A FINITE MATRIX WITH GROTHENDIECK RATIO GREATER THAN $\sqrt{2}$

J. A. REEDS AND N. J. A. SLOANE

One form of Grothendieck's inequality states there exists a smallest constant K with $\pi/2 \le K \le \sinh(\pi/2)$, such that for any any finite real matrix $M = (m_{i,j})$, and any real Hilbert space H with inner product $(\cdot, \cdot)$, we must have

$$r(M) \le K$$

where

$$r(M) = \frac{\sup_{x_i, y_j \in B(H)} \sum_{i,j} (x_i, y_j) m_{i,j}}{\sup_{\varepsilon_i, \delta_j \in \{-1,1\}} \sum_{i,j} \varepsilon_i m_{i,j} \delta_j},$$

where $B(H)$ denotes the unit ball in H.

Because $\pi/2 > \sqrt{2}$ there exist matrices M for which $r(M) > \sqrt{2}$, but until April 1990 we were unaware of any particular examples. Our first example was of size 120×120. This discovery was anounced but not explicitly stated a few years later in [1]; recently Sébastien Designolle asked for details. The point of this current note is to give our example, and sketch why $r(M) > \sqrt{2}$.

We use some of the terminology and facts about root vectors and Weyl chambers found in [2, Chapter III].

Take the system $\mathcal{R}$ of root vectors of the lattice E_8 to be the $112 = 2^2\binom{8}{2}$ vectors in $\mathbb{R}^8$ of form

$$(\pm 1, \pm 1, 0, 0, 0, 0, 0, 0)$$

and all coordinate permutations thereof, together with the 128 vectors of form $(\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2},\pm\frac{1}{2})$ with an even number of minus signs; 240 in all. The set

$$\bigcap_{v\in\mathcal{R}} \left\{x \in \mathbb{R}^8 : (v,x) \neq 0\right\}$$

is open; its connected components are known as the *Weyl chambers.* Interior points in the Weyl chambers are called *regular points.*

Of the 240 elements of $\mathcal{R}$, 120 are "positive" in the sense that the leftmost non-zero coordinate is positive; let them be denoted by $v_1,\ldots,v_{120}$; write $\mathcal{R}_+ = \{v_1,\ldots,v_{120}\}$. Note that $\mathcal{R} = \mathcal{R}_+ \cup (-\mathcal{R}_+)$.

Define $\overline{\alpha} = \sum_{v\in\mathcal{R}_+} v = \sum_{i=1}^{120} v_i$. One checks that

$$\overline{\alpha} = (46, 12, 10, 8, 6, 4, 2, 0)$$

so $\|\overline{\alpha}\|^2 = 2480$, and that $(v_i, \overline{\alpha}) > 0$ for all $v_i \in \mathcal{R}_+$, and hence that $\overline{\alpha}$ is regular.

Our matrix entries are the inner products

$$m_{i,j} = (v_i, v_j);$$

one can write $M = VV'$, where V is the 120×8 matrix whose i-th row is v_i. One can check that M is 30 times the matrix of the orthogonal projection onto the column space of V. A histogram of the matrix values is

value	count
-1	2240
0	7560
1	4480
2	120
total	14400.

Since the vectors v_i all have length $\sqrt{2}$ we note that $v_i/\sqrt{2} \in B(\mathbb{R}^8)$, and hence

$$\begin{aligned}\sup_{x_i,y_j\in B(\mathbb{R}^8)} \sum_{i,j}(x_i,y_j)m_{i,j} &\geq \sum_{i,j}(v_i,v_j)m_{i,j}/2 \\ &= \sum_{i,j} m_{i,j}^2/2 \\ &= 3600.\end{aligned}$$

Also

$$\begin{aligned}\sup_{\varepsilon_i,\delta_j\in\{-1,1\}} \sum_{i,j}\varepsilon_i m_{i,j}\delta_j &= \sup_{\varepsilon_i,\delta_j\in\{-1,1\}} \left(\sum_i \varepsilon_i v_i, \sum_j \delta_j v_j\right) \\ &= \sup_{\varepsilon_i} \left\|\sum_i \varepsilon_i v_i\right\|^2 = \left\|\sum_i v_i\right\|^2 \\ &= \|\overline{\alpha}\|^2 = 2480.\end{aligned} \tag{1}$$

Hence

$$r(M) \geq \frac{3600}{2480} \approx 1.45 > \sqrt{2}.$$

The key step here lies in showing (1), namely, that for any choice of signs we have

$$\left\| \sum_i \pm v_i \right\| \le \left\| \sum_i v_i \right\| , \tag{2}$$

which is not true for arbitrary collections of vectors, but is for the root vectors v_i in $\mathcal{R}_+$.

