On Dissecting Polygons into Rectangles

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Abstract

What is the smallest number of pieces that you can cut an n-sided regular polygon into so that the pieces can be rearranged to form a rectangle? Call it $r(n)$. The rectangle may have any proportions you wish, as long as it is a rectangle. The rules are the same as for the classical problem where the rearranged pieces must form a square. Let $s(n)$ denote the minimum number of pieces for that problem. For both problems the pieces may be turned over and the cuts must be simple curves. The conjectured values of $s(n), 3 \le n \le 12$, are $4, 1, 6, 5, 7, 5, 9, 7, 10, 6$. However, only $s(4) = 1$ is known for certain. The problem of finding $r(n)$ has received less attention. In this paper we give constructions showing that $r(n)$ for $3 \le n \le 12$ is at most $2, 1, 4, 3, 5, 4, 7, 4, 9, 5$, improving on the bounds for $s(n)$ in every case except $n = 4$. For the 10-gon our construction uses three fewer pieces than the bound for $s(10)$. Only $r(3)$ and $r(4)$ are known for certain.

1 Introduction

Two polygons are said to be *equidecomposable* if one can be cut into a finite number of pieces that can be rearranged to form the other. Pieces may be turned over, and the cuts must be along simple plane curves. The Bolyai-Gerwien theorem from the 1830s states that any two polygons of the same area are equidecomposable, and the dissection can be carried out using only triangular pieces. Furthermore, the dissection can be carried out using only a straightedge and compass. Boltianskii [3] gives an excellent survey.

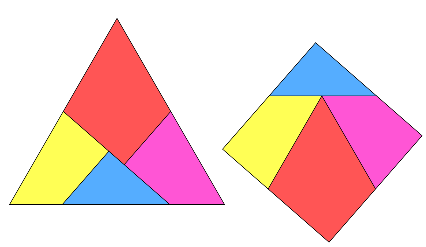

Figure 1: A 4-piece dissection of an equilateral triangle into a square.

A much-studied special case of this problem asks for the minimum number of pieces ($s(n)$, say) of any shape needed to dissect a regular polygon with n sides into a square of the same area. Despite its long history [2, 6, 8, 9, 10, 11, 12, 15, 16, 19, 21], surprisingly little is known about this problem. The best upper bounds currently known for $s(n), n = 3, 4, 5, \dots, 16$, are shown in Table 1. The only exact value known appears to be the trivial result that $s(4) = 1$. Even the value of $s(3)$ is not known. The four-piece dissection of an equilateral triangle into a square shown in Fig. 1 is at least 120 years old (see the discussions in [2], [8, Ch. 12]), but there is no proof that it is optimal. The conjecture that it is impossible to dissect an equilateral triangle into three pieces that can be rearranged to form a square must be one of the oldest unsolved problems in geometry.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n) \le$	4	1	6	5	7	5	9	7	10	6	11	9	11	10
$r(n) \le$	2	1	4	3	5	4	7	4	9	5	10	7	10	9

Table 1: $s(n)$ (resp. $r(n)$) is the minimum number of pieces needed to dissect a regular n-sided polygon into a square (resp. rectangle). Only the values of $s(4)$, $r(3)$, and $r(4)$ are known to be exact.

In the present article we consider a weaker constraint: what is the minimum number of pieces ($r(n)$, say) needed to dissect a regular polygon with n sides into a *rectangle* of the same area? The proportions of the rectangle can be anything you want. It is clear that $r(3) = 2$ and $r(6) \le 3$ (Figs. 2, 3). (Surely it should be possible to prove that no two-piece dissection of a regular hexagon into a rectangle is possible?)

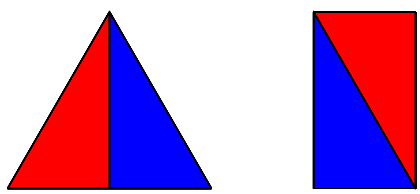

Figure 2: A 2-piece dissection of an equilateral triangle into a rectangle. One piece must be turned over.

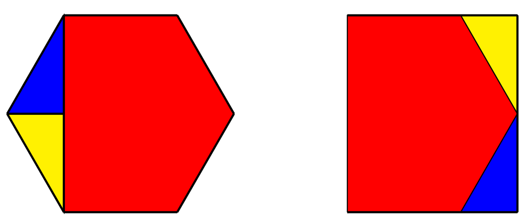

Figure 3: A 3-piece dissection of a regular hexagon into a rectangle.

Lindgren [15, 16] gives examples of regular n-gon to non-square rectangle dissections, but none have fewer pieces than the corresponding n-gon to square dissections. Frederickson [8, pp. 150-151] mentions that in 1926 H. E. Dudeney found a 4-piece octagon to rectangle dissection (probably that shown in Fig. 17).

In April 2023, before the current investigation was begun, the second author (G.A.T.)'s *Geometric Dissections* database [21] contained several examples of regular n-gon to rectangle dissections that had fewer pieces than the best square dissections known, including a 5-piece dissection of a pentagon, a 4-piece dissection of an octagon, a 6-piece dissection of a 10-gon, and a 5-piece dissection of a 12-gon. The database also contained the star polygon and Greek cross dissections shown in §11 and §12.[1]

In May 2023, Adam Gsellman [13] wrote to N.J.A.S. enclosing dissections showing that $r(5) \le 4$, $r(7) \le 6$, and $r(8) \le 4$. As mentioned above, the third of these results was already known, and G.A.T. was aware that $r(5) \le 4$, although that result had not yet been mentioned in the literature, but the upper bound on $r(7)$ appeared to be new. We have since shown that $r(7) \le 5$ (see §4), but Gsellman's dissections of the pentagon and octagon are shown in §3 and §5.

Table 1 shows the current best upper bounds on $s(n)$ and $r(n)$ for $n \le 16$, although to avoid making the paper too long we shall say very little about the cases when $n > 12$.

The paper is arranged as follows. The remainder of this section gives some general remarks about our dissections. Section 2 defines some parameters and coordinates that will be generally useful for our dissections. Subsequent sections deal in turn with pentagons, heptagons, octagons, up through 16-gons, followed by sections on star polygons and the Greek cross. Finally, Section 13 gives some examples of dissections where curved cuts appear essential if one wishes to minimize the number of pieces. The dissections given without attribution are believed to be new.

Remark 1.1. *Certifying the dissections.*

We have attempted to give detailed descriptions of most of the dissections in the main body of the paper (§3-§10), enough at any rate to convince the reader that the dissections are correct. If the dissection begins by cutting up the regular polygon, for example, we have to make sure that the rearranged pieces form a proper rectangle. The pieces must not overlap, there can be no holes; when pieces fit together at a vertex, the sum of the angles must be 2π at an interior point, or π or $\pi/2$ at a boundary point, and so on.

In simple cases the the correctness can be checked "by hand", like solving a jigsaw puzzle. The first pentagon dissection, in §3.1, is an example.

Many of our dissections were obtained by one of the standard strip or superposition constructions. There are a great many versions of these constructions, and they are described in most of the books on the subject, and in the *Methods* section of [21], so we shall not say much about them here.[2]

A simple example of a superposition, used to dissect a polygon A into a polygon B, is to overlay a strip tiled with pieces from A and a second strip tiled with pieces from B. Then with luck the intersection of the two strips will provide the desired dissection (see for example [8, Ch. 11], [15, Chaps. 2-5].) Since for our problem we can assume B is a rectangle, we can often dispense with the second strip. All we need then is a strip tiled with pieces from A. We obtain the dissection by cutting out a rectangle of the correct length from the strip. Examples are shown in Figs. 14 and 22.

[1] [21] now includes all the dissections mentioned in this article.

[2] For the mathematical theory underlying these constructions, see [1, 17, 20].

If a dissection is obtained by one of these standard constructions, it can generally be assumed to be correct. However, one must be careful: with complicated strips like those shown in sections §6 onwards, it is easy to be mistaken about points coinciding, or when triangular regions shrink to a point (in order to save a piece).

For this and other reasons, we have therefore tried to give *ab initio* descriptions of the dissections. In most cases we are able to give a straightedge and compass construction, and to sketch a proof that it is correct.

Remark 1.2. *Straightedge and compass constructions.*

Given an initial drawing of a regular n-gon, most of our dissections[3] can be constructed using only a straightedge and compass. That is, there is no need for a ruler: the construction does not require locating a point which is at some arbitrary irrational distance from another point.[4]

Besides its aesthetic appeal, the advantage of a straightedge and compass construction is that it enables us to give explicit coordinates for every vertex in the construction.[5] We usually start from from the vertices (2.1) of the n-gon, and every subsequent vertex is then determined. In §7 we start from the rectangle, which we assume has width $\sqrt{5}$ and height $\cos(\pi/10)$, and again all subsequent vertices are determined.

Although in theory we could find exact expressions for all the coordinates in a straightedge and compass construction in this way, the expressions rapidly become unwieldy. In practice we have found it better to use computer algebra systems such as WolframAlpha and Maple to guess expressions for the coordinates based on 20-digit decimal expansions, knowing that they could be justified if necessary. We shall see examples of this is §6.

Remark 1.3. *Turning pieces over.*

Although the definitions of $s(n)$ and $r(n)$ allow pieces to be turned over, this is deprecated by purists. Fortunately all the dissections $s(n)$ and $r(n)$ for $3 \le n \le 16$ mentioned in Table 1 can be accomplished without turning pieces over, with the single exception of $r(3)$ (see Fig. 2), which seems to require three pieces if turning over is forbidden.

Another example where turning pieces over appears to be essential to achieve the minimum number of pieces is the seven-piece dissection of $\{6\}$ into $\{8\}$ given in [21].

Remark 1.4. *Convex pieces.*

The regular polygon dissections in Figs. 1-4, 7-9, 17, and 39 use only convex pieces. Other things being equal, we prefer convex pieces, of roughly equal size, that do not need to be turned over. The primary goal however is always to minimize the number of pieces.

Remark 1.5. *Improving on classic dissections.*

We were surprised to find that $r(8)$ is apparently less than $s(8)$, and $r(12)$ apparently less than $s(12)$, since the best octagon to square and 12-gon to square constructions are so striking (Figs. 4, 5). One feels that they could not possibly be improved on. Yet if we only want a rectangle, there is a four-piece dissection of the octagon that has essentially the same symmetry as Fig. 4, as we shall see in §5. Likewise, for the 12-gon, we can save a piece if we only want a rectangle, at the cost however of giving up all symmetry (see §9).

[3] The dissections of the 11-gon in §8 are at present exceptions.

[4] Hadlock [14] contains an excellent introduction to straightedge and compass constructions.

