On Dissecting Polygons into Rectangles

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Abstract

What is the smallest number of pieces that you can cut an n-sided regular polygon into so that the pieces can be rearranged to form a rectangle? Call it $r(n)$. The rectangle may have any proportions you wish, as long as it is a rectangle. The rules are the same as for the classical problem where the rearranged pieces must form a square. Let $s(n)$ denote the minimum number of pieces for that problem. For both problems the pieces may be turned over and the cuts must be simple curves. The conjectured values of $s(n), 3 \le n \le 12$, are $4, 1, 6, 5, 7, 5, 9, 7, 10, 6$. However, only $s(4) = 1$ is known for certain. The problem of finding $r(n)$ has received less attention. In this paper we give constructions showing that $r(n)$ for $3 \le n \le 12$ is at most $2, 1, 4, 3, 5, 4, 7, 4, 9, 5$, improving on the bounds for $s(n)$ in every case except $n = 4$. For the 10-gon our construction uses three fewer pieces than the bound for $s(10)$. Only $r(3)$ and $r(4)$ are known for certain. We also briefly discuss $q(n)$, the minimum number of pieces needed to dissect a regular n-gon into a monotile.

1 Introduction

Two polygons are said to be *equidecomposable* if one can be cut into a finite number of pieces that can be rearranged to form the other. Pieces may be turned over, and the cuts must be along simple plane curves. The Bolyai-Gerwien theorem from the 1830s states that any two polygons of the same area are equidecomposable, and the dissection can be carried out using only triangular pieces. Furthermore, the dissection can be carried out using only a straightedge and compass. Boltianskii [3] gives an excellent survey.

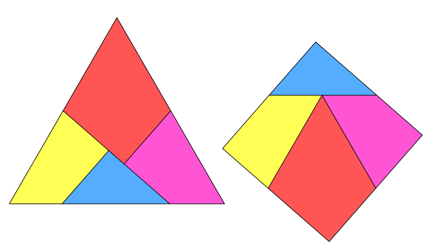


Figure 1: A 4-piece dissection of an equilateral triangle into a square.

A much-studied special case of this problem asks for the minimum number of pieces ($s(n)$, say) of any shape needed to dissect a regular polygon with n sides into a square of the same area. Despite its long history [2, 7, 9, 10, 11, 12, 13, 18, 19, 23, 26], surprisingly little is known about this problem. The best upper bounds currently known for $s(n), n = 3, 4, 5, \ldots, 16$, are shown in Table 1. The only exact value known appears to be the trivial result that $s(4) = 1$. Even the value of $s(3)$ is not known. The four-piece dissection of an equilateral triangle into a square shown in Fig. 1 is at least 120 years old (see the discussions in [2], [9, Ch. 12]), but there is no proof that it is optimal. The conjecture that it is impossible to dissect an equilateral triangle into three pieces that can be rearranged to form a square must be one of the oldest unsolved problems in geometry.

n	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$s(n) \leq$	4	1	6	5	7	5	9	7	10	6	11	9	11	10
$r(n) \leq$	2	1	4	3	5	4	7	4	9	5	10	7	10	9
$q(n) \leq$	1	1	2	1	3	2	3	2	4	3	4	3	5	4

Table 1: $s(n)$ (resp. $r(n)$, $q(n)$) is the minimum number of pieces needed to dissect a regular n-sided polygon into a square (resp. rectangle, monotile). Only the values of $s(4)$, $r(3)$, $r(4)$ and $q(n)$ for $n = 3, 4, 5, 6, 8, 10$ are known to be exact.

In the present article we consider a weaker constraint: what is the minimum number of pieces ($r(n)$, say) needed to dissect a regular polygon with n sides into a *rectangle* of the same area? The proportions of the rectangle can be anything you want. It is clear that $r(3) = 2$ and $r(6) \leq 3$ (Figs. 2, 3). (Surely it should be possible to prove that no two-piece dissection of a regular hexagon into a rectangle is possible?)

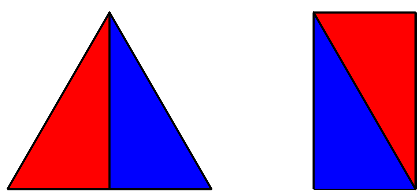


Figure 2: A 2-piece dissection of an equilateral triangle into a rectangle. One piece must be turned over.

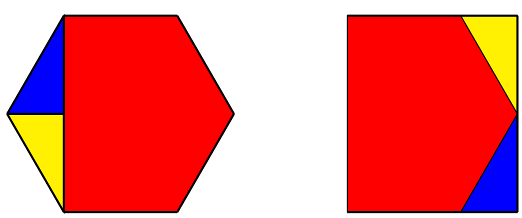


Figure 3: A 3-piece dissection of a regular hexagon into a rectangle.

Lindgren [18, 19] gives examples of regular n-gon to non-square rectangle dissections, but none have fewer pieces than the corresponding n-gon to square dissections. Frederickson [9, pp.

150-151] mentions that in 1926 H. E. Dudeney found a 4-piece octagon to rectangle dissection (probably that shown in Fig. 21).

In April 2023, before the current investigation was begun, the second author (G.A.T.)'s *Geometric Dissections* database [26] contained several examples of regular n-gon to rectangle dissections that had fewer pieces than the best square dissections known, including a 5-piece dissection of a pentagon, a 4-piece dissection of an octagon, a 6-piece dissection of a 10-gon, and a 5-piece dissection of a 12-gon. The database also contained the star polygon and Greek cross dissections shown below in §11 and §12.[1]

In May 2023, Adam Gsellman [15] wrote to N.J.A.S. enclosing dissections showing that $r(5) \le 4$, $r(7) \le 6$, and $r(8) \le 4$. As mentioned above, the third of these results was already known, and G.A.T. was aware that $r(5) \le 4$, although that result had not yet been mentioned in the literature, but the upper bound on $r(7)$ appeared to be new. We have since shown that $r(7) \le 5$ (see §4), but Gsellman's dissections of the pentagon and octagon are shown in §3 and §5.

Table 1 shows the current best upper bounds on $s(n)$ and $r(n)$ for $n \le 16$, although to avoid making the paper too long we shall say very little about the cases when $n > 12$.

The recent remarkable discovery [25] of a single polygon, or *monotile*, that tiles the plane, but can only do so in a non-periodic way, reminded us of a question asked by Grünbaum and Shephard in 1986 [14, §2.6]: what is the minimum number of pieces needed to dissect a regular n-gon into a monotile that tiles the plane (allowing periodic tilings)? Call this number $q(n)$. Of course squares and rectangles themselves are monotiles, so $q(n) \le r(n) \le s(n)$. We have included bounds on $q(n)$ in Table 1. All of $s(n)$, $r(n)$, and $q(n)$ are fundamental geometric quantities, with $q(n)$ perhaps the most basic of the three. We will give some examples of these monotiles in Remark 1.6, including a dissection of a 9-gon which improves on Grünbaum and Shephard's.

The paper is arranged as follows. The remainder of this section gives some general remarks about our dissections. Section 2 defines some parameters and coordinates that will be generally useful for our dissections. Subsequent sections deal in turn with pentagons, heptagons, octagons, up through 16-gons, followed by sections on star polygons and the Greek cross. Finally, Section 13 gives some examples of dissections where curved cuts appear essential if one wishes to minimize the number of pieces. Dissections given without attribution are believed to be new.

Remark 1.1. *Certifying the dissections.*

We have attempted to give detailed descriptions of most of the dissections in the main body of the paper (§3-§10), enough at any rate to convince the reader that the dissections are correct. If the dissection begins by cutting up the regular polygon, for example, we have to make sure that the rearranged pieces form a proper rectangle. The pieces must not overlap, there can be no holes; when pieces fit together at a vertex, the sum of the angles must be 2π at an interior point, or π or $\pi/2$ at a boundary point, and so on.

In simple cases the the correctness can be checked "by hand", like solving a jigsaw puzzle. The first pentagon dissection, in §3.1, is an example.

Many of our dissections were obtained by one of the standard strip or superposition constructions. There are a great many versions of these constructions, and they are described in

[1] [26] now includes all the dissections mentioned in this article.

most of the books on the subject, and in the *Methods* section of [26], so we shall not say much about them here.[2]

A simple example of a superposition, used to dissect a polygon A into a polygon B, is to overlay a strip tiled with pieces from A and a second strip tiled with pieces from B. Then with luck the intersection of the two strips will provide the desired dissection (see for example [9, Ch. 11], [18, Chaps. 2-5].) Since for our problem we can assume B is a rectangle, we can often dispense with the second strip. All we need then is a strip tiled with pieces from A. We obtain the dissection by cutting out a rectangle of the correct length from the strip. Examples are shown in Figs. 18 and 26.

If a dissection is obtained by one of these standard constructions, it can generally be assumed to be correct. However, one must be careful: with complicated strips like those shown in sections §6 onwards, it is easy to be mistaken about points coinciding, or when triangular regions shrink to a point (in order to save a piece).

For this and other reasons, we have therefore tried to give *ab initio* descriptions of the dissections. In most cases we are able to give a straightedge and compass construction, and to sketch a proof that it is correct.

Remark 1.2. *Straightedge and compass constructions.*

Given an initial drawing of a regular n-gon, our dissections can be constructed using only a straightedge and compass. That is, there is no need for a ruler: the construction does not require locating a point which is at some arbitrary irrational distance from another point.[3] We need to be given the initial n-gon, since, for example, a regular heptagon cannot be obtained with only a straightedge and compass.

Besides its aesthetic appeal, the advantage of a straightedge and compass construction is that it enables us to give explicit coordinates for every vertex in the construction.[4] We usually start from from the vertices (2.1) of the n-gon, and every subsequent vertex is then determined. In §7 we start from the rectangle, which we assume has width $\sqrt{5}$ and height $\cos(\pi/10)$, and again all subsequent vertices are determined.

Although in theory we could find exact expressions for all the coordinates in a straightedge and compass construction in this way, the expressions rapidly become unwieldy. In practice we have found it better to use computer algebra systems such as WolframAlpha and Maple to guess expressions for the coordinates based on 20-digit decimal expansions, knowing that they could be justified if necessary. We shall see examples of this is §6.

Remark 1.3. *Turning pieces over.*

Although the definitions of $s(n)$ and $r(n)$ allow pieces to be turned over, this is deprecated by purists. Fortunately all the dissections $s(n)$ and $r(n)$ for $3 \le n \le 16$ mentioned in Table 1 can be accomplished without turning pieces over, with the single exception of $r(3)$ (see Fig. 2), which seems to require three pieces if turning over is forbidden.

Another example where turning pieces over appears to be essential to achieve the minimum number of pieces is the seven-piece dissection of $\{6\}$ into $\{8\}$ given in [26].

[2] For the mathematical theory underlying these constructions, see [1, 21, 24].

[3] Hadlock [16] contains an excellent introduction to straightedge and compass constructions.

[4] Although the Bolyai-Gerwien theorem guarantees that a straightedge and compass dissection of a regular n-gon to a square exists, we don't know that this is true for a dissection with the minimum number of pieces.

Remark 1.4. *Convex pieces.*

The dissections in Figs. 1-4, 11, 12, 21, 43 and 50 use only convex pieces. Other things being equal, we prefer convex pieces, of roughly equal size, that do not need to be turned over. The primary goal however is always to minimize the number of pieces.

Remark 1.5. *Improving on classic dissections.*

We were surprised to find that $r(8)$ is apparently less than $s(8)$, and $r(12)$ apparently less than $s(12)$, since the best octagon to square[5] and 12-gon to square constructions are so striking (Figs. 4, 5). One feels that they could not possibly be improved on. Yet if we only want a rectangle, there is a four-piece dissection of the octagon that has essentially the same symmetry as Fig. 4, as we shall see in §5. Likewise, for the 12-gon, we can save a piece if we only want a rectangle, at the cost however of giving up all symmetry (see §9).