One way to see (2) is by considering the vectors

$$\begin{aligned} S(\alpha) &= \sum_{v \in \mathcal{R}} \operatorname{sign}(\alpha, v)\, v \\ &= 2 \sum_{i=1}^{120} \operatorname{sign}(\alpha, v_i)\, v_i \end{aligned}$$

for regular $\alpha \in \mathbb{R}^8$; these are constant on Weyl chambers. We first show that the left hand side of (2) is bounded by $\|S(\alpha)\|/2$ for *some* regular α, and then that in fact the function $\alpha \mapsto \|S(\alpha)\|$ is constant, so $\|S(\alpha)\| = \|S(\overline{\alpha})\|$ for all regular α.

Let $\mathcal{S} = \{\sigma : \mathcal{R}_+ \to \{-1, +1\}\}$ be the set of all possible assignments of signs to the elements of $\mathcal{R}_+$.

Lemma 1 ([1, Lemma 2]). *There exists a regular α such that*

$$\sup_{\sigma \in \mathcal{S}} \left\| \sum_{v \in \mathcal{R}_+} \sigma(v) v \right\| = \frac{1}{2} \|S(\alpha)\|.$$

Proof. Let $\sigma \in \mathcal{S}$ maximize $\| \sum_{v \in \mathcal{R}_+} \sigma(v) v \|$, let $\alpha = \sum_{v \in \mathcal{R}_+} \sigma(v) v$. For each individual $w \in \mathcal{R}_+$ consider the effect of flipping just the sign of w, that is, replacing $\sigma(w)$ with $-\sigma(w)$, but leaving the other $\sigma(v)$ values unchanged. Then $\|\alpha\|^2 \ge \|\alpha - 2\sigma(w)w\|^2 = \|\alpha\|^2 - 4\sigma(w)(\alpha, w) +$

$4\|w\|^2$, so

$$\sigma(w)(\alpha, w) \geq \|w\|^2 > 0,$$

so $\sigma(w) = \text{sign}(\alpha, w)$. Since this is true for all $w \in \mathcal{R}_+$, we see that α is regular, and that $\alpha = S(\alpha)/2$. □

Lemma 2. *Let $\alpha, \beta \in \mathbb{R}^8$ both be regular. Then*

$$\|S(\alpha)\| = \|S(\beta)\|.$$

Proof. For $\tau \in \mathcal{R}$ let

$$r_\tau : x \mapsto x - 2\frac{(\tau, x)}{\|\tau\|^2}\tau$$

be the reflection through the hyperplane perpendicular to τ. Since $\|\tau\|^2 = 2$ for all $\tau \in \mathcal{R}$, the simpler formula $r_\tau(x) = x - (\tau, x)\tau$ also holds. The map $r_\tau : \mathbb{R}^8 \to \mathbb{R}^8$ is a linear isometry: it preseves the norm and the inner product: $\|a\| = \|r_\tau(a)\|$ and $(a, b) = (r_\tau(a), r_\tau(b))$, for $a, b \in \mathbb{R}^8$. And r_τ permutes the elements of $\mathcal{R}$. So

$$\begin{aligned}
\|S(\alpha)\| &= \|r_\tau(S(\alpha))\| && \text{(by isometry)}\\
&= \left\|\sum_{v\in\mathcal{R}} \text{sign}(\alpha, v)\, r_\tau(v)\right\| && \text{(by linearity)}\\
&= \left\|\sum_{v\in\mathcal{R}} \text{sign}(r_\tau(\alpha), r_\tau(v))\, r_\tau(v)\right\| && \text{(by isometry)}\\
&= \left\|\sum_{w\in\mathcal{R}} \text{sign}(r_\tau(\alpha), w)\, w\right\| && \text{(by permutation)}\\
&= \|S(r_\tau(\alpha))\| && \text{(by definition of } S\text{).}
\end{aligned}$$

The group generated by $\{r_v : v \in \mathcal{R}\}$ is transitive on the set of Weyl chambers (see [2, [p.51]), so for given regular $\alpha, \beta \in \mathbb{R}^8$ there exists a finite sequence of reflections r_v whose composition maps the chamber of α to the chamber of β. $\square$

References

[1] P.C. Fishburn, J.A. Reeds, "Bell Inequalities, Grothendieck's Constant and Root Two", *SIAM J. Disc. Math.* 7(1994)48–56.

[2] James E. Humphreys, *Introduction to Lie Algebras and Representation Theory.* New York: Springer, 1972.