[5] Although the Bolyai-Gerwien theorem guarantees that a straightedge and compass dissection of a regular

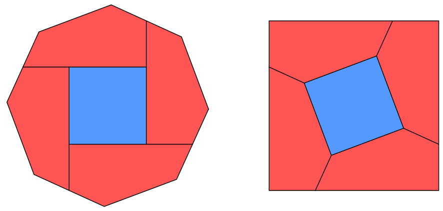

Figure 4: A 14-th century 5-piece regular octagon to square dissection, with cyclic 4-fold symmetry in both the octagon and the square.

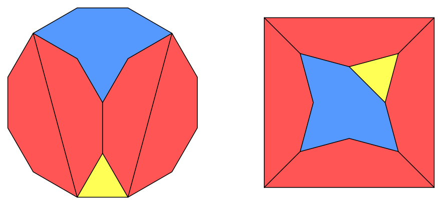

Figure 5: Lindgren's astonishing 6-piece 12-gon to square dissection, with mirror symmetry [15, Ch. 9].

Remark 1.6. *Deplorable absence of lower bounds.*

It is deplorable that we have no (nontrivial) lower bounds. The only negative results we know of are the easily-proved fact[6] that $s(3)$ cannot be 2, and the nontrivial theorem [5] that a circular disk cannot be dissected into a polygon.

Remark 1.7. *The OEIS entry.*

The best upper bounds currently known for the classic sequence $s(n)$ are listed in the *On-Line Encyclopedia of Integer Sequences* (or *OEIS*) database [18] as sequence A110312. This is an exception to the usual OEIS policy of requiring that all terms in sequences must be known exactly, but this sequence is included because of its importance and in the hope that someone will establish the truth of some of the conjectured values. The values of $r(n)$ are included in the comments in A110312. If it could be proved that $r(5) = 4$ and $r(6) = 3$, this would be enough for the $r(n)$ sequence to have its own OEIS entry.

Remark 1.8. *Applications*

These polygon to rectangle dissections have potential applications to lossless source coding (cf. [20]). If a source (a lens, perhaps) repeatedly produces an output which is a point in a regular 12-gon, say, then the dissection could be used to map the point to a more convenient pair (x, y) of rectangular coordinates (compare Figs 39 and 40).

Remark 1.9. *For further information.*

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n-gon to a square exists, we don't know that this is true for a dissection with the minimum number of pieces.

[6] An equilateral triangle of area 1 has side-length $2/3^{1/4} = 1.519$, which won't fit into a square of area 1. So each vertex must be in a different piece.

The database [21] is the best reference for drawings of dissections mentioned in Table 1 but not included in the present article.

Remark 1.10. *Notation*

For up to eight sides we will use the names triangle, ..., hexagon, heptagon, and octagon, but for nine or more sides we will say 9-gon, 10-gon, ... The symbol $\{n\}$ denotes a regular n-sided polygon, and $\{n/m\}$ is a regular star polygon (cf. [4]). $L_{k,n}$ is the length of the chord joining two vertices of $\{n\}$ that are k edges apart (2.3). Decimal expansions have been truncated not rounded.

2 Regular polygons: coordinates and metric properties

In later sections we will usually begin with a regular n-sided polygon, with $n \geq 3$, having sides of length 1, with the goal of cutting it into as few pieces as possible which can be rearranged to form a rectangle.

We often take the polygon to have center $P_0 = (0, 0)$ at the origin:, and to have vertices $P_1, P_2, \ldots, P_n$ labeled counterclockwise, starting with P_1 at the apex of the figure (Fig. 6). The angle subtended at the center by an edge is $2\pi/n$, and we define $\theta = \pi/n$ and $\phi = \pi/2 - \theta = \frac{n-2}{2n}\pi$, which will be the principal angles used in the formulas.

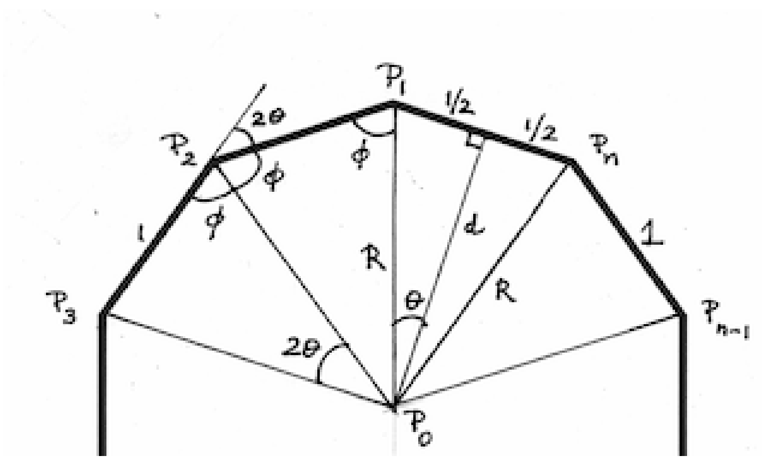


Figure 6: Vertices and principal angles and distances for a regular n-gon.

Since the sides have length 1, the radius of the polygon is $R = \frac{1}{2\sin\theta}$, and the distance from the center to the midpoint of an edge is $d = \frac{1}{2\tan\theta}$. The vertices have coordinates

$$\begin{aligned} P_1 &= (0, R)\,, \\ P_k &= (-R\sin(2(k-1)\theta), R\cos(2(k-1)\theta))\,, \\ P_{n+2-k} &= (R\sin(2(k-1)\theta), R\cos(2(k-1)\theta))\,, \end{aligned} \tag{2.1}$$

for $k = 2, \ldots, \lfloor (n+2)/2 \rfloor$. The polygon has area

$$\frac{nd}{2} = \frac{n}{4}\cot\frac{\pi}{n}. \tag{2.2}$$

Many dissections involve a cut along a chord P_iP_{i+k} joining two vertices. Let $L_k = L_{k,n}$ denote the length of the chord joining two vertices that are $k \geq 1$ edges apart in $\{n\}$ and are in the same semicircle. Then $L_1 = 1$ for $n \geq 3$, and for $t \geq 1$,

$$L_{2t} = 2\sum_{j=0}^{t-1} \cos(2j+1)\theta\,, \quad L_{2t+1} = 1 + 2\sum_{j=1}^{t} \cos 2j\theta\,. \tag{2.3}$$

3 Four-piece dissections of a pentagon

In 1891 Robert Brodie discovered a six-piece dissection of a regular pentagon into a square ([8, p. 120], [15, Fig. 3.1], [21]), and there has been no improvement since then, so it seems likely that $s(5) = 6$. In this section we give four different four-piece dissections of a regular pentagon into a rectangle, showing that $r(5) \leq 4$. Rather surprisingly, this result does not seem to have been mentioned in the literature before now. The fact that there are at least four ways to get $r(5) \leq 4$ makes us wonder if $r(5)$ might actually be equal to 3.

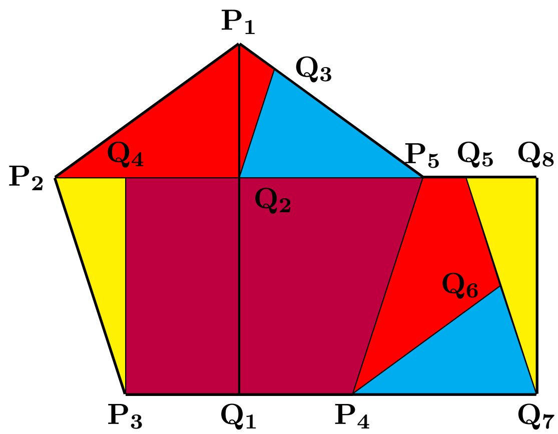


Figure 7: The pentagon $P_1P_2\dots P_5$ is cut into four convex pieces which can be rearranged to form the rectangle $Q_4P_3Q_7Q_8$.

The first two four-piece dissections (§3.1, §3.2) were found by the authors (although they can hardly be new), and have the property that the pieces are convex; the other two (§3.3, §3.4) are due to Adam Gsellman [13].

We use the notation introduced in the previous section (taking $n = 5$). The pentagon has vertices $P_1, \dots, P_5$ with sides of length 1. The key angles are $\theta = \angle P_1P_2P_5 = 36°$, $\phi = 54°$, $\angle P_2P_1P_5 = 2\phi$, and $\angle P_2P_3Q_3 = \theta/2$. We note the values of

$$\begin{aligned} \sin 36° &= \cos 54° = \sqrt{\frac{5-\sqrt{5}}{8}}\,, & \cos 36° &= \sin 54° = \frac{\sqrt{5}+1}{4}\,, \\ \sin 72° &= \cos 18° = \sqrt{\frac{5+\sqrt{5}}{8}}\,, & \cos 72° &= \sin 18° = \frac{\sqrt{5}-1}{4}\,. \end{aligned} \tag{3.1}$$

3.1 Pentagon #1

To construct this dissection (see Fig. 7) we draw a perpendicular from P_1 to the mid-point Q_1 of the opposite side, and draw the chord $P_2 - P_5$. Let Q_2 be the intersection of these two lines, and place Q_3 on $P_1 - P_5$ so that $Q_2P_5Q_3$ is an isosceles triangle. Finally, draw a perpendicular $P_3 - Q_4$ from P_3 to $P_2 - P_5$.

To form the rectangle, we first rotate the quadrilateral $P_1P_2Q_2Q_3$ by 36° and move it to $P_5P_4Q_6Q_5$, then the triangle $Q_2P_5Q_3$ is moved to $Q_6P_4Q_7$, and the triangle $P_2P_3Q_4$ is moved horizontally to $Q_5Q_7Q_8$. The two isosceles triangles (blue) have long sides of length $L_{2,5}/2 = (1+\sqrt{5})/4$ and base $1/2$. The two right triangles (yellow) have sides of lengths $\sin\theta$, $\cos\theta$, and 1.

To prove that this dissection is correct, we must check that, after the rearrangement, the result is indeed a rectangle, with area equal to that of the pentagon. In particular, we must check that the points $P_2, Q_4, Q_2, P_5, Q_5, Q_8$ are collinear, as are P_3, P_4, Q_7, and Q_5, Q_6, Q_7, that Q_6 bisects $Q_5 - Q_7$, and also that the difference between the x-coordinates of P_2 and P_3 is equal to the difference between the x-coordinates of Q_5 and Q_8. Since we have complete information about the points, these checks are easily carried out. The pentagon has area $(5^{3/4}/(8\sqrt{2}))(\sqrt{5}+1)^{3/2}$, and we can check that this is equal to $|P_3Q_4| \cdot |P_3Q_7|$. This completes the proof of the dissection.

It is worth pointing out that the trapezoid $P_5P_4Q_7Q_5$ is symmetric about its vertical axis.

We chose this relatively simple example to illustrate the steps needed to prove that a dissection is correct. In later examples we will just give the basic information needed for the proof and leave the detailed verification to the reader.