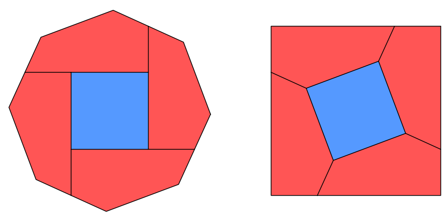


Figure 4: A 14-th century 5-piece regular octagon to square dissection, with cyclic 4-fold symmetry in both the octagon and the square.

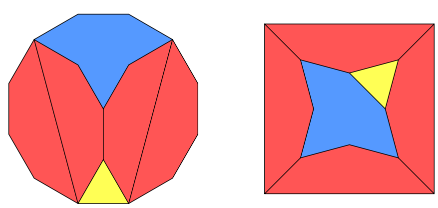


Figure 5: Lindgren's astonishing 6-piece 12-gon to square dissection, with mirror symmetry [18, Ch. 9].

Remark 1.6. *Dissecting a regular n-gon into a monotile.*

As mentioned above, the problem of finding $q(n)$ is discussed in Section 2.6 of Grünbaum and Shephard [14], where a dissection attaining $q(n)$ is called a *minimal dissection tiling*. Figure 2.6.1 of [14] shows dissections achieving $q(5) = q(8) = q(10) = 2$ and conjectured solutions for $n = 7, 9, 12$, with a reference to Lindgren [19]. Our Figures 6, 7, 8, and 9 show examples of monotiles for $n = 5, 7, 9$, and 10. Many further examples with $n \leq 17$ can be seen in OEIS entry A362938.

[5]The octagon in the well-known Chase Bank logo is different from the octagon in Fig. 4. The Chase octagon has a square surrounded by four trapezoids, whereas Fig. 4 has a square surrounded by four pentagons. The pieces in the Chase logo can be rearranged to form a rectangle, but not a square.

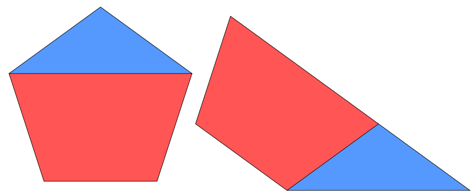


Figure 6: A two-piece dissection of a pentagon into a monotile, illustrating $q(5) = 2$.

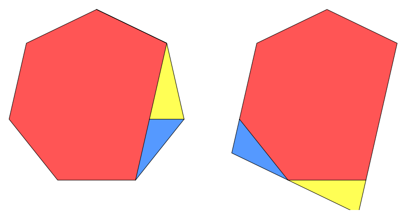


Figure 7: A three-piece dissection of a heptagon into a monotile, illustrating $q(7) \le 3$ [18].

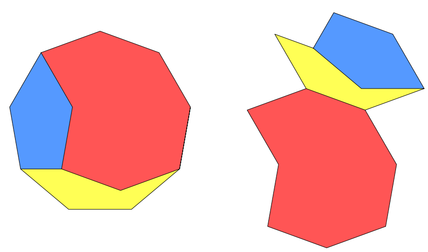


Figure 8: A three-piece dissection of a 9-gon into a monotile, illustrating $q(9) \le 3$. This improves on the dissection in [14, Fig. 2.6.1].

By generalizing the constructions for $n = 8, 10, 12, 14, \ldots$, we can show that $q(2t) \le \lfloor t/2 \rfloor$ for $t \ge 2$, and it may even be true that $q(2t) = \lfloor t/2 \rfloor$ holds for all $t \ge 2$. No similar conjecture is known for $q(2t+1)$.

Remark 1.7. *Lower bounds.*

It seems to be difficult to obtain nontrivial lower bounds on any of $s(n)$, $r(n)$, or $q(n)$.[6] References [5, 17] do give some lower bounds, but only for more restricted classes of dissections (only allowing polygonal cuts, or alternatively what are called "glass cuts"). Two negative results we do know of are the easily-proved fact[7] that $s(3)$ cannot be 2, and the nontrivial

[6] G.A.T. conjectures that $1/4 \le q(n)/n \le r(n)/n \le s(n)/n$ for $n > 10$, and that all three of these ratios approach $1/4$ as n increases.

[7] An equilateral triangle of area 1 has side-length $2/3^{1/4} = 1.519$, which won't fit into a square of area 1. So each vertex of the triangle must be in a different piece of the dissection.

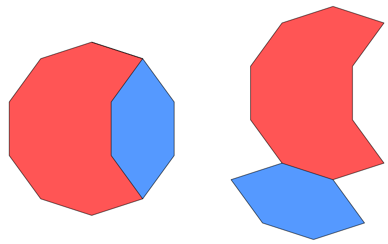


Figure 9: A two-piece dissection of a 10-gon into a monotile, illustrating $q(10) = 2$.

theorem [6] that a circular disk cannot be dissected into a polygon.

We cannot resist mentioning that the latter question has been in the news recently, because of progress on the problem of dissecting a circular disk into a square *if fractal pieces are allowed* [20]. The new construction involves at most 10^{200} pieces.

Remark 1.8. *The OEIS entry.*

The best upper bounds currently known for $s(n)$, $r(n)$, and $q(n)$ for n up to around 16 or 20 are listed in the *On-Line Encyclopedia of Integer Sequences* (or *OEIS*) database [22] as sequences A110312, A362939, and A362938, respectively. This is an exception to the usual OEIS policy of requiring that all terms in sequences must be known exactly, but these sequence are included because of their importance and in the hope that someone will establish the truth of some of the conjectured values.

Remark 1.9. *Applications*

These polygon to rectangle dissections have potential applications to lossless source coding (cf. [24]). If a source (a lens, perhaps) repeatedly produces an output which is a point in a regular 12-gon, say, then the dissection could be used to map the point to a more convenient pair (x, y) of rectangular coordinates (compare Figs 43 and 44).

Remark 1.10. *For further information.*

The database [26] is the best reference for drawings of dissections mentioned in Table 1 but not included in the present article.

Remark 1.11. *Notation*

For up to eight sides we will use the names triangle, ..., hexagon, heptagon, and octagon, but for nine or more sides we will say 9-gon, 10-gon, ... The symbol $\{n\}$ denotes a regular n-sided polygon, and $\{n/m\}$ is a regular star polygon (cf. [4]). $L_{k,n}$ is the length of the chord joining two vertices of $\{n\}$ that are k edges apart (2.3). Decimal expansions have been truncated not rounded.

2 Regular polygons: coordinates and metric properties

In later sections we will usually begin with a regular n-sided polygon, with $n \geq 3$, having sides of length 1, with the goal of cutting it into as few pieces as possible which can be rearranged to form a rectangle.

We often take the polygon to have center $P_0 = (0,0)$ at the origin:, and to have vertices $P_1, P_2, \ldots, P_n$ labeled counterclockwise, starting with P_1 at the apex of the figure (Fig. 10). The angle subtended at the center by an edge is $2\pi/n$, and we define $\theta = \pi/n$ and $\phi = \pi/2 - \theta = \frac{n-2}{2n}\pi$, which will be the principal angles used in the formulas.

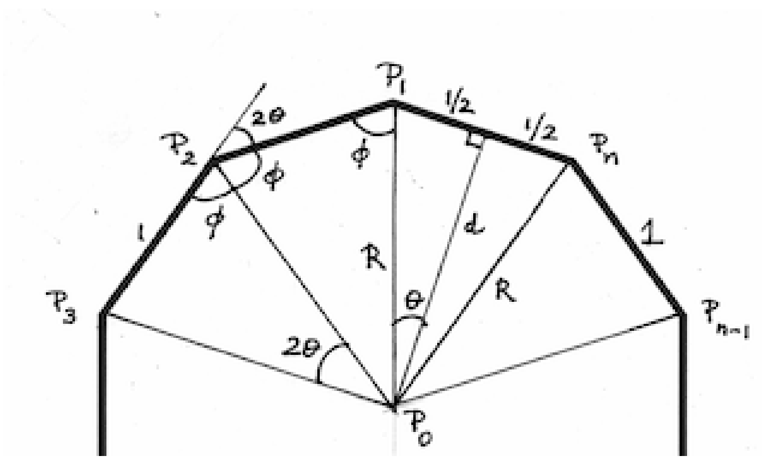


Figure 10: Vertices and principal angles and distances for a regular n-gon.

Since the sides have length 1, the radius of the polygon is $R = \frac{1}{2\sin\theta}$, and the distance from the center to the midpoint of an edge is $d = \frac{1}{2\tan\theta}$. The vertices have coordinates

$$
\begin{aligned}
P_1 &= (0, R)\,, \\
P_k &= (-R\sin(2(k-1)\theta), R\cos(2(k-1)\theta))\,, \\
P_{n+2-k} &= (R\sin(2(k-1)\theta), R\cos(2(k-1)\theta))\,,
\end{aligned}
\tag{2.1}
$$

for $k = 2, \ldots, \lfloor (n+2)/2 \rfloor$. The polygon has area

$$\frac{nd}{2} = \frac{n}{4}\cot\frac{\pi}{n}. \tag{2.2}$$

Many dissections involve a cut along a chord $P_i P_{i+k}$ joining two vertices. Assuming $n \geq 3$, $k \geq 0$, let $L_k = L_{k,n}$ denote the length of the chord joining two vertices that are k edges apart in $\{n\}$ and are in the same semicircle. Then $L_0 = 0, L_1 = 1$, $L_k = L_{k-2} + 2\cos(k-1)\theta$, and for $t \geq 1$,

$$L_{2t} = 2\sum_{j=0}^{t-1}\cos(2j+1)\theta\,, \quad L_{2t+1} = 1 + 2\sum_{j=1}^{t}\cos 2j\theta\,. \tag{2.3}$$

3 Four-piece dissections of a pentagon

In 1891 Robert Brodie discovered a six-piece dissection of a regular pentagon into a square ([9, p. 120], [18, Fig. 3.1], [26]), and there has been no improvement since then, so it seems likely that $s(5) = 6$. In this section we give four different four-piece dissections of a regular pentagon into a rectangle, showing that $r(5) \le 4$. Rather surprisingly, this result does not seem to have been mentioned in the literature before now. The fact that there are at least four ways to get $r(5) \le 4$ makes us wonder if $r(5)$ might actually be equal to 3.

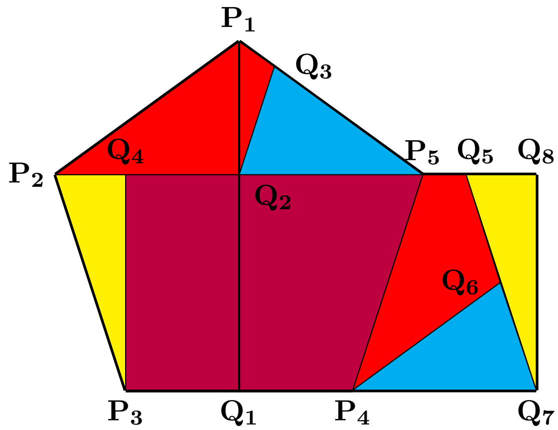


Figure 11: The pentagon $P_1P_2\dots P_5$ is cut into four convex pieces which can be rearranged to form the rectangle $Q_4P_3Q_7Q_8$.

The first two four-piece dissections (§3.1, §3.2) were found by the authors (although they can hardly be new), and have the property that the pieces are convex; the other two (§3.3, §3.4) are due to Adam Gsellman [15].

