The Antipode Construction for Sphere Packings

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Summary

A construction for sphere packings is introduced that is parallel to the "anticode" construction for codes. This provides a simple way to view Vardy's recent 20-dimensional sphere packing, and also produces packings in dimensions 22, 44–47 that are denser than any previously known.

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1. Introduction

Vardy [3] recently found a remarkable 20-dimensional sphere packing that is denser than any previously known, and asked if the affine automorphism group of the packing is transitive on spheres. We will give an alternative simple construction which yields Vardy's packing and some other new packings. We will also show that the group of Vardy's packing does act transitively.

For undefined terms from sphere packing, see [1, Chap. 1].

2. The antipode construction

The following construction is an analogue of the "anticode" construction for codes [2, Chap. 17, §6]. (The common theme of these two constructions is that instead of looking for well-separated points, which is what most constructions do, now we look for points somewhere else that are close together and factor them out.)

Let Λ be a lattice with minimal norm (i.e. squared length) μ in an n-dimensional Euclidean space W. Let U, V be respectively k- and l-dimensional vector spaces with $W = U \oplus V$, $n = k + l$, such that $\Lambda \cap U = K$ and $\Lambda \cap V = L$ are k- and l-dimensional lattices, respectively. The projection maps from W to U and V will be denoted by π_U, π_V, respectively, and we define the lattices $M := \pi_U(\Lambda)$, $N := \pi_V(\Lambda)$. In the examples below, Λ will be self-dual, and so M and N will be the lattices K^* and L^* dual to K and L [1, p. 166].

Suppose we can find a subset $S = \{u_1, \ldots, u_s\}$ of M such that $dist^2(u_i, u_j) \leq \beta$ for all i, j, for some constant β. We define $\mathcal{A}(S)$ to consist of the points

$$\{\pi_V(w) : \ w \in \Lambda, \ \pi_U(w) \in S\} \ .$$

We say that the sets S and $\mathcal{A}(S)$ are *antipodal* to each other - this explains our name for this construction.

Theorem 1. *By drawing spheres of radius $\rho = \frac{1}{2}\sqrt{\mu - \beta}$ around the points of $\mathcal{A}(S)$ we obtain an l-dimensional sphere packing of center density*

$$\delta = \frac{s \cdot \rho^l}{\sqrt{\det L}} \ . \tag{1}$$

Proof. $\mathcal{A}(S)$ is the union of s translates of L, and by construction the squared distance between the points of $\mathcal{A}(S)$ is at least $\mu - \beta$. The result follows from Eq. (13) of [1, Chap. 1]. ∎

In many cases (in particular, in dimensions just below 8, 24 and 48) the densest packings known up to now are obtained by taking a self-dual lattice Λ (E_8, Λ_{24} and P_{48p} or P_{48q}, respectively) and using suitable sections $L = \Lambda \cap V$. Some of these densities can now be improved by the antipode construction, since $\mathcal{A}(S)$ is denser than L whenever

$$s\left(1 - \frac{\beta}{\mu}\right)^{\frac{l}{2}} > 1 \ , \tag{2}$$

the left-hand side of this formula being the ratio of their densities.

This construction improves on the old records in the following cases. In the first two examples we take $n = 24$ and Λ to be the Leech lattice Λ_{24} in $\mathbb{R}^{24}$, with minimal norm $\mu = 4$. We write the vectors of Λ_{24} in the form $\frac{1}{\sqrt{8}}(a_1, \ldots, a_{24})$, where the a_i are the coordinates of a MOG diagram [1, Chap. 11].