3.2 Pentagon #2

This is very similar to the first dissection. Two of the pieces are the same, only now the rest of the pentagon is divided into two pieces that are reflections of each other. The trapezoid from Fig. 7 has moved to the center of the rectangle.

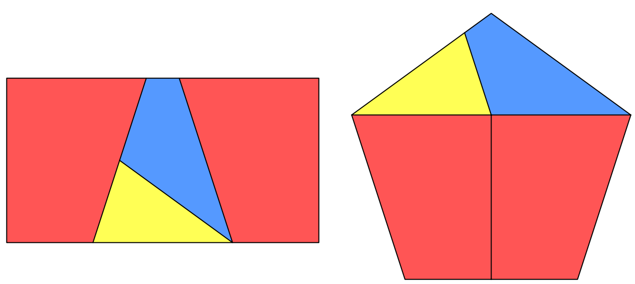


Figure 8: Similar to the first pentagon dissection, only now the pieces are more nearly equal in size.

Although logically we are dissecting the polygon *into* a rectangle, many of our colored illustrations have the rectangle on the left of the picture, as in Figs. 8, 15, 17, etc. This is because of the convention in [21] of starting with the figure having the smaller number of vertices. Also, in the case of strip or tessellation constructions, one often proceeds from the rectangle to the polygon.

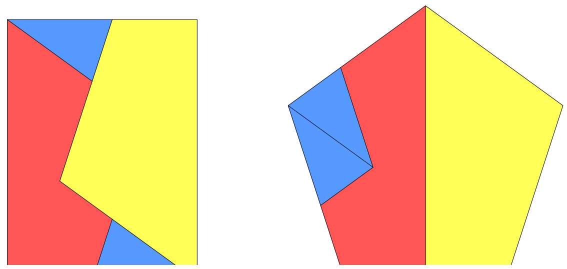


Figure 9: Gsellman's first pentagon dissection.

3.3 Pentagon #3

The first of Adam Gsellman's four-piece dissections of a pentagon is shown in Fig. 9. We do not know how Gsellman discovered it, but we have found that it can be obtained by a simple slide construction. (It can also be obtained from a strip dissection.)

Figure 10: A construction that produces the dissection in Fig. 9 (see text for details).

Cut the pentagon down the middle into two quadrilaterals A and B, reflect A in a vertical mirror, and rotate B by 180° (see Fig. 10). Now slide the pieces towards each other until they overlap in a parallelogram whose diagonal is equal in length to the gap in the top (and the bottom) edge. Cut the parallelogram in the A piece into two equal isosceles triangles which can be rotated to complete the rectangle.

For the proof that this dissection is correct we label the points as in Fig. 11. Q_1 is the midpoint of the side P_3P_4. The key parameters are $\theta = \pi/5$, $R = 1/(2\sin\theta)$, and $d = R\cos\theta = (\sqrt{5}+1)^{3/2}/(4\sqrt{2}\,5^{1/4})$. The pentagon has height $h = R + d = (5^{1/4}/(4\sqrt{2}))(\sqrt{5}+1)^{3/2}$. This is also the height of the final rectangle, which (since we know the area) has width $w = \sqrt{5}/2$.

We label the vertices of the rectangle $a_1, \ldots, a_4$, and let $b_1, \ldots, b_5$, a_5 denote the points in the center of the figure. Finally, let s and b denote the side and base of the small isosceles triangles.

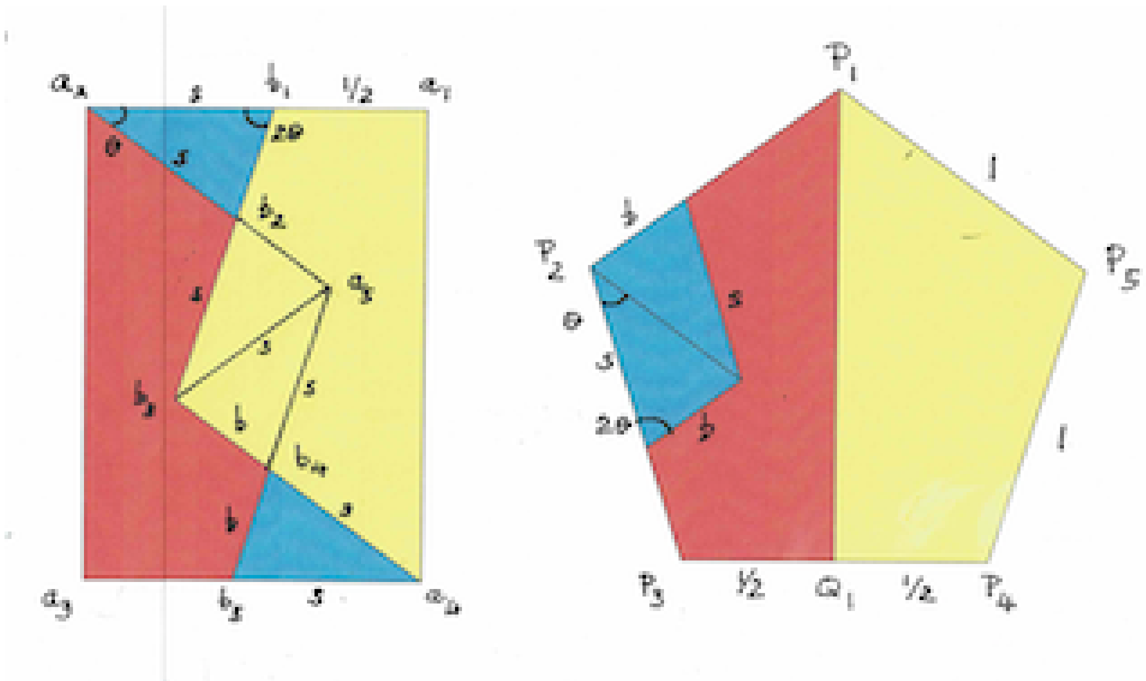


Figure 11: Names for points and lengths in dissection of Fig. 9.

Note that $a_1b_1b_3a_4$ is a rotated copy of $Q_1P_4P_5P_1$, and $a_2a_3b_5a_5$ is a reflected copy of $P_1P_2P_3Q_1$. In particular, $|a_1b_1| = 1/2$, so $s = w - 1/2 = (\sqrt{5} - 1)/2$. The base of the isosceles triangle is therefore $b = (3 - \sqrt{5})/2$. This implies $s + b = 1$, and we can now check that all the pieces fit together correctly.

3.4 Pentagon #4

Gsellman's second pentagon dissection is shown in Fig. 12. It can be obtained by a similar slide construction.

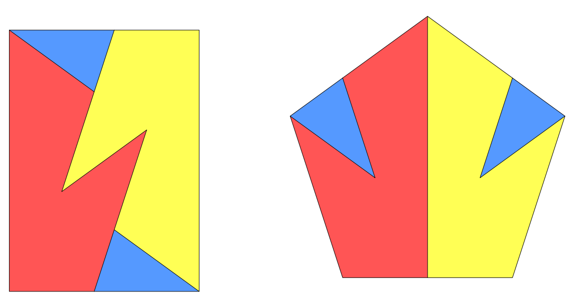


Figure 12: Gsellman's second pentagon dissection.

There is a third version of Gsellman s dissection which has the zigzag cut through the center of the rectangle going the other way (Fig. 13). These are elegant dissections, but all three have the slight defect of requiring a piece to be turned over.

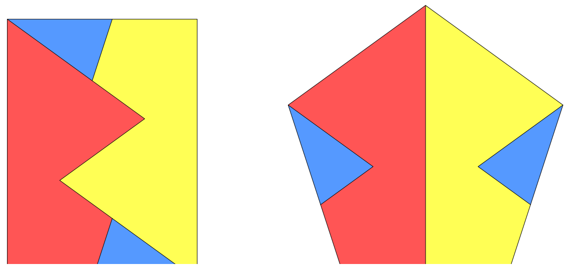


Figure 13: Another version of Fig. 12.

4 A five-piece dissection of a heptagon

Even the great master Harry Lindgren [15] could only show that $s(7) \leq 9$, but around 1995 G.A.T. found a seven-piece dissection of a heptagon into a square, and it is reasonable to conjecture that indeed $s(7) = 7$. This dissection is described in [8, pp. 128–129] and [21].

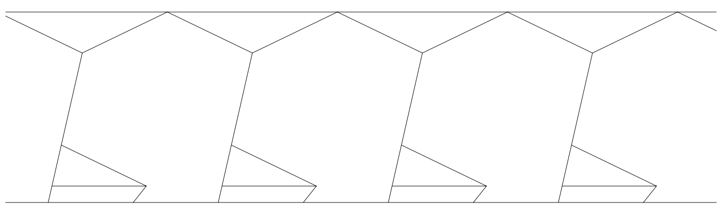


Figure 14: A heptagon strip used for the heptagon to rectangle dissection.

For the rectangle problem, we start from a heptagon strip (shown in Fig. 14) that is a modification of a strip used in the square case (compare [8, Fig. 11.28]). By cutting a rectangle from this strip, we obtain a five-piece dissection of a heptagon to a rectangle, shown in Fig. 15.

Rather than following the path by which it was discovered, we will construct this dissection directly from the heptagon, something that can be done using only a straightedge and compass. The vertices of the 7-gon are labeled $P_1, \ldots, P_7$ (see Fig. 16). Drop a perpendicular from P_1 to the midpoint Q_6 of P_4P_5, and draw chords P_3P_5, P_4P_7, and P_5P_7. Let Q_2 be the intersection of P_3P_5 and P_4P_7, and let Q_3 be the midpoint of P_4Q_2. Draw a line $Q_3Q_5Q_4$ through Q_3 parallel to P_4P_5.

Using these these lines as guides, we get the five pieces by cutting $P_7P_5P_6P_7$; $P_1Q_5Q_4P_7P_1$; $P_1P_2P_3Q_2Q_3Q_5P_1$; $P_3P_4Q_2P_3$; and $Q_3P_4P_5Q_4Q_3$.

These pieces can then be rearranged to form the rectangle as shown in Fig. 15, left.

To help anyone who wishes to verify the correctness of this dissection, we list some key angles and lengths. We set $\theta = \pi/7$ and note that $\cos(\theta) = .0.9009\ldots$ has minimal polynomial

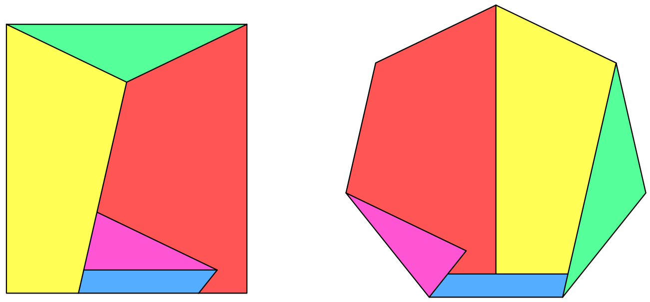


Figure 15: A five-piece dissection of a heptagon to a rectangle.