We use the notation introduced in the previous section (taking $n = 5$). The pentagon has vertices $P_1, \dots, P_5$ with sides of length 1. The key angles are $\theta = \angle P_1P_2P_5 = 36°$, $\phi = 54°$, $\angle P_2P_1P_5 = 2\phi$, and $\angle P_2P_3Q_4 = \theta/2$. We note the values of

$$\begin{aligned} \sin 36° &= \cos 54° = \sqrt{\frac{5-\sqrt{5}}{8}}, \qquad & \cos 36° &= \sin 54° = \frac{\sqrt{5}+1}{4}, \\ \sin 72° &= \cos 18° = \sqrt{\frac{5+\sqrt{5}}{8}}, \qquad & \cos 72° &= \sin 18° = \frac{\sqrt{5}-1}{4}. \end{aligned} \tag{3.1}$$

3.1 Pentagon #1

To construct this dissection (see Fig. 11) we draw a perpendicular from P_1 to the mid-point Q_1 of the opposite side, and draw the chord $P_2 - P_5$. Let Q_2 be the intersection of these two lines, and place Q_3 on $P_1 - P_5$ so that $Q_2P_5Q_3$ is an isosceles triangle. Finally, draw a perpendicular $P_3 - Q_4$ from P_3 to $P_2 - P_5$.

To form the rectangle, we first rotate the quadrilateral $P_1P_2Q_2Q_3$ by 36° and move it to $P_5P_4Q_6Q_5$, then the triangle $Q_2P_5Q_3$ is moved to $Q_6P_4Q_7$, and the triangle $P_2P_3Q_4$ is moved horizontally to $Q_5Q_7Q_8$. The two isosceles triangles (blue) have long sides of length $L_{2,5}/2 = (1+\sqrt{5})/4$ and base $1/2$. The two right triangles (yellow) have sides of lengths $\sin\theta$, $\cos\theta$, and 1.

To prove that this dissection is correct, we must check that, after the rearrangement, the result is indeed a rectangle, with area equal to that of the pentagon. In particular, we must check that the points $P_2, Q_4, Q_2, P_5, Q_5, Q_8$ are collinear, as are P_3, P_4, Q_7, and Q_5, Q_6, Q_7, that Q_6 bisects $Q_5 - Q_7$, and also that the difference between the x-coordinates of P_2 and P_3 is equal to the difference between the x-coordinates of Q_5 and Q_8. Since we have complete information about the points, these checks are easily carried out. The pentagon has area $(5^{3/4}/(8\sqrt{2}))(\sqrt{5}+1)^{3/2}$, and we can check that this is equal to $|P_3Q_4|\cdot|P_3Q_7|$. This completes the proof of the dissection.

It is worth pointing out that the trapezoid $P_5P_4Q_7Q_5$ is symmetric about its vertical axis.

We chose this relatively simple example to illustrate the steps needed to prove that a dissection is correct. In later examples we will just give the basic information needed for the proof and leave the detailed verification to the reader.

3.2 Pentagon #2

This is very similar to the first dissection. Two of the pieces are the same, only now the rest of the pentagon is divided into two pieces that are reflections of each other. The trapezoid from Fig. 11 has moved to the center of the rectangle.

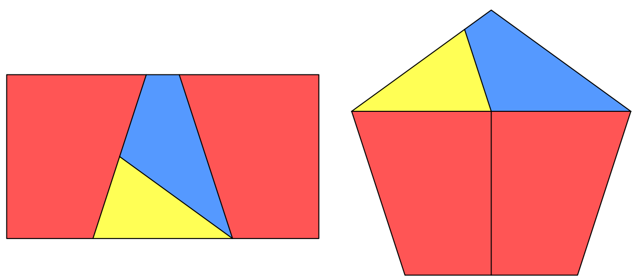


Figure 12: Similar to the first pentagon dissection, only now the pieces are more nearly equal in size.

Although logically we are dissecting the polygon *into* a rectangle, many of our colored illustrations have the rectangle on the left of the picture, as in Figs. 12, 19, 21, etc. This is because of the convention in [26] of starting with the figure having the smaller number of vertices. Also, in the case of strip or tessellation constructions, one often proceeds from the rectangle to the polygon.

3.3 Pentagon #3

The first of Adam Gsellman's four-piece dissections of a pentagon is shown in Fig. 13. We do not know how Gsellman discovered it, but we have found that it can be obtained by a simple slide construction. (It can also be obtained from a strip dissection.)

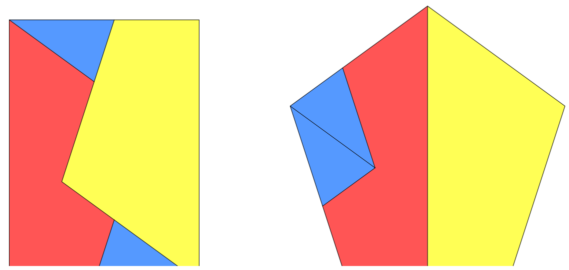


Figure 13: Gsellman's first pentagon dissection.

Figure 14: A construction that produces the dissection in Fig. 13 (see text for details).

Cut the pentagon down the middle into two quadrilaterals A and B, reflect A in a vertical mirror, and rotate B by 180° (see Fig. 14). Now slide the pieces towards each other until they overlap in a parallelogram whose diagonal is equal in length to the gap in the top (and the bottom) edge. Cut the parallelogram in the A piece into two equal isosceles triangles which can be rotated to complete the rectangle.

For the proof that this dissection is correct we label the points as in Fig. 15. Q_1 is the midpoint of the side P_3P_4. The key parameters are $\theta = \pi/5$, $R = 1/(2\sin\theta)$, and $d = R\cos\theta = (\sqrt{5}+1)^{3/2}/(4\sqrt{2}\,5^{1/4})$. The pentagon has height $h = R + d = (5^{1/4}/(4\sqrt{2}))(\sqrt{5}+1)^{3/2}$. This is also the height of the final rectangle, which (since we know the area) has width $w = \sqrt{5}/2$.

We label the vertices of the rectangle $a_1, \ldots, a_4$, and let $b_1, \ldots, b_5$, a_5 denote the points in the center of the figure. Finally, let s and b denote the side and base of the small isosceles triangles.

Note that $a_1b_1b_3a_4$ is a rotated copy of $Q_1P_4P_5P_1$, and $a_2a_3b_5a_5$ is a reflected copy of $P_1P_2P_3Q_1$. In particular, $|a_1b_1| = 1/2$, so $s = w - 1/2 = (\sqrt{5}-1)/2$. The base of the isosceles triangle is therefore $b = (3-\sqrt{5})/2$. This implies $s + b = 1$, and we can now check that all the pieces fit together correctly.

3.4 Pentagon #4

Gsellman's second pentagon dissection is shown in Fig. 16. It can be obtained by a similar slide construction.

There is a third version of Gsellman s dissection which has the zigzag cut through the

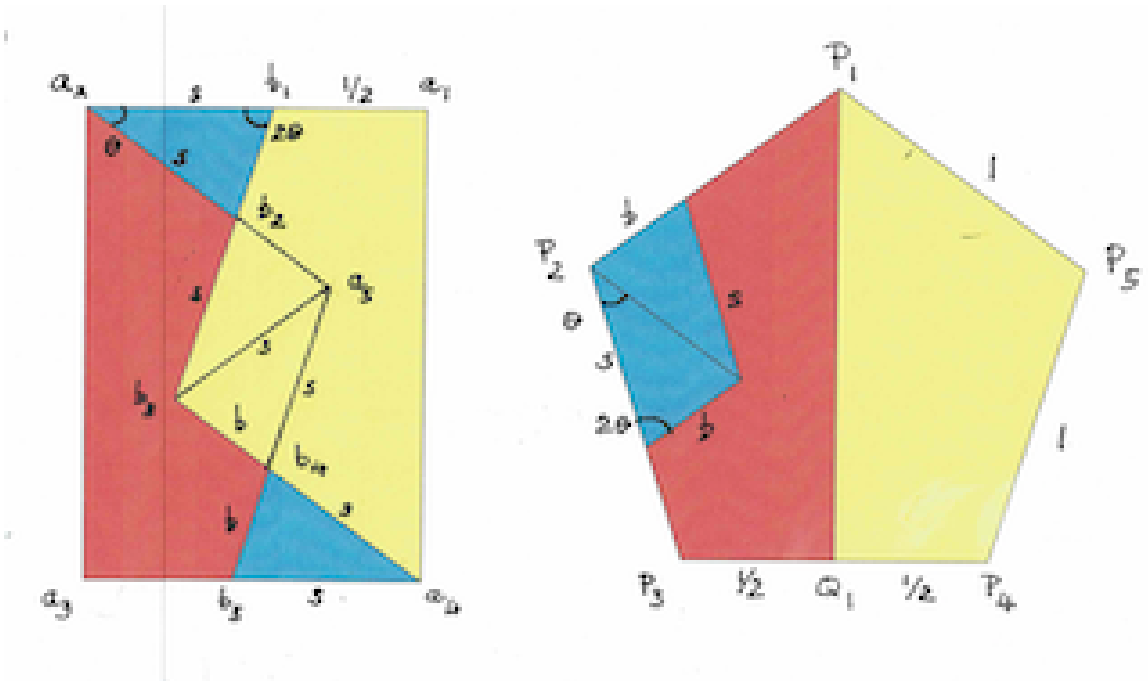


Figure 15: Names for points and lengths in dissection of Fig. 13.

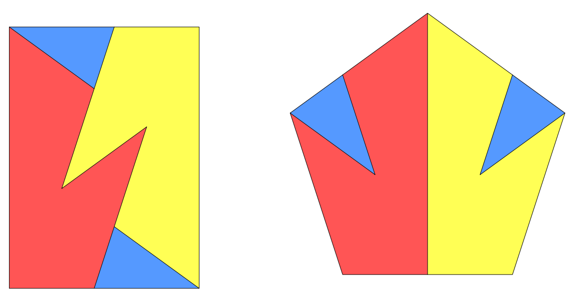


Figure 16: Gsellman's second pentagon dissection.

center of the rectangle going the other way (Fig. 17). These are elegant dissections, but all three have the slight defect of requiring a piece to be turned over.

4 A five-piece dissection of a heptagon

Even the great master Harry Lindgren [18] could only show that $s(7) \leq 9$, but around 1995 G.A.T. found a seven-piece dissection of a heptagon into a square, and it is reasonable to conjecture that indeed $s(7) = 7$. This dissection is described in [9, pp. 128–129] and [26].

For the rectangle problem, we start from a heptagon strip (shown in Fig. 18) that is a modification of a strip used in the square case (compare [9, Fig. 11.28]). By cutting a

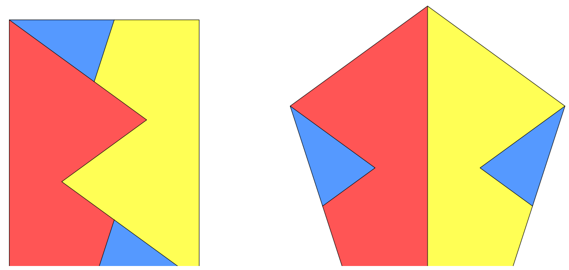


Figure 17: Another version of Fig. 16.

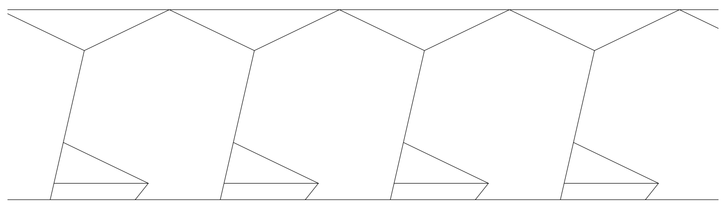


Figure 18: A heptagon strip used for the heptagon to rectangle dissection.

rectangle from this strip, we obtain a five-piece dissection of a heptagon to a rectangle, shown in Fig. 19.

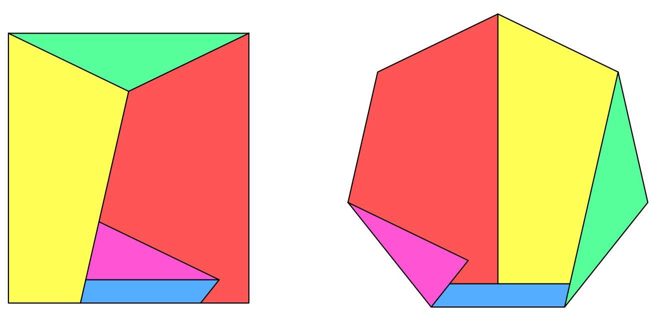


Figure 19: A five-piece dissection of a heptagon to a rectangle.