Vardy's 20-dimensional packing [3]. We take $l = 20$, $V =$ vectors in $\mathbb{R}^{24}$ beginning with four zeros, $L =$ laminated lattice Λ_{20} with determinant 64, $K = \sqrt{2}D_4$ (the root lattice D_4, rescaled so as to have minimal norm 4), $M = \frac{1}{\sqrt{2}}D_4^*$ (the dual lattice D_4^*, rescaled so as to have minimal norm $\frac{1}{2}$), $s = 4$, $S = \frac{1}{\sqrt{8}}\{0\ 0\ 0\ 0, \ \ 1\ 1\ 1\ 1, \ \ 2\ 0\ 0\ 0, \ \ 1\ 1\ 1\ -1\}$, $\beta = \frac{1}{2}$, obtaining a packing V_{20} with center density

$$\frac{7^{10}}{2^{31}} = .1315\ldots \ ,$$

compared with the old record of $1/8 = .125$ for Λ_{20}. It is easy to show that this construction is equivalent to Vardy's (compare Eq. (9) of [3]). Vardy shows that each sphere touches 15360 others.

Dimension 22. Again using the Leech lattice, we take $l = 22$, V = vectors in $\mathbb{R}^{24}$ beginning with three equal coordinates, $L = \Lambda_{22}$ with determinant 12, $K = \sqrt{2}A_2$, $M = \frac{1}{\sqrt{2}}A_2^*$, and S to consist of the $s = 3$ vertices of an equilateral triangle in M, with $\beta = \frac{1}{3}$, obtaining a packing V_{22} with center density

$$\delta = \frac{11^{11}}{2^{23}3^{10.5}} = .3325\ldots \ ,$$

a new record, compared with $1/\sqrt{12} = .2886\ldots$ for Λ_{22}.

Explicit coordinates for V_{22} are as follows. The three translates of Λ_{22} may be obtained by taking all vectors of Λ_{24} in which the first three coordinates have the form $\frac{1}{\sqrt{8}}(a,a,a)$, $\frac{1}{\sqrt{8}}(a+2,a,a)$, or $\frac{1}{\sqrt{8}}(a,a-2,a)$. Their projections onto U are then proportional to $(0,0,0)$, $\left(\frac{2}{3}, -\frac{1}{3}, -\frac{1}{3}\right)$, and $\left(\frac{1}{3}, -\frac{2}{3}, \frac{1}{3}\right)$, which do indeed form an equilateral triangle. The corresponding vectors in V_{22}, which are the projections onto V, are $\frac{1}{\sqrt{8}}(a,a,a,\ldots)$, $\frac{1}{\sqrt{8}}\left(a+\frac{2}{3}, a+\frac{2}{3}, a+\frac{2}{3},\ldots\right)$, and $\frac{1}{\sqrt{8}}\left(a-\frac{2}{3}, a-\frac{2}{3}, a-\frac{2}{3},\ldots\right)$.

As we will see in §3, the affine groups of both V_{20} and V_{22} act transitively on the spheres. To find the kissing number of V_{22}, consider the sphere centered at the zero vector. No sphere in this copy of Λ_{22} touches it. The number in the second translate that touch it is equal to the number of minimal vectors in Λ_{24} which begin $\frac{1}{\sqrt{8}}(a+2,a,a,\ldots)$. These vectors are

$$\begin{array}{lll} \text{for } \ a=-2: & 0,-2,-2;\pm 2^6 0^{15} & (\# = 2^5\cdot 56)\ , \\ \text{for } \ a=-1: & 1,-1,-1;\pm 1^{20}\pm 3 & (\# = 2^9\cdot 21)\ , \\ \text{for } \ a=0: & 2,0,0;\pm 2^7 0^{14} & (\# = 2^6\cdot 120)\ , \\ \text{for } \ a=1: & 3,1,1;\pm 1^{21} & (\# = 2^9\cdot 1) \end{array}$$

(cf. [1], p. 278), for a total of 20736. Therefore each sphere in this packing touches 41472 others.