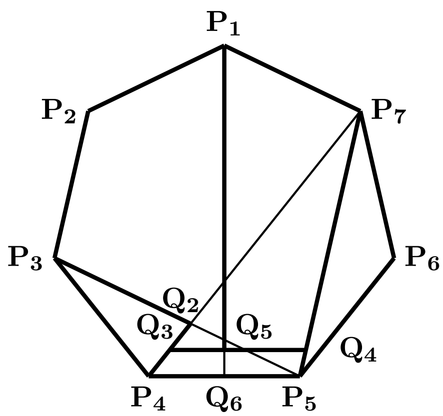


Figure 16: Heptagon dissection (Fig. 15, right) showing labels for points.

$8x^3 - 4x^2 - 4x + 1$. Then $\angle P_1P_2P_3 = 5\theta$, $\angle P_7P_4P_5 = 2\theta$, $\angle P_4P_3P_5 = \angle P_5P_7P_6 = \theta$, and $\angle P_4P_5P_7 = 4\theta$. The chord P_7P_5 has length $L_{2,7} = 2\cos\theta$, and $|P_4Q_3| = |Q_3Q_2| = 1/2 - (\sec\theta)/4$. The trapezoidal piece has cross-section $|Q_5Q_6| = (4\cos\theta - 3)/(8\sin\theta)$. The rectangle has height $7/(8\sin\theta)$ and width $2\cos\theta$.

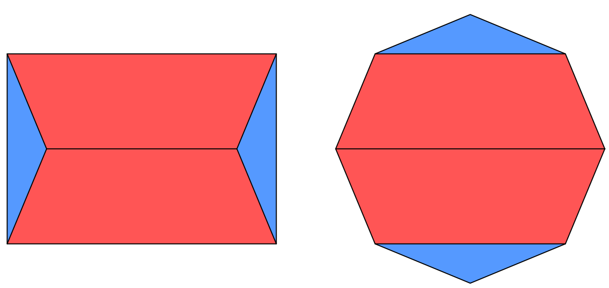


Figure 17: The classic four-piece octagon to rectangle dissection [21].

5 Four-piece dissections of an octagon

It appears that five pieces are needed to dissect an octagon into a square (see Fig. 4), whereas four pieces are enough if we only want a rectangle (Fig. 17). The former has cyclic four-fold symmetry, while the latter has the symmetry of a Klein 4-group.

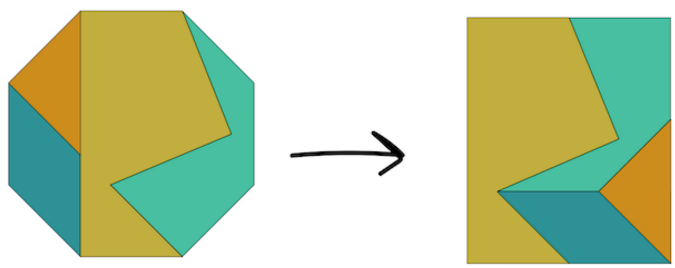


Figure 18: Gsellman's first four-piece dissection of an octagon. See Fig. 19 for details.

Adam Gsellman [13] has found two other four-piece dissections, shown in Figs. 18-20. The description in Fig. 19 is self-explanatory (the angles are multiples of $\pi/8$ and the only irrationality needed is $\sqrt{2}$). The second dissection (Fig. 20) is very similar.

These two dissections are admittedly less elegant than that in Fig. 17, and require pieces to be turned over, but we include them because, as the example in Fig. 24 shows, complicated non-convex dissections may be needed to get the minimum number of pieces.

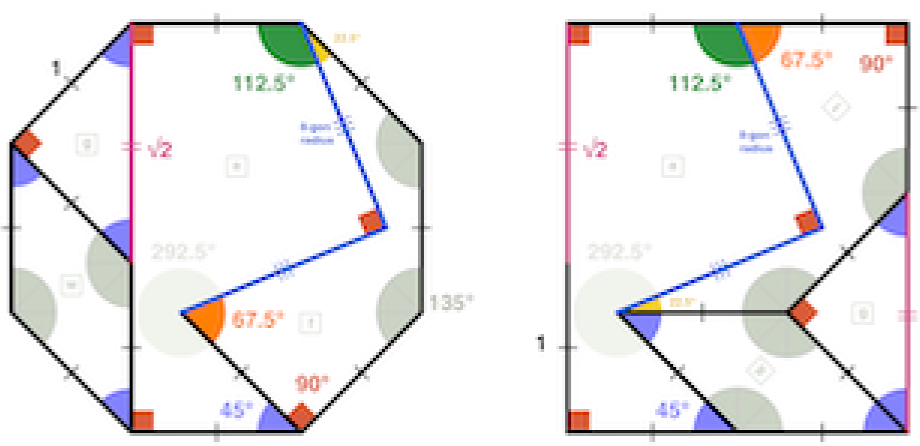


Figure 19: Gsellman's detailed description of the dissection in Fig. 18.

6 A seven-piece dissection of a 9-gon

Figure 21 shows a seven-piece dissection of a 9-gon into a rectangle, which has two fewer pieces than the best dissection into a square presently known. It was obtained from the strip

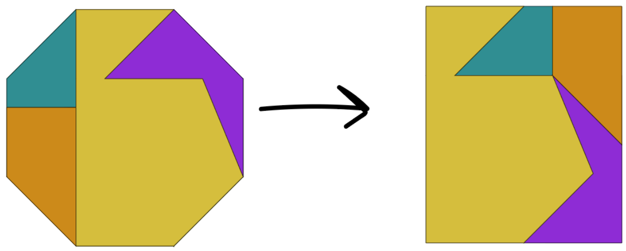


Figure 20: Gsellman's second four-piece dissection of an octagon.

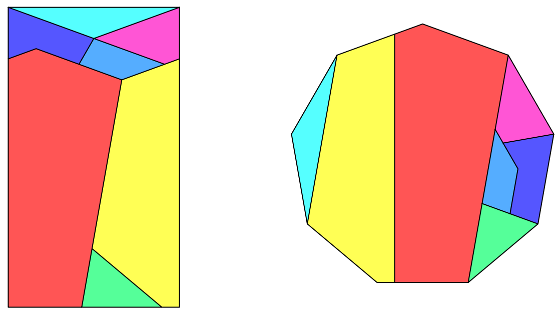


Figure 21: A seven-piece dissection of a 9-gon into a rectangle.

dissection of a 9-gon shown in Fig. 22, by cutting a rectangle from the strip.

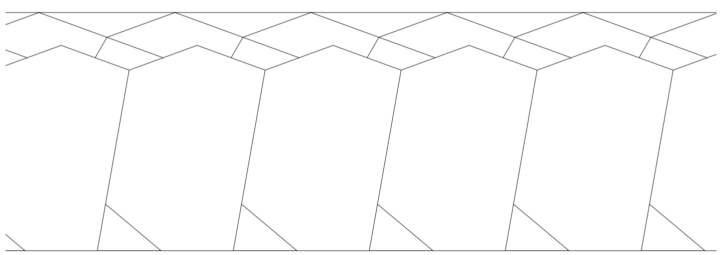


Figure 22: The strip dissection of a 9-gon which led to Fig. 21.

As we did with the heptagon, we will construct this dissection directly from the 9-gon, using only a straightedge and compass.

The vertices of the 9-gon are labeled $P_1, \ldots, P_9$ (see Fig. 23), and we use the coordinates established in §2. Also $\theta = \pi/9$, $C_1 = \cos\theta$ has minimal polynomial $8x^3 - 6x - 1$, and $S_1 = \sin\theta$ has minimal polynomial $64x^6 - 96x^4 + 36x^2 - 3$. (For this reason, we write our expressions as rational functions of $\cos\theta$, with at most linear terms in $\sin\theta$.)

To obtain the dissection we first draw chords $P_2 - P_4$, $P_3 - P_7$, and $P_6 - P_9$. Then Q_2 is the intersection of $P_3 - P_7$ and $P_6 - P_9$, Q_3 is the midpoint of $Q_2 - P_9$, and Q_7 is the midpoint of $Q_2 - P_7$. Draw a line segment $Q_7 - Q_6$ of length $1/2$ parallel to $P_7 - P_8$, join Q_3 to Q_6, and

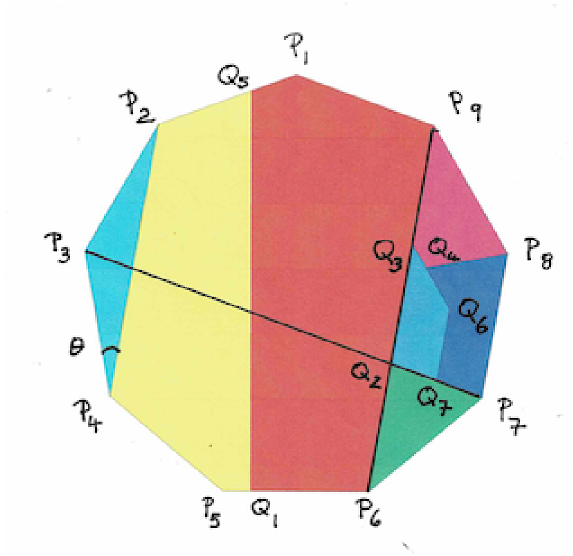


Figure 23: Labels for points in the 9-gon.

locate Q_4 at the intersection of $Q_3 - Q_6$ and a perpendicular drawn from P_8 to the midpoint of $P_3 - P_4$. Finally Q_5 is on $P_1 - P_2$ at distance $|Q_4Q_6|$ from P_1, and $Q_5 - Q_1$ is perpendicular to $P_5 - P_6$.

To assist in the analysis we define a further point Q_8 (not shown in Fig. 23), at the intersection of $Q_7 - Q_6$ (extended) and $Q_4 - P_8$. Then $Q_4Q_6Q_8$ is an isosceles triangle and $Q_3Q_2Q_7Q_8$ is a parallelogram.

The seven pieces of the dissection can now be found by making the cuts indicated by the colored regions on the right of Fig. 21. To complete the proof that the dissection is correct, we must verify that the pieces can be rearranged to form the rectangle on the left of Fig. 21. We will not take the space to do that here, but to assist the reader we give two key lengths. The length

$$|Q_2Q_7| = |Q_2P_7|/2 = |Q_4Q_6| = |Q_4Q_8| = |Q_8Q_3| = |P_1Q_5| = |Q_2Q_3| - \frac{1}{2} = \frac{C_1}{2C_1+1} = 0.3263\ldots \tag{6.1}$$

plays a central role, as does $|P_8Q_4| = 3/(8S_1(C_1+1)) = 0.5652$. The rectangle has width $2C_1 = 1.8793\ldots$ and height $9/(8S_1) = 3.2892\ldots$.