Rather than following the path by which it was discovered, we will construct this dissection directly from the heptagon, something that can be done using only a straightedge and compass. The vertices of the 7-gon are labeled $P_1, \ldots, P_7$ (see Fig. 20). Drop a perpendicular from P_1 to the midpoint Q_6 of P_4P_5, and draw chords P_3P_5, P_4P_7, and P_5P_7. Let Q_2 be the intersection of P_3P_5 and P_4P_7, and let Q_3 be the midpoint of P_4Q_2. Draw a line $Q_3Q_5Q_4$ through Q_3 parallel to P_4P_5.

Using these these lines as guides, we get the five pieces by cutting $P_7P_5P_6P_7$; $P_1Q_5Q_4P_7P_1$; $P_1P_2P_3Q_2Q_3Q_5P_1$; $P_3P_4Q_2P_3$; and $Q_3P_4P_5Q_4Q_3$.

These pieces can then be rearranged to form the rectangle as shown in Fig. 19, left.

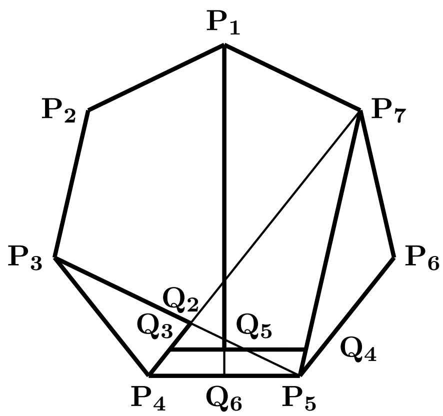


Figure 20: Heptagon dissection (Fig. 19, right) showing labels for points.

To help anyone who wishes to verify the correctness of this dissection, we list some key angles and lengths. We set $\theta = \pi/7$ and note that $\cos(\theta) = .0.9009\ldots$ has minimal polynomial $8x^3 - 4x^2 - 4x + 1$. Then $\angle P_1P_2P_3 = 5\theta$, $\angle P_7P_4P_5 = 2\theta$, $\angle P_4P_3P_5 = \angle P_5P_7P_6 = \theta$, and $\angle P_4P_5P_7 = 4\theta$. The chord P_7P_5 has length $L_{2,7} = 2\cos\theta$, and $|P_4Q_3| = |Q_3Q_2| = 1/2 - (\sec\theta)/4$. The trapezoidal piece has cross-section $|Q_5Q_6| = (4\cos\theta - 3)/(8\sin\theta)$. The rectangle has height $7/(8\sin\theta)$ and width $2\cos\theta$.

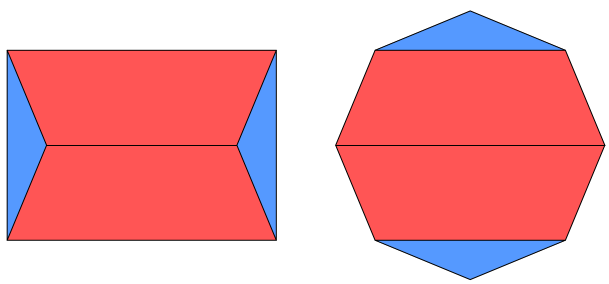


Figure 21: The classic four-piece octagon to rectangle dissection [26].

5 Four-piece dissections of an octagon

It appears that five pieces are needed to dissect an octagon into a square (see Fig. 4), whereas four pieces are enough if we only want a rectangle (Fig. 21). The former has cyclic four-fold symmetry, while the latter has the symmetry of a Klein 4-group.

Adam Gsellman [15] has found two other four-piece dissections, shown in Figs. 22-24. The description in Fig. 23 is self-explanatory (the angles are multiples of $\pi/8$ and the only irrationality needed is $\sqrt{2}$). The second dissection (Fig. 24) is very similar.

These two dissections are admittedly less elegant than that in Fig. 21, and require pieces to be turned over, but we include them because, as the example in Fig. 28 shows, complicated

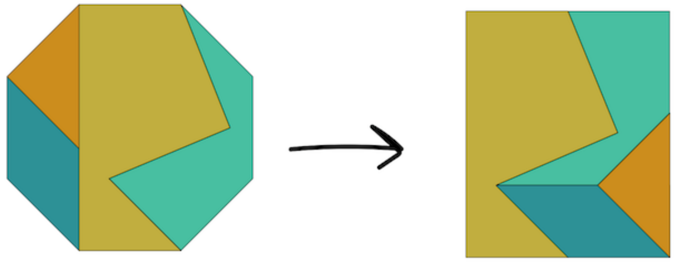


Figure 22: Gsellman's first four-piece dissection of an octagon. See Fig. 23 for details.

non-convex dissections may be needed to get the minimum number of pieces.

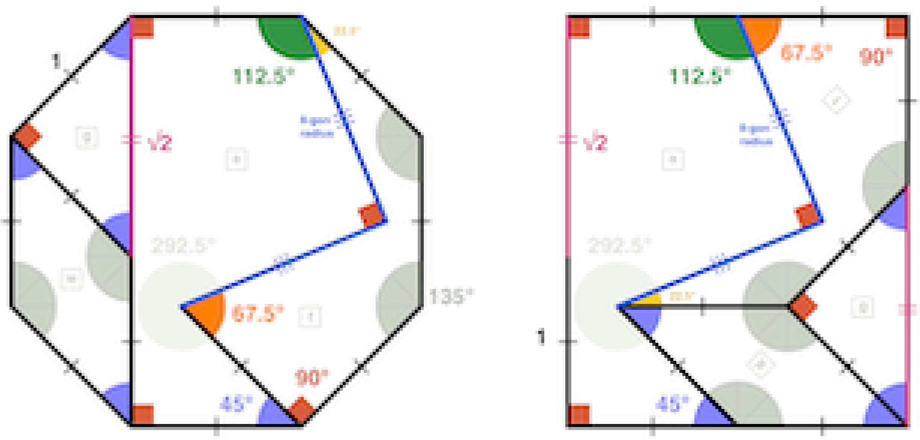


Figure 23: Gsellman's detailed description of the dissection in Fig. 22.

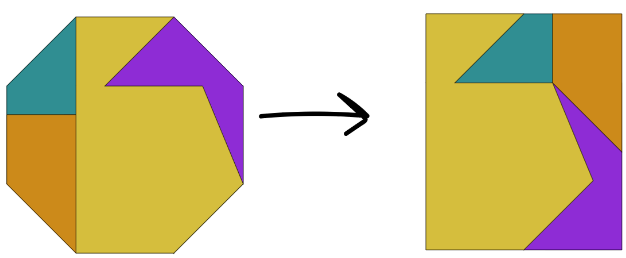


Figure 24: Gsellman's second four-piece dissection of an octagon.

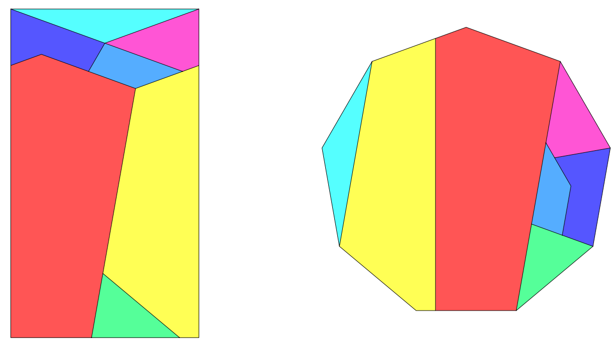


Figure 25: A seven-piece dissection of a 9-gon into a rectangle.

6 A seven-piece dissection of a 9-gon

Figure 25 shows a seven-piece dissection of a 9-gon into a rectangle, which is two fewer pieces than the best dissection into a square presently known. It was obtained from the strip dissection of a 9-gon shown in Fig. 26, by cutting a rectangle from the strip.

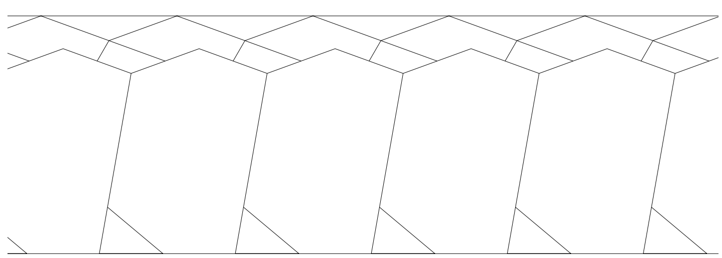


Figure 26: The strip dissection of a 9-gon which led to Fig. 25.

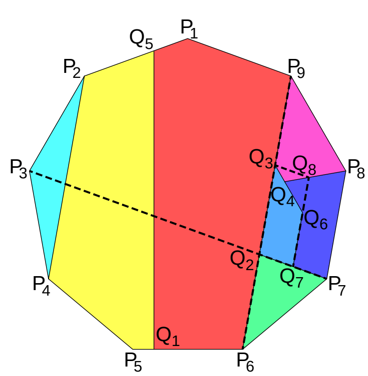


Figure 27: Labels for points in the 9-gon.

As with the heptagon, we will construct this dissection directly from the 9-gon, using only a straightedge and compass.

The vertices of the 9-gon are labeled $P_1, \ldots, P_9$ (see Fig. 27), and we use the coordinates established in §2. Also $\theta = \pi/9$, $C_1 = \cos\theta$ has minimal polynomial $8x^3 - 6x - 1$, and $S_1 = \sin\theta$ has minimal polynomial $64x^6 - 96x^4 + 36x^2 - 3$. (For this reason, we write our expressions as rational functions of $\cos\theta$, with at most linear terms in $\sin\theta$.)

To obtain the dissection we first draw chords $P_2 - P_4$, $P_3 - P_7$, and $P_6 - P_9$. Then Q_2 is the intersection of $P_3 - P_7$ and $P_6 - P_9$, Q_3 is the midpoint of $Q_2 - P_9$, and Q_7 is the midpoint of $Q_2 - P_7$. Draw a line segment $Q_7 - Q_6$ of length 1/2 parallel to $P_7 - P_8$, join Q_3 to Q_6, and locate Q_4 at the intersection of $Q_3 - Q_6$ and a perpendicular drawn from P_8 to the midpoint of $P_3 - P_4$. Finally Q_5 is on $P_1 - P_2$ at distance $|Q_4Q_6|$ from P_1, and $Q_5 - Q_1$ is perpendicular to $P_5 - P_6$.

To assist in the analysis we define a further point Q_8 at the intersection of $Q_7 - Q_6$ (extended) and $Q_4 - P_8$. Then $Q_4Q_6Q_8$ is an isosceles triangle and $Q_3Q_2Q_7Q_8$ is a parallelogram.

The seven pieces of the dissection can now be found by making the cuts indicated by the colored regions on the right of Fig. 25. To complete the proof that the dissection is correct, we must verify that the pieces can be rearranged to form the rectangle on the left of Fig. 25. We will not take the space to do that here, but to assist the reader we give two key lengths. The length

$$|Q_2Q_7| = |Q_2P_7|/2 = |Q_4Q_6| = |Q_4Q_8| = |Q_8Q_3| = |P_1Q_5| = |Q_2Q_3| - \frac{1}{2} = \frac{C_1}{2C_1+1} = 0.3263\ldots \tag{6.1}$$

plays a central role, as does $|P_8Q_4| = 3/(8S_1(C_1+1)) = 0.5652$. The rectangle has width $2C_1 = 1.8793\ldots$ and height $9/(8S_1) = 3.2892\ldots$.

