the rest of (16), (17) follow from (2), (5), (7). Let $\theta = 2\pi/p$, $\omega = e^{i\theta}$. Since p is a prime, the numbers $1, \omega, \omega^2, \ldots, \omega^{p-2}$ are independent over the integers. Consider the 2^{r+1} sums

$$a_0 + 2a_1 \cos\theta + 2a_2 \cos 2\theta + \cdots + 2a_r \cos r\theta,$$

where $r = [p/4]$ and each $a_i \in \{0, 1\}$. These sums are distinct and lie in the range $[0, \frac{1}{2}p + 1]$. By the pigeonhole principle, there is a pair of such sums X, Y (say) with

$$|X - Y| \leqslant \frac{\frac{1}{2}p + 1}{2^{r+1} - 1} \leqslant \frac{cn}{2^{n/8}}.$$

Since -1 is an nth root of unity, $X - Y$ is (after canceling common terms) a nonzero sum of distinct nth roots of unity, as required.

The following is an explicit construction for a circulant with a nonzero eigenvalue of magnitude

$$O\left(n^{-\frac{1}{2}\log_2 n}\right) \tag{42}$$

(which of course is not as good as the circulants guaranteed by the preceding argument). Again we take $n = 2p$, p prime, and let $\psi = 2\pi/n$. Also let $w(k)$ denote the number of 1's in the binary expansion of k. Then it is an amusing exercise to show that the sum

$$\sum_{k=1}^{[n/4]} (-1)^{w(k-1)} \cos k\phi$$

is bounded above by (42).

VIII. OPEN PROBLEMS

(1) Improve the bounds of Theorems 1–5. In particular, is the true rate of growth of $\mu(n)$ equal to $O(c^n)$ or $O(n^n)$?

(2) Is it true that $(+1, -1)$ anti-Hadamard matrices are given by (21)?

(3) Show that a matrix attaining any of $\mu(n)$, $\mu_s(n)$, or $\mu_T(n)$ has determinant ± 1. Must it have a well-signed inverse?

(4) Find other infinite sequences of (0, 1) Toeplitz matrices with determinant ± 1 [besides (33)].

(5) If an invertible (0, 1) matrix A is chosen at random, what is the expected value of $\mu(A)$?

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