Some of the equalities in (6.1) are by no means obvious. They do not follow directly from the geometry, but depend on the fact that $\cos\theta$ satisfies a cubic equation. As discussed in Remark 1.2, we can find exact expressions for the coordinates of the points. We *could* do this by solving the appropriate equations, using a computer algebra system such as Maple, but we have found it a lot easier to use another computer algebra system, WolframAlpha, and ask it to find the coordinates for us.

For example, the first step in finding the present dissection is to find Q_2. Using Maple, and working to 20 decimal places, we find that

$$Q_2 = (0.65270364466613930216, -0.50771330594287249271)\,.$$

We now ask WolframAlpha to express these two numbers in terms of $\cos\theta$, $1/\cos\theta$, $\sin\theta$, and $1/\sin\theta$. The result (setting $C_1 = \cos\theta$, $S_1 = \sin\theta$) is

$$Q_2 \;=\; \left(\frac{2C_1}{2C_1+1},\; -(2S_1 - \frac{1}{S_1} + \sqrt{3})\right).$$

We do this for all the points. Another example is

$$\begin{aligned} Q_8 \;&=\; (1.1028685319524432095, 0.19446547835755153996) \\ &=\; \left(\frac{C_1}{2} + \frac{1}{8C_1} + \frac{1}{2}.\ \frac{1}{8S_1} - \frac{S_1}{2}\right). \end{aligned} \tag{6.2}$$

The verification of the equalities in (6.1) is then a routine calculation (using Maple's `simplify(..., trig);` command).

7 A four-piece dissection of a 10-gon

In the final appendix ("Recent Progress") to his 1964 book [15], Lindgren gave a new strip based on the 10-gon (Fig. 25), and used it to obtain several new dissections, including an eight-piece dissection of a 10-gon to a square, and a seven-piece dissection to a golden rectangle. As Frederickson reports in [8, Ch. 11], G.A.T. was then able to show that in the dissection to a square, two of Lindgren's pieces could be merged, leading to a seven-piece dissection to a square, still the record. This dissection is also described in the *Variable Strips* section of [21]. If we draw vertical lines across Lindgren's strip (without changing it), we obtain a five-piece dissection of a 10-gon to a (non-golden) rectangle, as shown in Fig. 26. There is a small range of possibilities for the positions of these vertical lines. In Fig. 26 they have been placed in the middle of their range, in order to obtain the most symmetric dissection.

Remarkably, if the goal is only to obtain a rectangle, it is possible to modify Lindgren's strip (Fig. 26) to get a four-piece dissection. The modified strip is shown in Fig. 27, and the dissection itself in Fig. 24. To go from Fig. 26 to Fig. 27 we merge the two right-most pieces of the rectangle, forming a church-shaped piece, and compensate by dividing the large piece into two by a zig-zag cut.

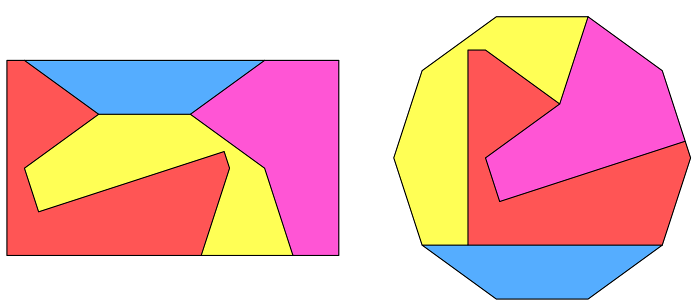


Figure 24: A four-piece dissection of a 10-gon into a rectangle.

This is one of the most complicated dissections in the article, and we give a precise straight-edge and compass construction starting from the rectangle.in Fig. 27.

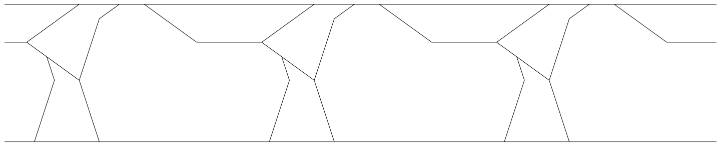


Figure 25: Lindgren's 1964 10-gon strip [15].

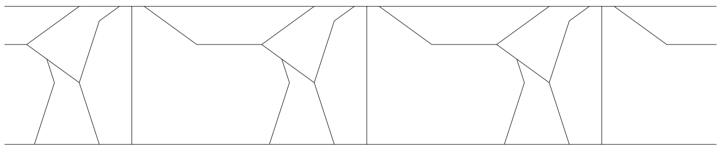


Figure 26: A 5-piece 10-gon to rectangle dissection obtained from Fig. 25.

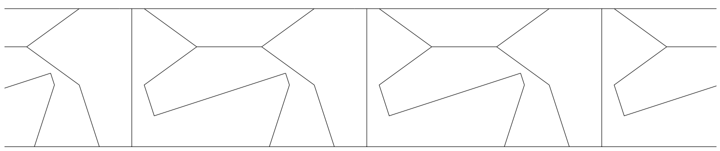


Figure 27: The strip which gives the four-piece dissection of the 10-gon to a rectangle.

We first construct an intermediate rectangle with five pieces, and then shift it slightly to save a piece. The angles involved are $\theta = \pi/10 = 18°$, 2θ, and $\phi = 4\theta$.

The intermediate rectangle has vertices labeled $2, 14, 19, 5$ in Fig. 28; the final rectangle has vertices $1, 13, 18, 4$. We place the origin of coordinates for the rectangle near the bottom left corner, at the point $14 = (0, 0)$. The 10-gon has area $\frac{5}{2\tan\theta}$ (see (2.2)), and we take the width of the strip to be $w = \sqrt{5}\cos\theta$. The other dimension of the rectangle is its height $h = 2\sqrt{5}\cos 2\theta$. (After a series of relabelings, the rectangle as drawn in Fig. 27 has ended up with height w and width h. We hope the reader will forgive us!)

The coordinates of the points $2, 14, 17, 5$ are therefore $(0, w)$, $(0, 0)$, $(h, 0)$, and $(h.w)$. We draw a network of lines as follows. Starting at point 2, we draw line segments of length 1 from 2 to 6 to 7 to 3, and from 6 to 8, 3 to 21, and 7 to 12 to 17 to 15 to 11 (the angles are indicated in the figure). We then complete the line 3 to 15. We also draw line segments of length 1/2 from 8 to 9 to 14. For the two final lines we join 9 to 21 and draw the perpendicular from 11 to 10. The coordinates have been chosen so that several coincidences occur.[7] The points 8, 11, 12, and 20 (in the adjacent rectangle in the strip) are collinear. Also the distance from 9 to 10 turns out to be equal to w. The angle $\angle 8, 9, 21$ is a right angle. The central point 21 has coordinates $(2 + \sin\theta, 2\sin 2\theta)$. The distance from 10 to 11 is $x := \frac{3-\sqrt{5}}{4}$, and we get the final rectangle by shifting the intermediate rectangle to the left by that amount.

[7] Similar to those in (6.1), but less dramatic, since $\sin\pi/10$ is only a quadratic irrationality.

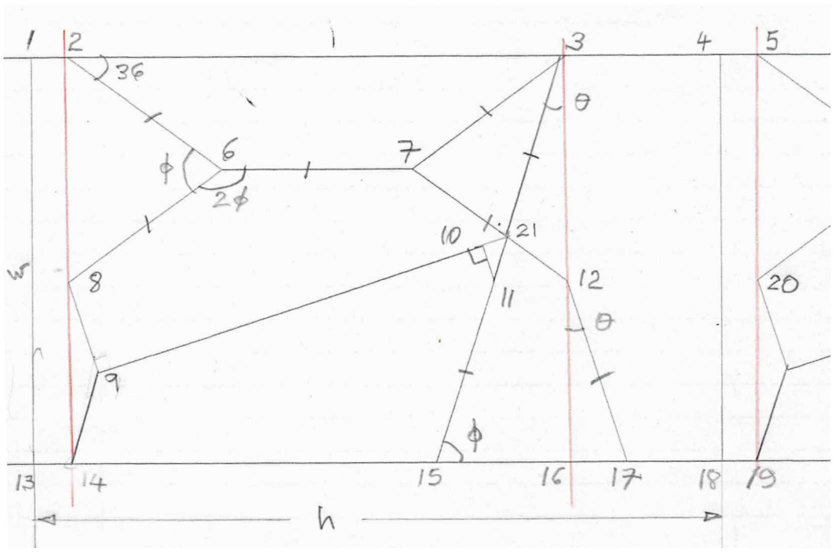


Figure 28: Labels for points used to construct the four-piece dissection of the 10-gon.

We get the four pieces in the dissection as follows. The quadrilateral piece (the "dish") is obtained by cutting along the path $2, 6, 7, 3, 2$. For the hexagon (the "church"), cut along $3, 7, 12, 17, 18, 4, 3$. For the first 9-gon (the "hammer"), cut along $6, 8, 9, 10, 11, 15, 17, 12, 7, 6$, and for the second 9-gon (the "triangle"), cut along $2, 1, 13, 15, 11, 10, 9, 8, 6, 2$. By moving the edge of the rectangle to the left so that it no longer passes through the point 8 we have reduced the number of pieces from five to four.

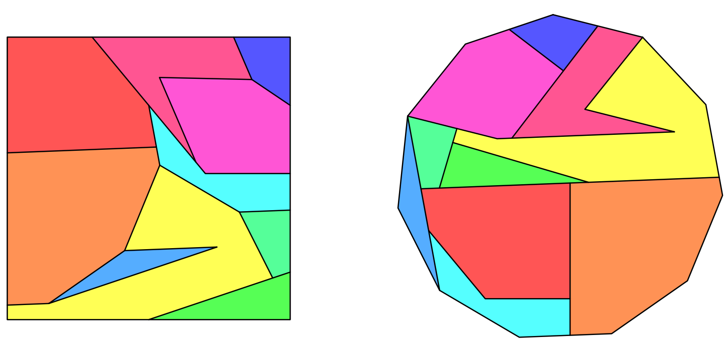


Figure 29: A ten-piece dissection of an 11-gon into a square [21].

8 Dissecting an 11-gon to a square and to a rectangle

8.1 A ten-piece dissection of an 11-gon to a square

Before the appearance of [21] there had been little work on dissections of the 11-gon: this polygon is not mentioned in any of [8, 15, 16]. G.A.T.'s ten-piece dissection of an 11-gon into a square was given in [21], and was described by Frederickson in [9, 11]. It is shown here in Fig. 29. It can be obtained by combining the two superpositions of strips shown in Figs. 30 and 31. When the quadrilateral outlined in red in Fig. 31 is combined with the hexagon outlined in red in Fig. 30, the result is the dissected square on the left of Fig. 29.