Some of the equalities in (6.1) are by no means obvious. They do not follow directly from the geometry, but depend on the fact that $\cos\theta$ satisfies a cubic equation. As discussed in Remark 1.2, we can find exact expressions for the coordinates of the points. We *could* do this by solving the appropriate equations, using a computer algebra system such as Maple, but we have found it a lot easier to use another computer algebra system, WolframAlpha, and ask it to find the coordinates for us.

For example, the first step in finding the present dissection is to find Q_2. Using Maple, and working to 20 decimal places, we find that

$$Q_2 \;=\; (0.65270364466613930216, -0.50771330594287249271)\,.$$

We now ask WolframAlpha to express these two numbers in terms of $\cos\theta$, $1/\cos\theta$, $\sin\theta$, and $1/\sin\theta$. The result (setting $C_1 = \cos\theta$, $S_1 = \sin\theta$) is

$$Q_2 \;=\; \left(\frac{2C_1}{2C_1+1},\; -(2S_1 - \frac{1}{S_1} + \sqrt{3})\right).$$

We do this for all the points. Another example is

$$\begin{aligned} Q_8 \;&=\; (1.10286853195244320 95, 0.19446547835755153996) \\ &=\; \left(\frac{C_1}{2} + \frac{1}{8C_1} + \frac{1}{2}.\; \frac{1}{8S_1} - \frac{S_1}{2}\right). \end{aligned} \tag{6.2}$$

The verification of the equalities in (6.1) is then a routine calculation (using Maple's `simplify(..., trig);` command).

7 A four-piece dissection of a 10-gon

In the final appendix ("Recent Progress") to his 1964 book [18], Lindgren gave a new strip based on the 10-gon (shown in Fig. 29 below), and used it to obtain several new dissections, including an eight-piece dissection of a 10-gon to a square, and a seven-piece dissection to a golden rectangle. As Frederickson reports in [9, Ch. 11], G.A.T. was then able to show that in the dissection to a square, two of Lindgren's pieces could be merged, leading to a seven-piece dissection to a square, still the record. This dissection is also described in the *Variable Strips* section of [26]. If we draw vertical lines across Lindgren's strip (without changing it), we obtain a five-piece dissection of a 10-gon to a (non-golden) rectangle, as shown in Fig. 30. There is a small range of possibilities for the positions of these vertical lines. In Fig. 30 they have been placed in the middle of their range, in order to obtain the most symmetric dissection.

Remarkably, if the goal is only to obtain a rectangle, it is possible to modify Lindgren's strip (Fig. 30) to get a four-piece dissection. The modified strip is shown in Fig. 31, and the dissection itself in Fig. 28. To go from Fig. 30 to Fig. 31 we merge the two right-most pieces of the rectangle, forming a church-shaped piece, and compensate by dividing the large piece into two by a zig-zag cut.

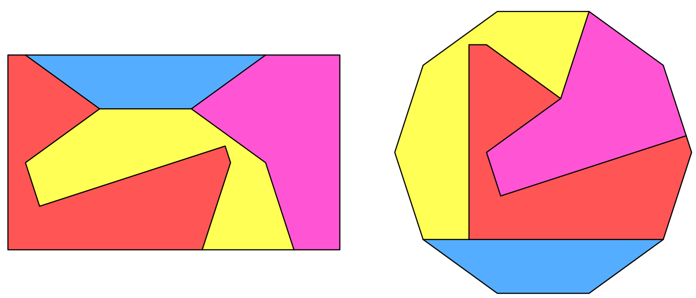


Figure 28: A four-piece dissection of a 10-gon into a rectangle.

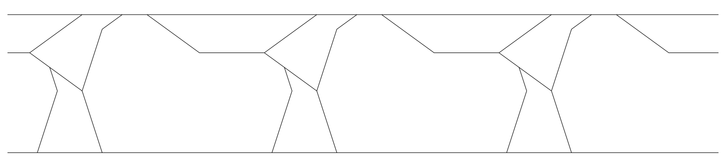


Figure 29: Lindgren's 1964 10-gon strip [18].

This is one of the most complicated dissections in the article, and we give a precise straight-edge and compass construction starting from the rectangle.in Fig. 31.

We first construct an intermediate rectangle with five pieces, and then shift it slightly to save a piece. The angles involved are $\theta = \pi/10 = 18°$, 2θ, and $\phi = 4\theta$.

The intermediate rectangle has vertices labeled $2, 14, 19, 5$ in Fig. 32; the final rectangle has vertices $1, 13, 18, 4$. We place the origin of coordinates for the rectangle near the bottom left corner, at the point $14 = (0, 0)$. The 10-gon has area $\frac{5}{2\tan\theta}$ (see (2.2)), and we take the

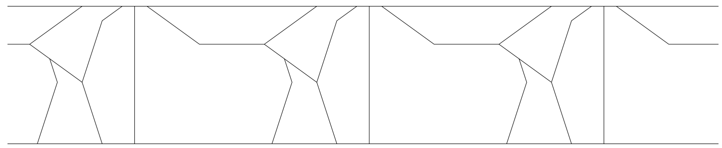


Figure 30: A five-piece 10-gon to rectangle dissection obtained from Fig. 29.

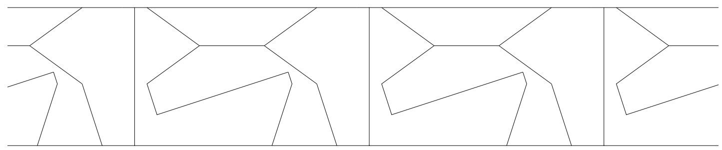


Figure 31: The strip which gives the four-piece dissection of the 10-gon to a rectangle.

width of the strip to be $w = \sqrt{5}\cos\theta$. The other dimension of the rectangle is its height $h = 2\sqrt{5}\cos 2\theta$. (After a series of relabelings, the rectangle as drawn in Fig. 31 has ended up with height w and width h. We hope the reader will forgive us!)

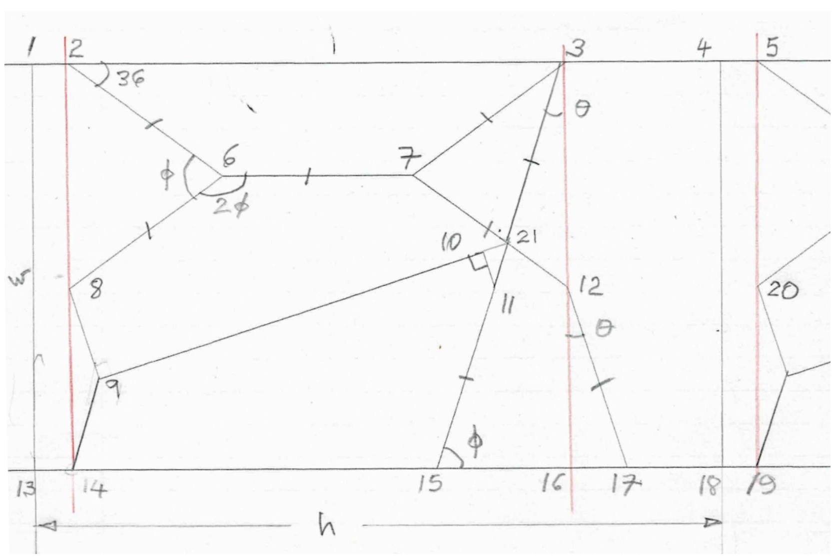


Figure 32: Labels for points used to construct the four-piece dissection of the 10-gon.

The coordinates of the points $2, 14, 17, 5$ are therefore $(0, w)$, $(0, 0)$, $(h, 0)$, and $(h.w)$. We draw a network of lines as follows. Starting at point 2, we draw line segments of length 1 from 2 to 6 to 7 to 3, and from 6 to 8, 3 to 21, and 7 to 12 to 17 to 15 to 11 (the angles are indicated in the figure). We then complete the line 3 to 15. We also draw line segments of length 1/2 from 8 to 9 to 14. For the two final lines we join 9 to 21 and draw the perpendicular from 11 to 10. The coordinates have been chosen so that several coincidences occur.[8] The points 8, 11, 12, and 20 (in the adjacent rectangle in the strip) are collinear. Also the distance from 9 to 10 turns out to be equal to w. The angle $\angle 8, 9, 21$ is a right angle. The central point 21 has coordinates $(2 + \sin\theta, 2\sin 2\theta)$. The distance from 10 to 11 is $x := \frac{3-\sqrt{5}}{4}$, and we get the final rectangle by shifting the intermediate rectangle to the left by that amount.

We get the four pieces in the dissection as follows. The quadrilateral piece (the "dish") is obtained by cutting along the path $2, 6, 7, 3, 2$. For the hexagon (the "church"), cut along $3, 7, 12, 17, 18, 4, 3$. For the first 9-gon (the "hammer"), cut along $6, 8, 9, 10, 11, 15, 17, 12, 7, 6$, and for the second 9-gon (the "triangle"), cut along $2, 1, 13, 15, 11, 10, 9, 8, 6, 2$. By moving the edge of the rectangle to the left so that it no longer passes through the point 8 we have reduced the number of pieces from five to four.

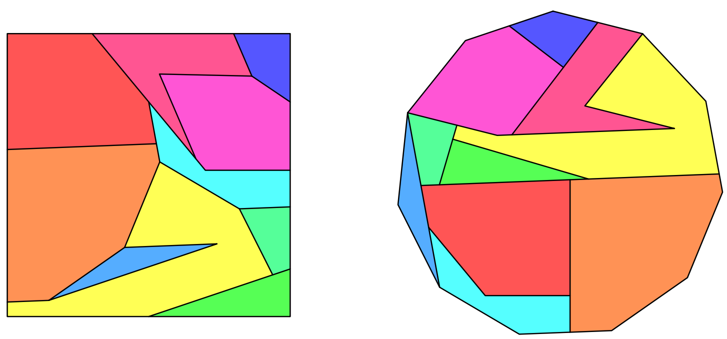


Figure 33: A ten-piece dissection of an 11-gon into a square [26].

8 Dissecting an 11-gon to a square and to a rectangle

8.1 A ten-piece dissection of an 11-gon to a square

Before the appearance of [26] there had been little work on dissections of the 11-gon: this polygon is not mentioned in any of [9, 18, 19]. G.A.T.'s ten-piece dissection of an 11-gon into a square was given in [26], and was described by Frederickson in [10, 12]. It is shown here in Fig. 33. It can be obtained by taking the 11-gon and constructing the two superpositions of strips shown in Figs. 34 and 35. When the quadrilateral outlined in red in Fig. 35 is combined with the hexagon outlined in red in Fig. 34, the result is the dissected square on the left of Fig. 33.

8.2 Our first nine-piece dissection of an 11-gon to a rectangle

A piece can be saved if our goal is only to dissect the 11-gon into a rectangle. We have found several examples of nine-piece 11-gon to rectangle dissections, two of which are described here and in the next section. Our first construction is similar to the 11-gon to square dissection of §8.1. The proof of correctness involves an interesting interplay between the two superpositions.

[8] Similar to those in (6.1), but less dramatic, since $\sin \pi/10$ is only a quadratic irrationality.

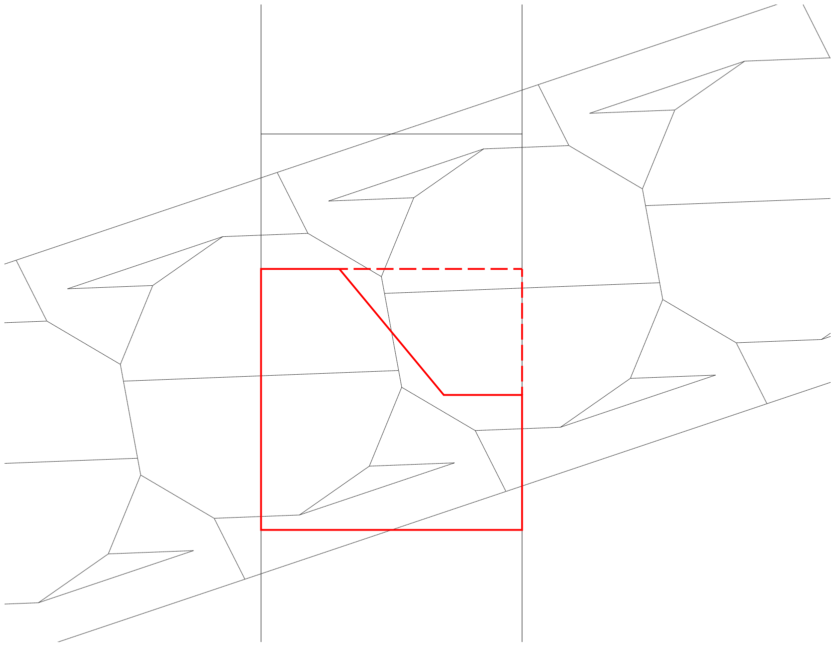


Figure 34: First of two strip superpositions used in construction of the 11-gon to square dissection of Fig. 33.