Dimensions 44–47. We take $n = 48$, and Λ = either P_{48p} or P_{48q} [1, p. 195] (so that there are at least two inequivalent packings in each of these cases), with $\mu = 6$. As on p. 168 of [1] we can find subspaces U in $\mathbb{R}^{48}$ of dimensions 1, 2, 3, 4 such that M is respectively $\frac{1}{\sqrt{3}}A_1^*$, $\frac{1}{\sqrt{3}}A_2^*$, $\frac{1}{\sqrt{3}}A_3^* \cong \frac{1}{\sqrt{3}}D_3^*$, $\frac{1}{\sqrt{3}}D_4^*$. In these four lattices it is easy to find $s = 2,3,4,4$ points, respectively, for which $\beta = \frac{1}{6}, \frac{2}{9}, \frac{1}{3}, \frac{1}{3}$. We then obtain packings in dimensions $l = 47,46,45,44$ with center densities $2^{-70}3^{-24}35^{23.5} = 5788.8\ldots$, $3^{-46.5}13^{23} = 2719.9\ldots$, $2^{-44}3^{-24}17^{22.5} = 974.6\ldots$, $2^{-43}3^{-24}17^{22} = 472.7\ldots$, which improve on the old records by factors of $1.031\ldots$, $1.259\ldots$, $1.105\ldots$ and 1.137, respectively.

We found no other improvements on the present records for density (cf. [1], Chap. 1), despite examining all likely cases in low dimensions.

3. The group of Vardy's packing

We will determine Aff(V_{20}), the full affine symmetry group of V_{20} and in particular show that it acts transitively on the spheres, thus answering a question raised in [3]. Similar but easier arguments apply to V_{22}.

Lemma 1. *Any affine symmetry of V_{20} is induced by an affine symmetry of Λ_{24}.*

Proof. V_{20} consists of four translates of Λ_{20}, which we shall label [0], [1], [2], [3], corresponding to the four members of S. Two points of V_{20} have integral squared distance if and only if they are in the same translate, so this decomposition into translates is intrinsic. In an abstract 4-space we take four vectors $v_0 = 0\ 0\ 0\ 0$, $v_1 = 1\ 1\ 1\ 1$, $v_2 = 2\ 0\ 0\ 0$, $v_3 = 1\ 1\ 1\ -1$, and glue v_i to $[i]$ (cf. [1, Chap. 4]). The lattice affinely generated by the result is the laminated lattice Λ_{23}, consisting of those vectors of Λ_{24} with second and third coordinates equal. So any affine symmetry of V_{20} extends uniquely to an affine symmetry of Λ_{23}. A similar argument shows that any affine symmetry of Λ_{23} extends uniquely to an affine symmetry of Λ_{24}. This completes the proof. ∎

So we may regard Aff(V_{20}) as a subgroup of the affine symmetry group Co_∞ of Λ_{24}. In fact Aff(V_{20}) is the subgroup of Co_∞ that preserves the partition of $W = \mathbb{R}^{24}$ into $U \oplus V$, and also fixes (setwise) the vertices of the tetrahedron formed by the elements of S. It is easy to see that this group acts as a symmetric group $\mathbf{S}_4$ on the vertices of the tetrahedron.

Lemma 2. *Aff(V_{20}) acts transitively on V_{20}.*

Proof. By the previous remark, Aff(V_{20}) acts as an $\mathbf{S}_4$ on the four cosets of Λ_{20}. But Aff(V_{20}) also includes translates by vectors of Λ_{20}, which act transitively in each coset. ∎

Lemma 3. *The subgroup of Aff(V_{20}) that fixes one sphere is a group of order $2^8\,|\mathbf{M}_{20}|\,2 \cdot 3! =$ 2949120, and acts transitively on the 15360 neighboring spheres.*

We omit the proof, which is a straightforward calculation using the results of [1, Chap. 11]. The factor $2^8|\mathbf{M}_{20}|$ is the order of the subgroup of $2^{12}\mathbf{M}_{24}$ that fixes the first four coordinates, the factor 2 corresponds to the interchange of the second and third coordinates, and the factor 3! refers to permutations of the remaining three vertices of the tetrahedron.

In other words, not only does every sphere in Vardy's packing look like every other one, but every pair of spheres in contact looks like every other such pair. However, there is more than one type of triple of mutually contiguous spheres.

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References

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