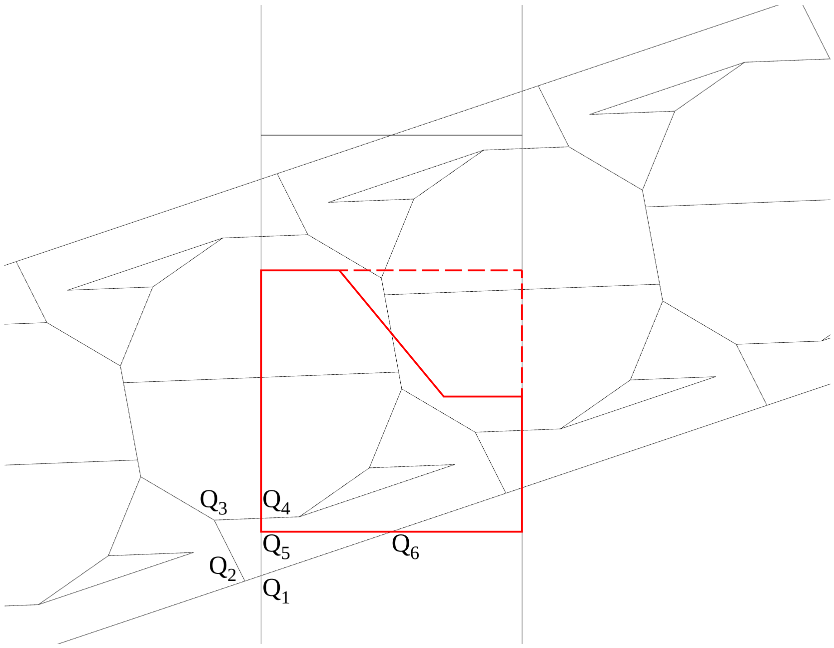


Figure 30: First of two strip superpositions used in construction of the 11-gon to square dissection of Fig. 29.

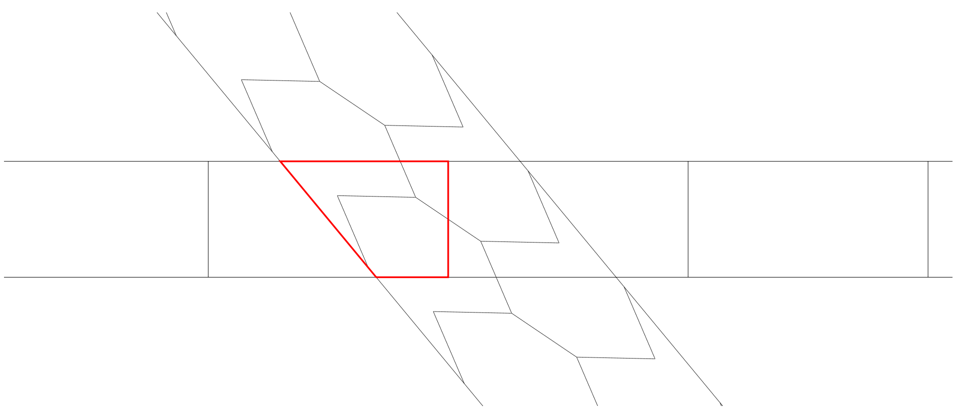


Figure 31: Second of two strip superpositions used in construction of the 11-gon to square dissection of Fig. 29.

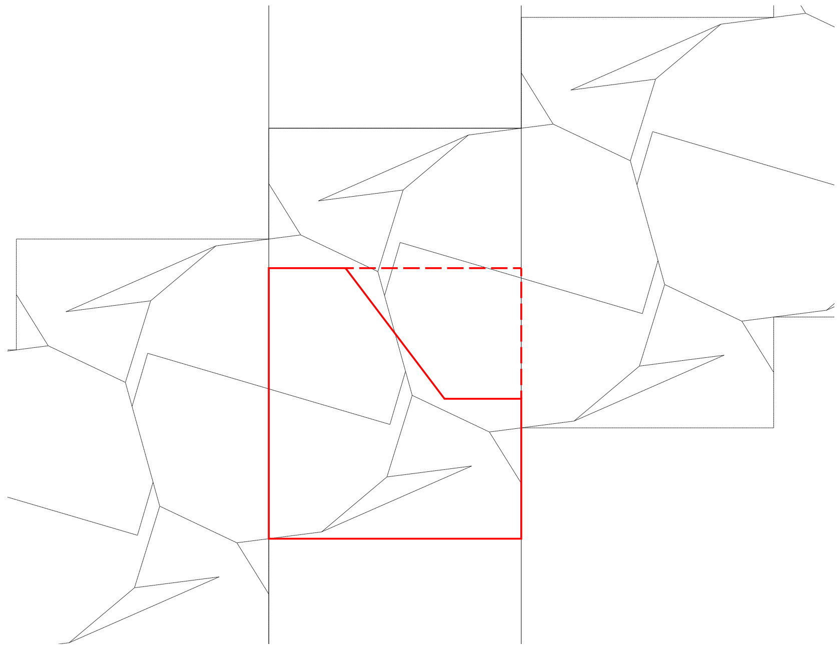


Figure 32: Superposition from Fig. 30 modified to produce the 9-piece rectangle dissection of Fig. 34.

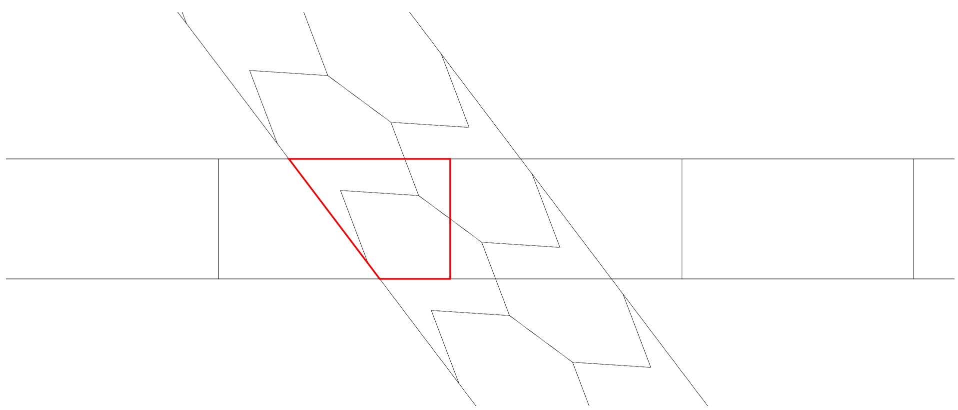
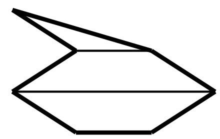


Figure 33: Superposition from Fig. 30 modified to be compatible with Fig. 32, and used together with Fig. 32 for the 9-piece rectangle dissection of Fig. 34.

8.2 A nine-piece dissection of an 11-gon to a rectangle

A piece can be saved if our goal is only to dissect the 11-gon into a rectangle. By adjusting the width of the vertical strip in Fig. 30 we can cause the point Q_1 to move along the line Q_2, Q_6 until it merges with Q_2, and by changing the length of the line Q_2, Q_3 we can make Q_4 merge with Q_5. The last step is to modify the horizontal line across the middle of the 11-gon so that the triangular piece Q_4, Q_1, Q_6 is no longer needed. The result is shown in Fig. 32. A considerable amount of experimenting was involved in finding Fig. 32. The second superposition (Fig. 31) has to be changed slightly to be compatible with Fig. 32 and is shown in Fig. 33.

Note the heptagonal tiles (of area 2.26123) used as a repeating element in Figs. 31 and 33 are formed from three pieces cut from the 11-gon along chords:



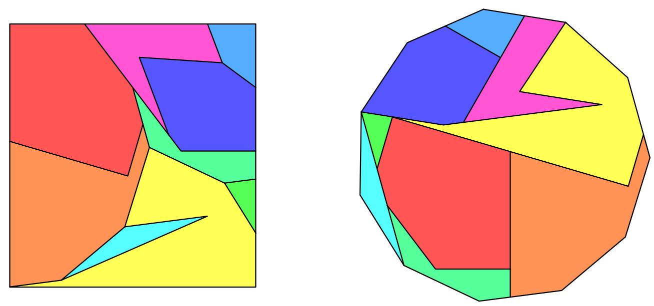


Figure 34: Nine-piece dissection of an 11-gon into a rectangle, obtained by combining superpositions in Figs. 32 and 33.

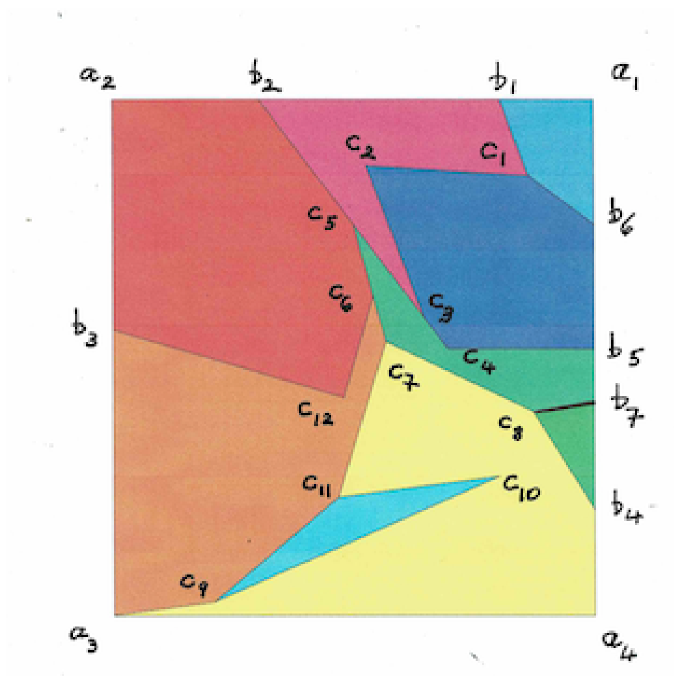


Figure 35: Rectangle in Fig. 34 with points labeled.

The resulting dissection, shown in Fig. 34, is very far from unique, and this is reflected in the fact that we have not been able to find a straightedge and compass description. To assist a reader who wishes to experiment further, we give numerical coordinates for the points of the rectangle, which we label as in Fig. 35. The rectangle has width 2.95644, height 3.16787, and area 9.36563.