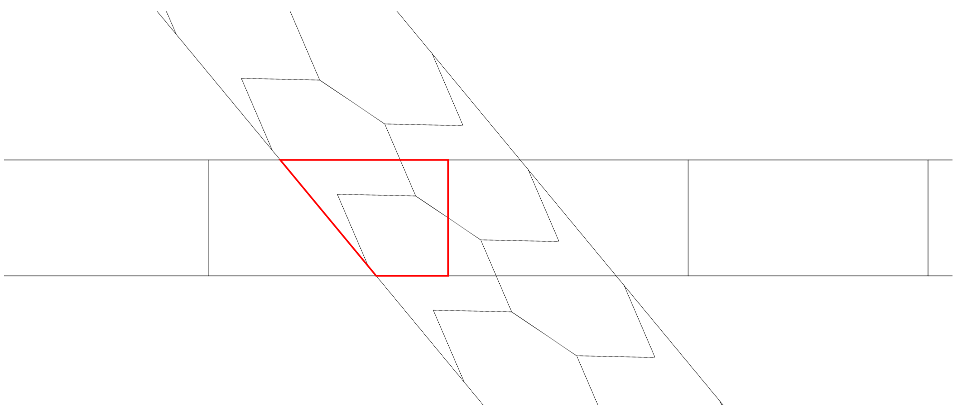


Figure 35: Second of two strip superpositions used in construction of the 11-gon to square dissection of Fig. 33. The large convex piece in the red region has one very short edge, and is actually a hexagon. This hexagon (colored pink) appears in both illustrations in Fig. 33.

A second reason for including this proof is that similar arguments can be used to give a proof of the 11-gon to square dissection mentioned above.

We start from the 11-gon and construct two superpositions of strips (see Figs. 37 and 38), and when the quadrilateral outlined in red in Fig. 37 is combined with the hexagon outlined in red in Fig. 38, the result is the dissected rectangle on the left of Fig. 39.

To build these superpositions, we first cut the 11-gon into five pieces, as shown in Fig. 36. The sides of the 11-gon have length 1, as usual, and the two long skinny triangles have a base

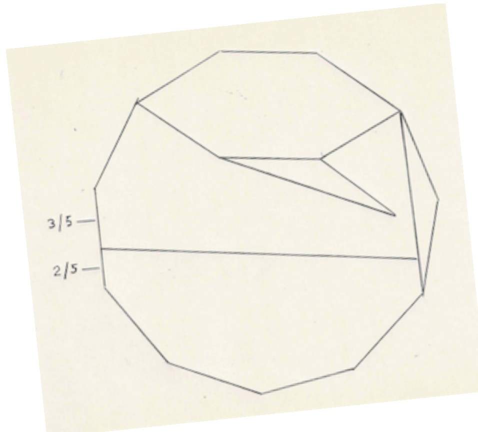
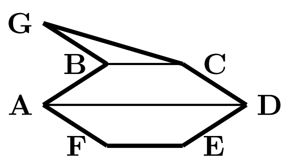


Figure 36: A 5-piece dissection of an 11-gon used to form the superposition of Fig. 37.

of length $L_{11,2} = 2\cos\theta$, where $\theta = \pi/11$. The long cut through the middle of Fig. 36 begins at a point three-fifths of the way along an edge. This parameter could be changed, but 3/5 seems to be a good choice. This cut is parallel to the top edge of the polygon, and has length $L_{11,4} = 2(\cos\theta + \cos 3\theta)$.

The heptagonal tadpole-shaped structure at the top of Fig. 36 was the repeating element in Fig. 35 and will be used again in the second superposition (Fig. 38). It can be formed from three pieces cut from the 11-gon along chords, as follows:



The edges in this "tadpole" have length 1, except for $|GC|$ and $|AD|$, which have lengths $L_{11,2}$ and $L_{11,3}$, respectively. The angle $\angle CGB = \theta$, and the angles $\angle GBA = \angle BAF$ are 4θ. The area of the tadpole is $\frac{5}{2}\sin 2\theta + \sin 4\theta$. The dissection of Fig. 36 and the remaining steps in the formation of Fig. 39 can all be carried out with straightedge and compass (assuming, of course, that we are given an 11-gon to start with) .

The wide, almost horizontal, strip in the first superposition (Fig. 37) is actually a double strip. Copies of the large heptagonal piece in Fig. 36 are placed along both the bottom edge of the strip (for example, $35, 30, 29, 28, 26, 25, 32$) and the top edge ($5, 9, 14, 15, 16, 12, 8$). The interior of the strip is filled with copies of two other pieces from Fig. 36. From Fig. 36 we see that $|32, 35| = |18, 22| = L_{11,4}$.

We then superimpose a vertical strip, bounded by the lines $26, 13$ and $24, 16$. The angle $\angle 26, 33, 36$ between the two strips is 5θ, and $\angle 36, 33, 37 = \pi/2 - \theta = \theta/2 = \angle 18, 20, 19$. Along the vertical line $26, 13$, the segment $33, 26$ is the side of a quadrilateral $33, 26, 25, 32$ with internal angles 2θ, 8θ, 6θ, and 6θ, so $|33, 26| = 4\cos^2\theta - 2\cos\theta - \frac{3}{5}$. From Fig. 36, $|26, 18| = L_{11,2} - (1 - 3/5)$, and we know $|18, 13| = 3/5$. Adding up the lengths of the segments, we get $|33, 4| = 8\cos^2\theta - 2\cos\theta - 1$.

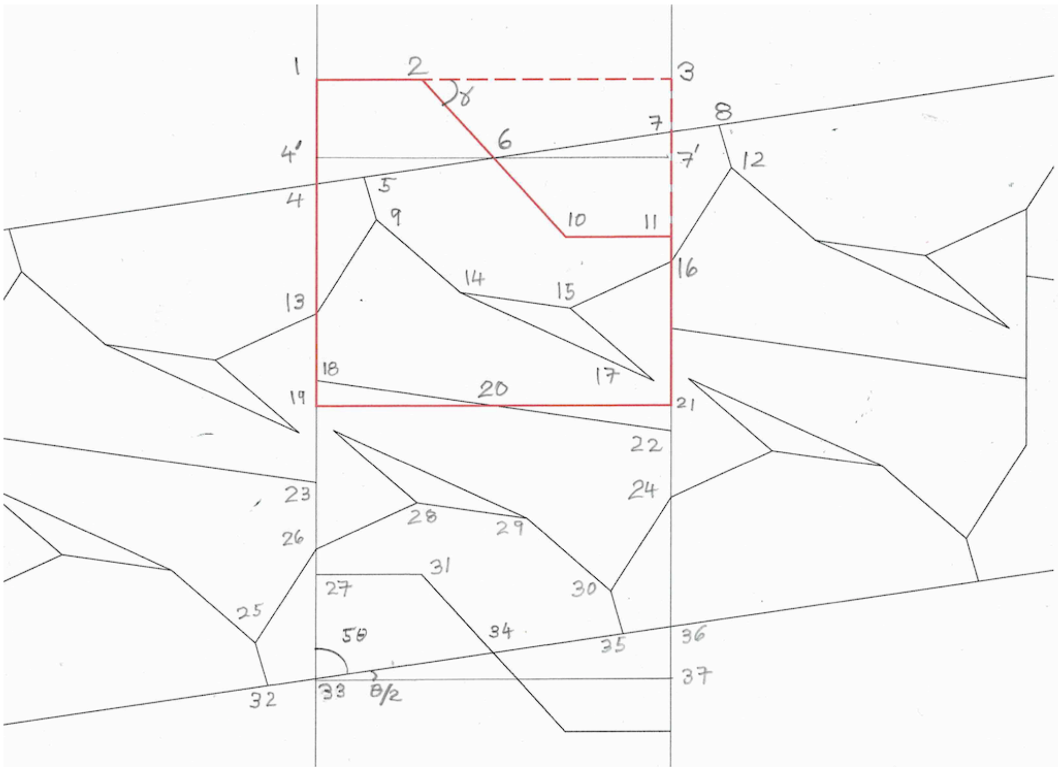


Figure 37: First superposition used to produce the 9-piece rectangle dissection of Fig. 39.

The vertical strip has width $|19, 21| = L_{11,4} \cos(\theta/2) = d_1$ (say) $= 3.1958\ldots$. This is the width of the final rectangle on the left of Fig. 39. The height $|1, 19|$ of this rectangle is then determined by the area of the 11-gon. This height is $11 \cot\theta/(4d_1) = d_2$ (say) $= 2.9305\ldots$. We easily determine the positions of the points $6, 4', 7', 20, 19$, and 21. The lengths $|2, 10|$ and $|3, 11|$ will be obtained from the second superposition.[9] We now have full information about the coordinates in Fig. 37.

The second superposition (Fig. 38) contains a diagonal double strip, formed from copies of the "tadpole", with a horizontal strip superimposed on it. We start by constructing the strip of tadpoles, and construct the horizontal strip from it. The points are labeled as in Fig. 38. The diagonal strip has period $2\cos\theta$ along the strip (e.g. $|21, 22|$), and width $3\sin\theta + 2\sin 3\theta$ (as can be easily found from the properties of the tadpole) in the perpendicular direction. The product A_2 (say) of these two quantities is the area of a fundamental region for the double strip.

The points 15, 16, and 17 are the midpoints of sides of the tadpoles. With 15 as center, we draw a circle of radius $d_1/2$ (taken from the first superposition), which meets the diagonal strip at points 1 and 24, The line $3, 26$ is constructed similarly. The trapezoid $1, 3, 28, 27$ will replace the region $2, 10, 11, 3$ in the first superposition. Let γ denote the angle $\angle 15, 1, 3$. The area of this trapezoid can now be found in two ways: it is half of A_2, that is, $\cos\theta(3\sin\theta + 2\sin 3\theta)$, and it is also $d_1 \cos\theta \sin\gamma$. It is also (area of 11-gon) $-\ d_1|6, 20|$. After some simplification, we

[9] We need the second superposition to find the angle γ.