$a_1 = (0, 0)$, $a_2 = (2.95644, 0)$, $a_3 = (2.95644, 3.16787)$, $a_4 = (0, 3.16787)$, $b_1 = (0.57934, 0)$, $b_2 = (2.057556, 0)$,
$b_3 = (2.95643, 1.41370)$, $b_4 = (0, 2.51903)$, $b_5 = (0, 1.52970)$, $b_6 = (0, 0.76485)$, $b_7 = (0, 1.87019)$,
$c_1 = (0.40196, 0.46747)$, $c_2 = (1.39980, 0.40178)$, $c_3 = (1.04504, 1.33674)$, $c_4 = (0.89888, 1.52970)$,
$c_5 = (1.47822, 0.76485)$, $c_6 = (1.35530, 1.20760)$, $c_7 = (1.27704, 1.48947)$, $c_8 = (0.37365, 1.91828)$,
$c_9 = (2.33827, 3.08830)$, $c_{10} = (0.58111, 2.317022)$, $c_{11} = (1.57292, 2.44469)$, $c_{12} = (1.53810, 1.83016)$.

8.3 A second nine-piece dissection of an 11-gon to a rectangle

(Added August 25, 2023.) We have very recently found another nine-piece dissection of an 11-gon to a rectangle, which appears to be more amenable to a straightedge and compass description. It is based on the two superpositions shown in Figs. 36 and 37, and is shown in Fig. 38. The rectangle is very nearly a square. If we succeed in finding a straightedge and compass construction for it we will replace the dissection in the previous subsection with this one.

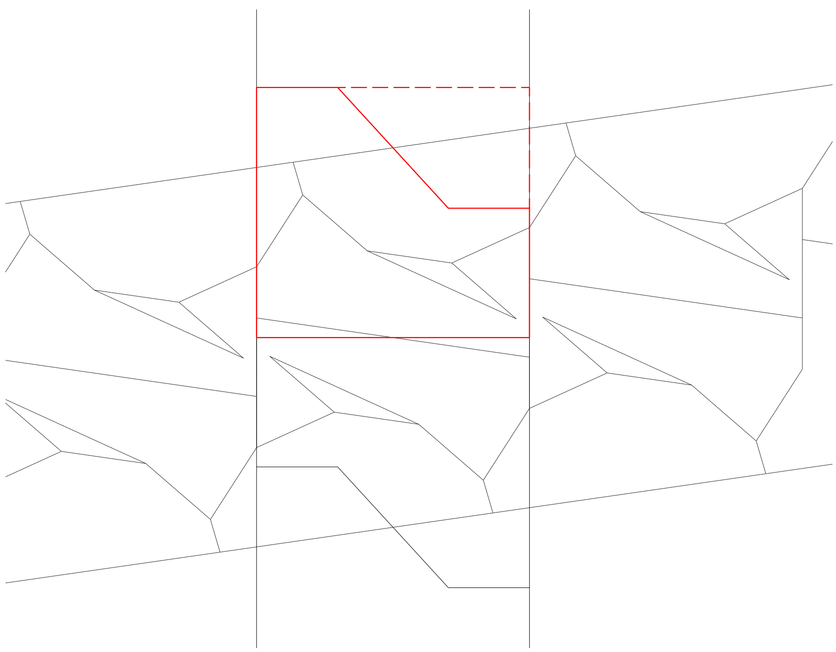


Figure 36: First of two superpositions used for the second 9-piece 11-gon to rectangle dissection of Fig. 38.

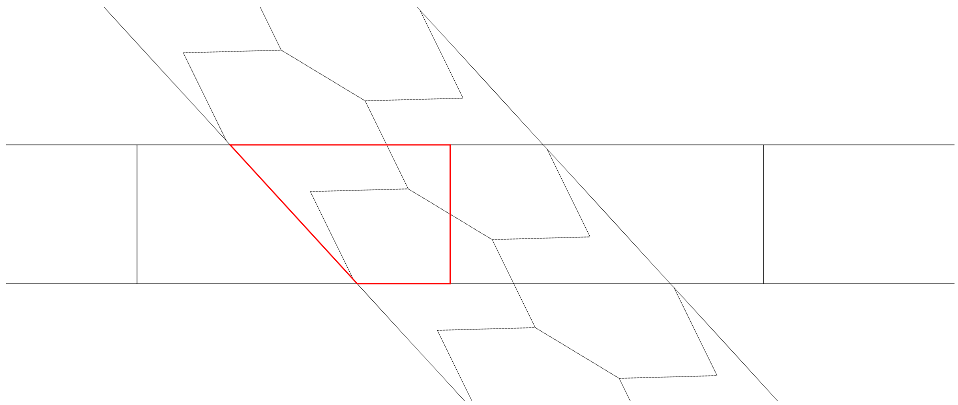


Figure 37: Second of two superpositions used for the second 9-piece 11-gon to rectangle dissection of Fig. 38.

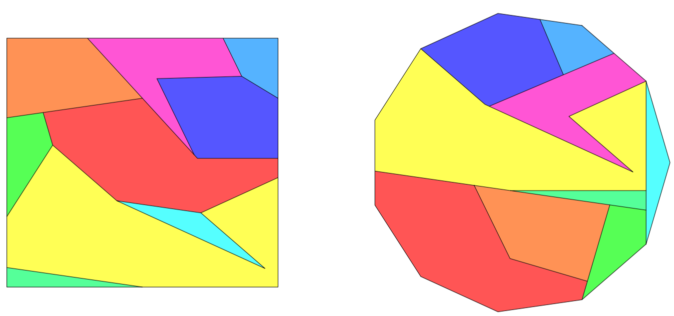


Figure 38: A second nine-piece dissection of an 11-gon into a rectangle, obtained by combining the superpositions in Figs. 36 and 37.

9 Five-piece dissections of a 12-gon

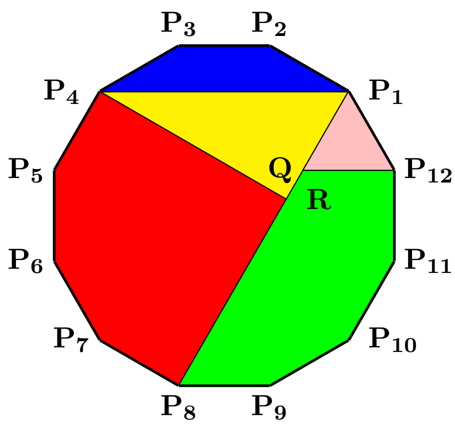


Figure 39: A five-piece dissection of a 12-gon.

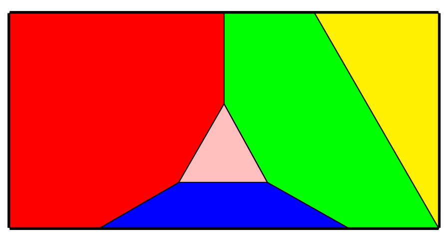


Figure 40: The pieces reassembled to form a rectangle.

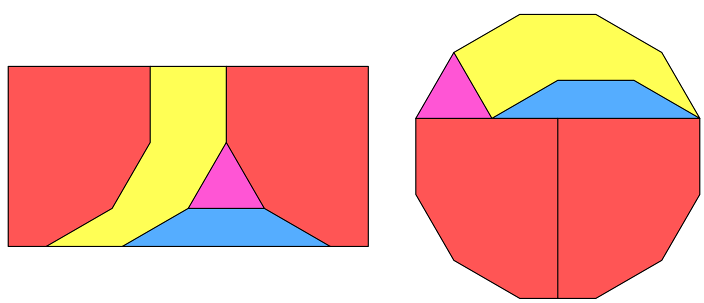


Figure 41: An alternative five-piece dissection of a dodecagon.

We start with a 12-gon with edge-length 1 (see Fig. 39). Draw chords from P_1 to P_4 and P_8. Draw a perpendicular from P_4 to $P_1 - P_8$, meeting it at R, and draw an equilateral triangle $P_1 Q P_{12}$ that touches $P_1 - P_8$.

The angle $\angle P_1P_4R$ is $\pi/12 = 15°$. The lengths of the line segments are as follows: $|P_1P_4| = |QP_8| = L_{3,12} = 1 + \sqrt{3}$, $|P_4R| = |RP_8| = (3 + \sqrt{3})/2$, and $|P_1R| = (1 + \sqrt{3})/2$.

After the pieces are rearranged (Fig. 40), the resulting rectangle has width $w = 3 + \sqrt{3}$ and height $h = (3 + \sqrt{3})/2$. We then easily check that the product wh is equal to the area $3 \cot 15°$.

A second dissection of the 12-gon (although with a non-convex piece) is shown in Fig. 41.

10 A seven-piece dissection of a 14-gon and a nine-piece dissection of a 16-gon

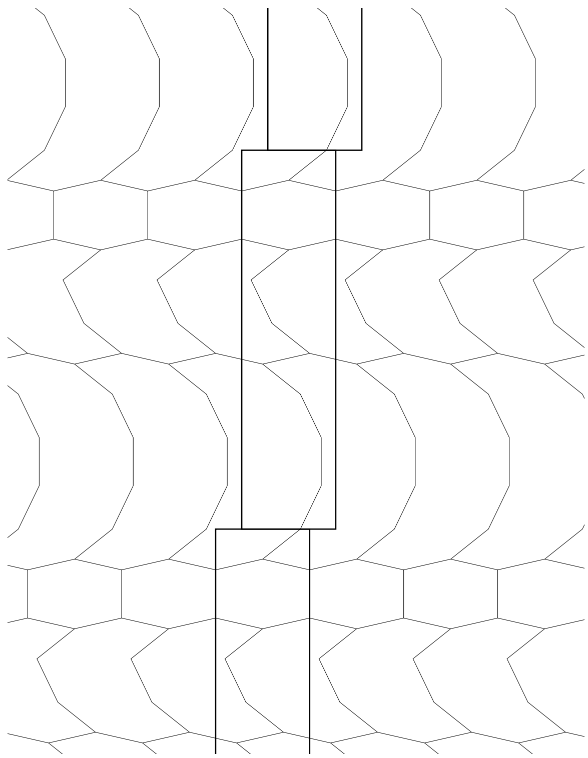


Figure 42: A tessellation based on a 14-gon, with the rectangle that leads to the dissection in Fig. 43.

It is known that $s(14) \leq 9$ and $s(16) \leq 10$ [21]. Figure 42 shows a tessellation of the plane based on a 14-gon, and a rectangle superimposed on it which leads to the seven-piece dissection shown in Fig. 43. Figures 44 and 45 play similar roles for the 16-gon.

These will be our final examples of a regular polygon to rectangle dissections. Other examples with larger numbers of sides may be found in the *Rectangle Dissections* section of [21], and in a projected sequel to the present work.

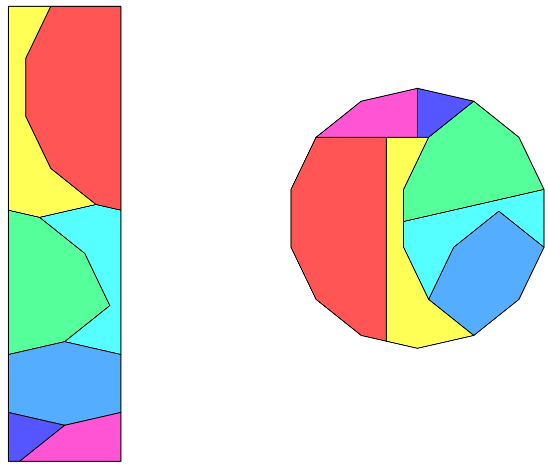


Figure 43: A seven-piece dissection of a 14-gon into a rectangle.

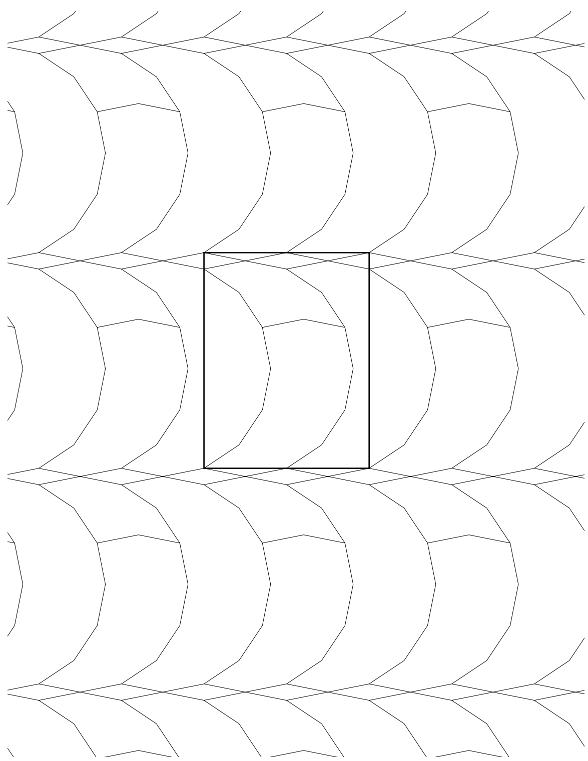


Figure 44: A tessellation based on a 16-gon, with the rectangle that leads to the dissection in Fig. 45.

11 Selected dissections of star polygons to rectangles

We give four examples of especially elegant dissections of star polygons to rectangles. These are taken from [21], where many further examples can be found.

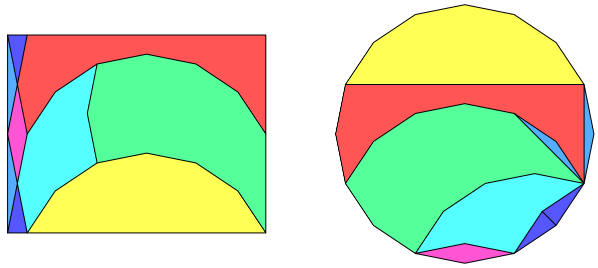


Figure 45: A nine-piece dissection of a 16-gon into a rectangle.

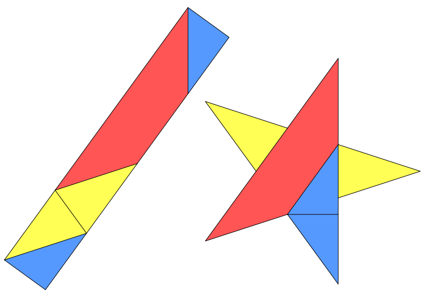


Figure 46: A five-piece dissection of a $\{5/2\}$ pentagram.

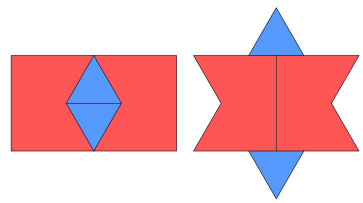


Figure 47: A four-piece dissection of a $\{6/2\}$ hexagram.

12 Three-piece dissections of a Greek cross

Most authors who study dissections of polygons include the Greek cross, so we briefly discuss it here. The classical four-piece dissection of a Greek cross into a square can be seen for example in [15, Fig. 9.1]. The dissected square has 4-fold rotational symmetry.

Three pieces seem to be the minimal number needed to form a rectangle from a Greek cross. The simplest three-piece construction cuts off two opposite arms from the cross and places them at the ends of the other two arms, forming a 1×5 rectangle. Eppstein [7] gives a three-piece dissection into non-convex pieces, shown in Fig. 50, and the database [21] gives another (Fig. 51), similar in spirit to the four-piece dissection into a square.

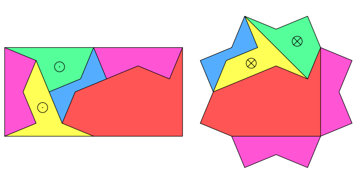


Figure 48: A six-piece dissection of an $\{8/2\}$ octagram. Two pieces must be turned over.

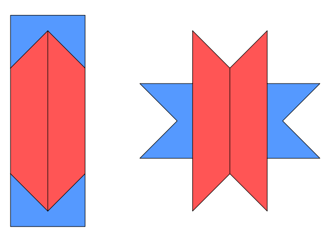


Figure 49: A four-piece dissection of an $\{8/3\}$ octagram.

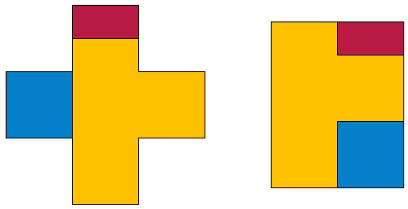


Figure 50: Eppstein's 3-piece dissection of a Greek cross into a rectangle [7].

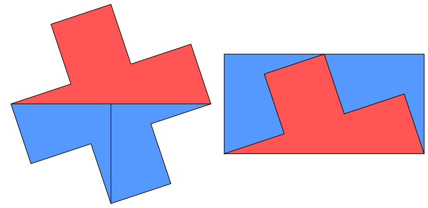


Figure 51: A 3-piece dissection of a Greek cross into a rectangle (from [21]).

13 Curved cuts are sometimes essential

We know of no theorem which will guarantee that polygonal cuts are sufficient to achieve $s(n)$ or $r(n)$. The following are three examples of other situations where it seems clear that minimal dissections can *not* be achieved using only polygonal cuts.

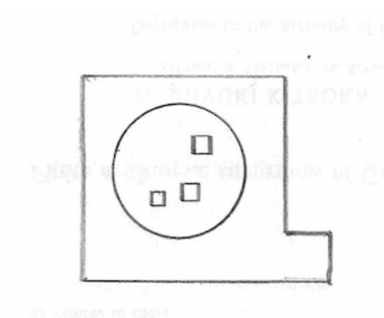


Figure 52: A two-piece dissection that can be accomplished by a single circular cut, but surely not by a polygonal cut. The polygon contains three small squares holes (Richard C. Schroeppel and Andy Latto).

1. Take a square with a smaller square attached to it, and cut out three small square holes at random positions in the interior. Call this figure A. For figure B, make a circular cut enclosing the three holes, and rotate the interior of the circle by a small random angle. This gives a two-piece dissection of A to B which surely cannot be accomplished with a single polygonal cut. This example was suggested by Richard C. Schroeppel and Andy Latto, (personal communication).

2. Figure 53 shows an example due to David desJardins (personal communication) of a three-piece dissection between two simply connected polygonal regions that appears to require a curved piece.

3. Greg Frederickson [8, Fig. 13.6] gives an example of 6-piece dissection of a hexagon into a hexagram which requires that two of the pieces be turned over. This can be modified to avoid turning the pieces over at the cost of adding an extra piece. But if a curved cuts is used, this can be accomplished without adding the extra piece, as shown in Fig. 54. We conjecture that a 6-piece hexagon to hexagram dissection that avoids turning pieces over cannot be constructed using only polygonal cuts.

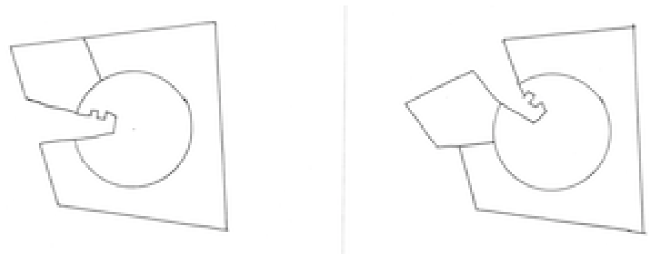


Figure 53: A three-piece dissection that appears to require a curved piece (David desJardins).

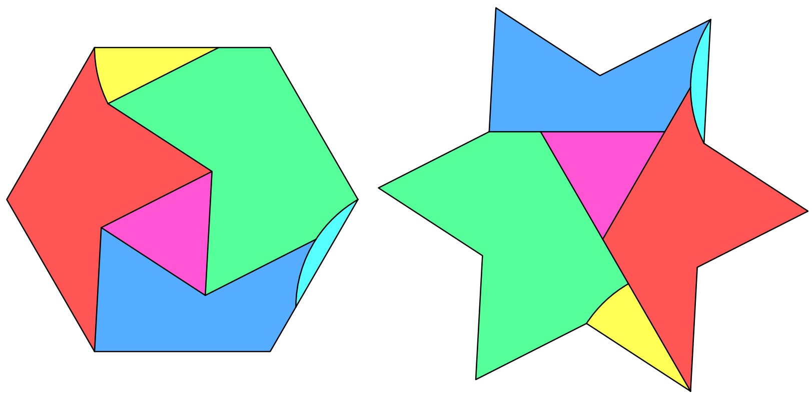


Figure 54: A 6-piece hexagon to hexagram dissection that avoids turning pieces over, but uses a curved cut.

4. In this regard, it is worth pointing out that a rotation by any desired angle that uses a single circular cut (see Fig. 55) can be accomplished by two square cuts and turning a piece over (Fig. 56).

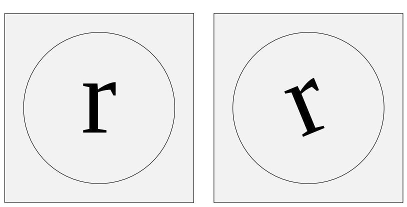


Figure 55: A rotation produced by a single circular cut ...

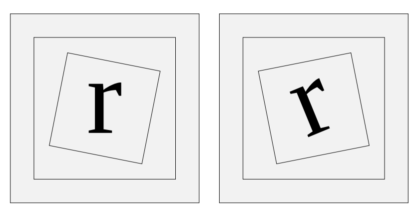


Figure 56: ... can also be achieved with two square cuts and turning a piece over.

14 Acknowledgments

Thanks to Adan Gsellman for telling us about his polygon to rectangle dissections, which were the seed that led to the present paper. Thanks also to David desJardins, Andy Latto, Richard C. Schroeppel, and Allan C. Wechsler for helpful comments. The writing of this paper depended heavily on Postscript, Latex, Tikz, Maple, WolframAlpha, and email.

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