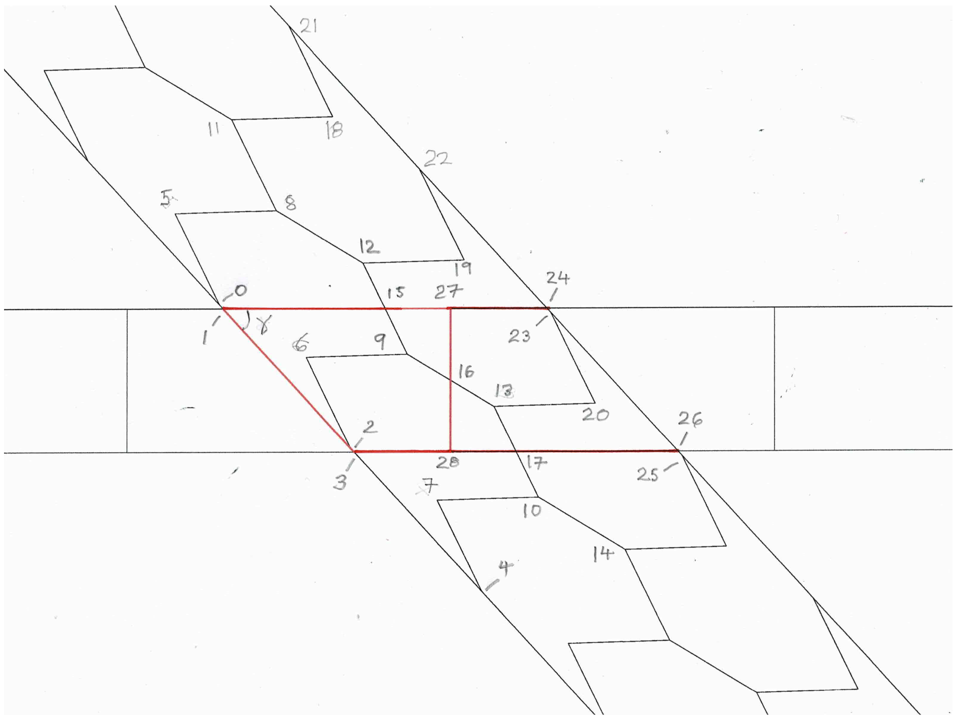


Figure 38: Second superposition used to produce the 9-piece rectangle dissection of Fig. 39.

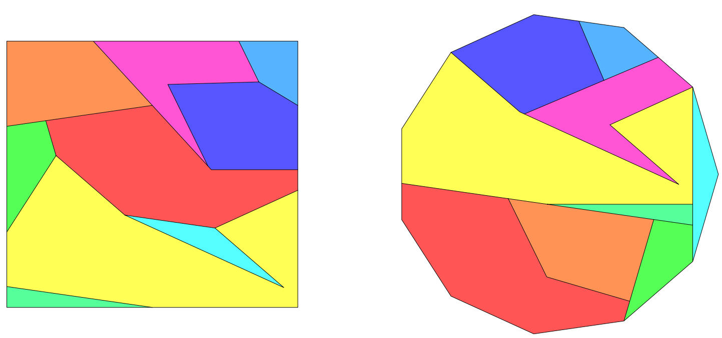


Figure 39: Nine-piece dissection of an 11-gon into a rectangle, obtained by combining superpositions in Figs. 37 and 38.

find that

$$\sin\gamma \;=\; \frac{2\cos^2\theta - 10\cos\theta + 11}{2\cos^2\theta + 3\cos\theta - 4} \;=\; 0.7374\ldots .$$

We can now give the two sides of the trapezoid that were needed for the first superposition: they are $|1,3| = 2\cos\theta$ and $|27,28| = |1,3|\sin\gamma$.

We now also have full information about the coordinates in Fig. 38. For example, by following around the boundary of the region $0,5,8,12,15,1$, we find that the very short side

0, 1 of that hexagon has length $\frac{1}{2}(\cos\theta + 2\cos 3\theta - d_1\cos\gamma) = 0.05532\ldots$. This is the (dark blue) hexagon at the top of the dissected 11-gon in Fig. 39.

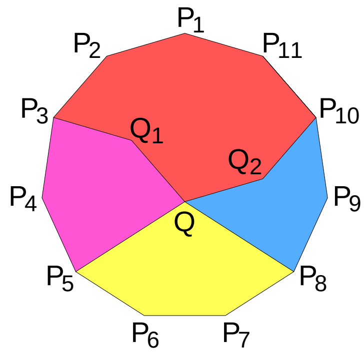


Figure 40: Four-piece dissection of an 11-gon used for tessellation in Fig. 41.

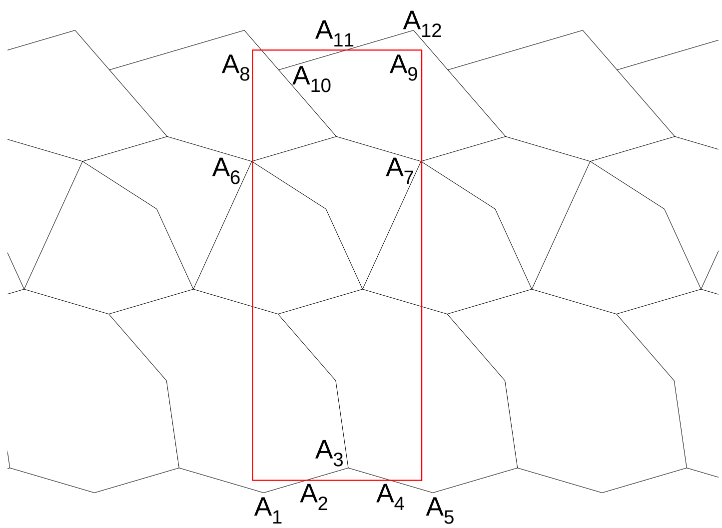


Figure 41: Tessellation of plane built from pieces from Fig. 40.

8.3 A second nine-piece dissection of an 11-gon to a rectangle

Our second nine-piece 11-gon to rectangle dissection is slightly simpler, as it only uses a single superposition. We describe the construction, and give some of the distances and angles, but leave the detailed verification of its correctness to the reader. The starting point is the simple four-piece dissection of the 11-gon shown in Fig. 40. We draw chords from P_3 to P_8 and from P_5 to P_{10}, intersecting at Q, say. Q is located at a distance $\sin(2\theta)/(\cos(\theta)+\cos(2\theta)) = 0.3002\ldots$ below the center of the 11-gon. The segments P_3Q and $P_{10}Q$ have length $2\cos\theta$. We replace P_3Q by a pair of line segments of length 1, P_3Q_1 and Q_1Q, where $\angle Q_1P_3Q = \angle Q_1QP_3 = \theta$, with a similar construction for Q_2.

The angles in Fig. 40 are remarkably nice, they are all multiples of θ: $\angle Q_1QP_5 = 5\theta$, $\angle P_5QP_8 = 7\theta$, $\angle P_8QQ_2 = 3\theta$, and so on. Q seems to be an auspicious interior point in the 11-gon.

We now use these four pieces to build a tessellation of the plane, as shown in Fig. 41, where some of the points have been labeled. We then cut out the rectangle outlined in red from the tessellation. The vertical edges of this rectangle pass through the points A_6 and A_7, so the width of the rectangle (see Fig. 40) is $2\cos\theta$. The height is therefore $11/(8\sin\theta)$. The rectangle is bounded at the top by a horizontal line through A_{11}, the midpoint of A_{10}, A_{12}, and at the bottom by a line through the midpoint A_2 of $A_1 - A_3$, and the midpoint A_4 of $A_3 - A_5$. This ise the rectangle on the left of Fig. 42. Finally, the nine pieces in the rectangle can be rearranged to form an 11-gon, as shown on the right in Fig. 42.

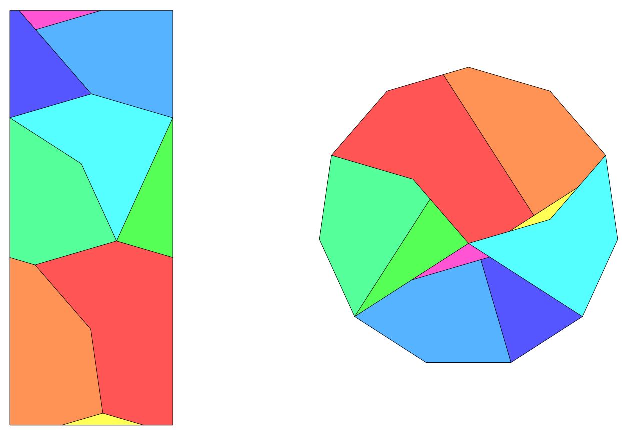


Figure 42: A second nine-piece dissection of an 11-gon into a rectangle.

9 Five-piece dissections of a 12-gon

We start with a 12-gon with edge-length 1 (see Fig. 43). Draw chords from P_1 to P_4 and P_8. Draw a perpendicular from P_4 to $P_1 - P_8$, meeting it at R, and draw an equilateral triangle P_1QP_{12} that touches $P_1 - P_8$.

The angle $\angle P_1P_4R$ is $\pi/6 = 30°$. The lengths of the line segments are as follows: $|P_1P_4| = |QP_8| = L_{3,12} = 1 + \sqrt{3}$, $|P_4R| = |RP_8| = (3 + \sqrt{3})/2$, and $|P_1R| = (1 + \sqrt{3})/2$.

After the pieces are rearranged (Fig. 44), the resulting rectangle has width $w = 3 + \sqrt{3}$ and height $h = (3 + \sqrt{3})/2$. We then easily check that the product wh is equal to the area $3\cot 15°$.

A second dissection of the 12-gon (although with a non-convex piece) is shown in Fig. 45.

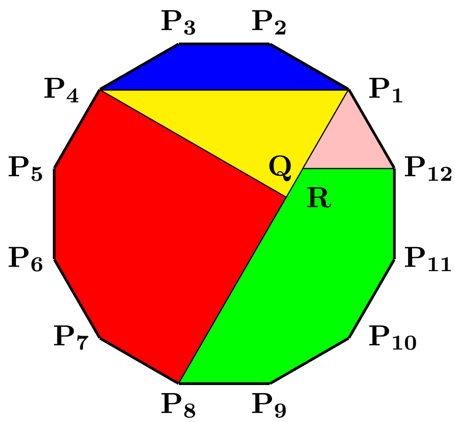


Figure 43: A five-piece dissection of a 12-gon.

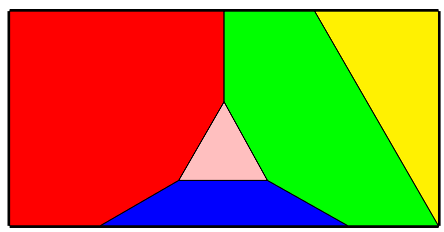


Figure 44: The pieces reassembled to form a rectangle.

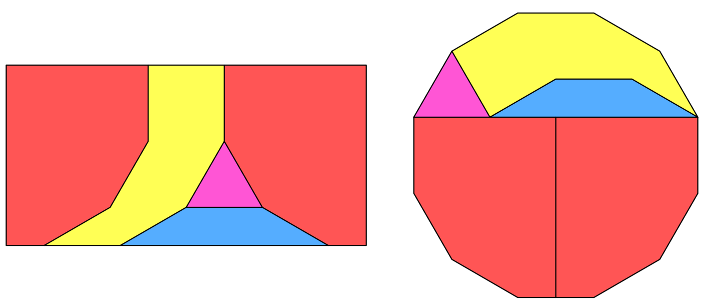


Figure 45: An alternative five-piece dissection of a dodecagon.

10 A seven-piece dissection of a 14-gon and a nine-piece dissection of a 16-gon

It is known that $s(14) \le 9$ and $s(16) \le 10$ [26]. Figure 46 shows a tessellation of the plane based on a 14-gon, and a rectangle superimposed on it which leads to the seven-piece dissection shown in Fig. 47. Figures 48 and 49 play similar roles for the 16-gon.

These will be our final examples of a regular polygon to rectangle dissections. Other

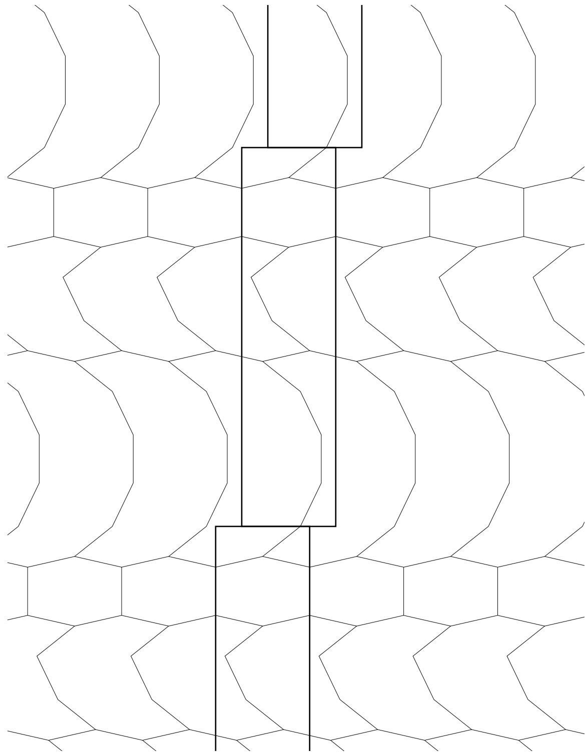


Figure 46: A tessellation based on a 14-gon, with the rectangle that leads to the dissection in Fig. 47.

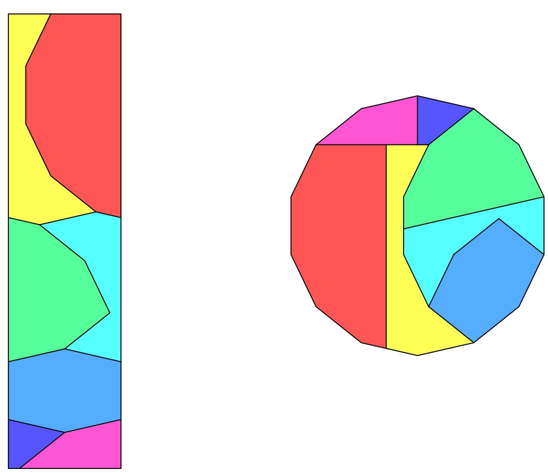


Figure 47: A seven-piece dissection of a 14-gon into a rectangle.

examples with larger numbers of sides may be found in the *Rectangle Dissections* section of [26], and in a projected sequel to the present work.

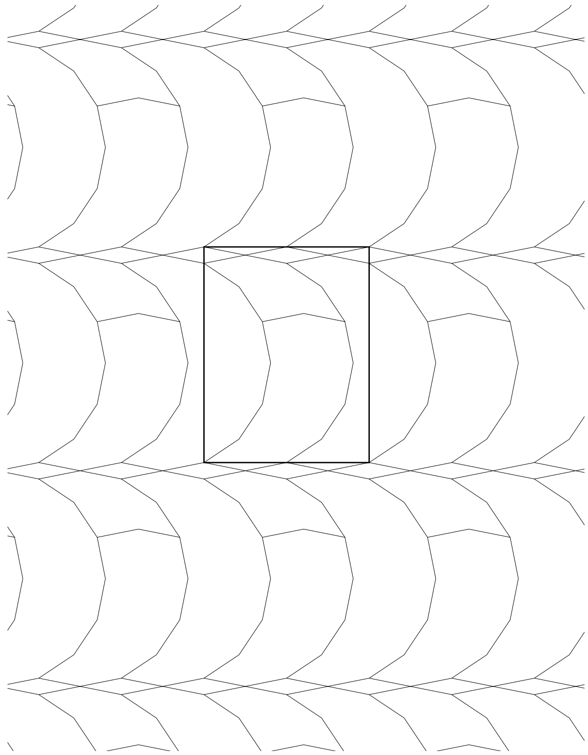


Figure 48: A tessellation based on a 16-gon, with the rectangle that leads to the dissection in Fig. 49.

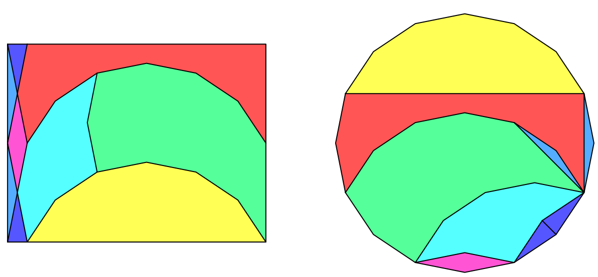


Figure 49: A nine-piece dissection of a 16-gon into a rectangle.

11 Selected dissections of star polygons to rectangles

We give four examples of especially elegant dissections of star polygons to rectangles. These are taken from [26], where many further examples can be found.

12 Three-piece dissections of a Greek cross

Most authors who study dissections of polygons include the Greek cross, so we briefly discuss it here. The classical four-piece dissection of a Greek cross into a square can be seen for example

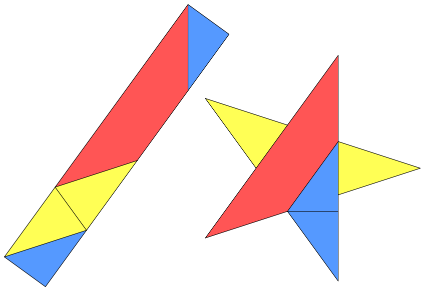


Figure 50: A five-piece dissection of a $\{5/2\}$ pentagram.

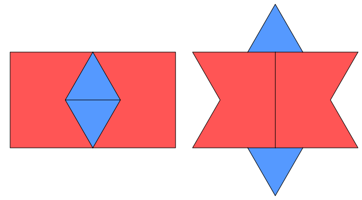


Figure 51: A four-piece dissection of a $\{6/2\}$ hexagram.

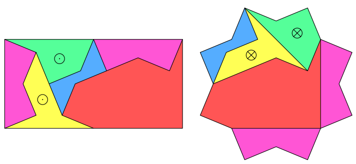


Figure 52: A six-piece dissection of an $\{8/2\}$ octagram. Two pieces must be turned over.

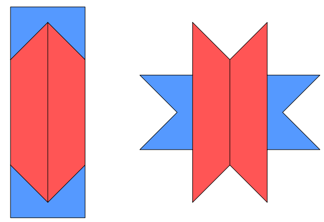
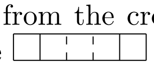


Figure 53: A four-piece dissection of an $\{8/3\}$ octagram.

in [18, Fig. 9.1]. The dissected square has 4-fold rotational symmetry.

Three pieces seem to be the minimal number needed to form a rectangle from a Greek cross. The simplest three-piece construction cuts off two opposite arms from the cross and places them at the ends of the other two arms, forming a 1×5 rectangle . Eppstein [8] gives a three-piece dissection into non-convex pieces, shown in Fig. 54, and the database [26] gives another (Fig. 55), similar in spirit to the four-piece dissection into a square.

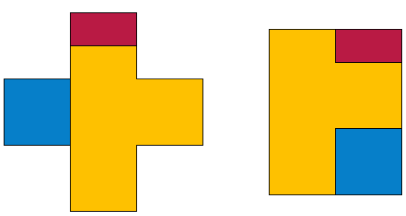


Figure 54: Eppstein's 3-piece dissection of a Greek cross into a rectangle [8].

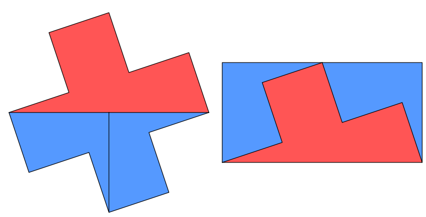


Figure 55: A 3-piece dissection of a Greek cross into a rectangle (from [26]).

13 Curved cuts are sometimes essential

We know of no theorem which will guarantee that polygonal cuts are sufficient to achieve $s(n)$ or $r(n)$. The following are three examples of other situations where it seems clear that minimal dissections can *not* be achieved using only polygonal cuts.

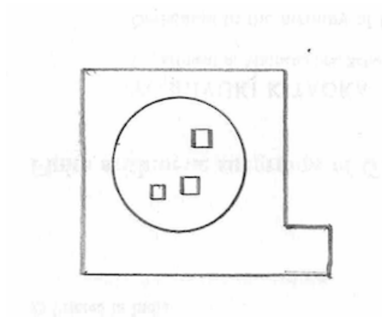


Figure 56: A two-piece dissection that can be accomplished by a single circular cut, but surely not by a polygonal cut. The polygon contains three small squares holes (Richard C. Schroeppel and Andy Latto).

1. Take a square with a smaller square attached to it, and cut out three small square holes at random positions in the interior. Call this figure A. For figure B, make a circular cut enclosing the three holes, and rotate the interior of the circle by a small random angle. This gives a two-piece dissection of A to B which surely cannot be accomplished with a single polygonal cut. This example was suggested by Richard C. Schroeppel and Andy Latto, (personal communication).

2. Figure 57 shows an example due to David desJardins (personal communication) of a three-piece dissection between two simply connected polygonal regions that appears to require a curved piece.

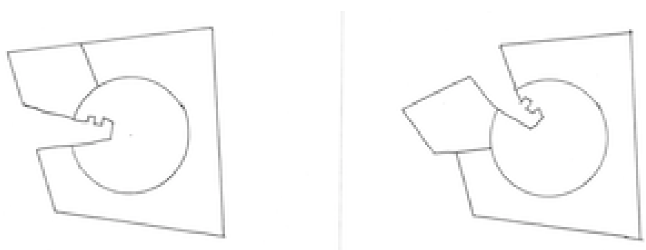


Figure 57: A three-piece dissection that appears to require a curved piece (David desJardins).

3. Greg Frederickson [9, Fig. 13.6] gives an example of 6-piece dissection of a hexagon into a hexagram which requires that two of the pieces be turned over. This can be modified to avoid turning the pieces over at the cost of adding an extra piece. But if curved cuts are used, this can be accomplished without adding the extra piece, as shown in Fig. 58. We conjecture that a 6-piece hexagon to hexagram dissection that avoids turning pieces over cannot be constructed using only polygonal cuts.

4. In this regard, it is worth pointing out that a rotation by any desired angle that uses a single circular cut (see Fig. 59) can be accomplished by two square cuts and turning a piece over (Fig. 60).

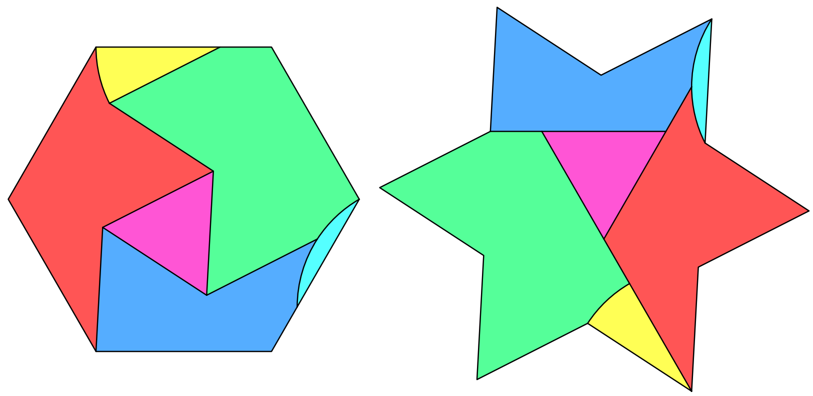


Figure 58: A 6-piece hexagon to hexagram dissection that avoids turning pieces over, but uses a curved cut.

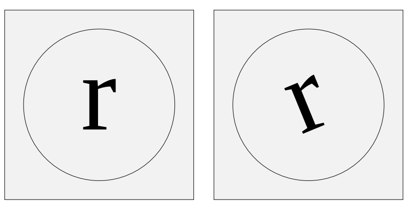


Figure 59: A rotation produced by a single circular cut ...

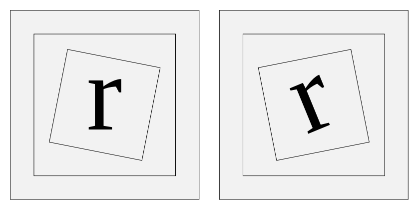


Figure 60: ... can also be achieved with two square cuts and turning a piece over.

14 Acknowledgments

Thanks to Adam Gsellman for telling us about his polygon to rectangle dissections, which was the seed that led to the present paper. Thanks also to David desJardins, Andy Latto, Richard C. Schroeppel, and Allan C. Wechsler for helpful comments. The writing of this paper depended heavily on PostScript, LaTeX, Tikz, Maple, WolframAlpha, and email.

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