A New Operation on Sequences: The Boustrophedon Transform*

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ABSTRACT

A generalization of the Seidel-Entringer-Arnold method for calculating the alternating permutation numbers (or secant-tangent numbers) leads to a new operation on sequences, the boustrophedon transform.

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1. Introduction

Let $E_{n,k}$ $(n \geq k \geq 0)$ denote the number of permutations of $\{1, 2, \ldots, n+1\}$ which alternately fall and rise (always starting with a fall), and start with $k+1$. These numbers have a long history (see the references), but we follow Poupard [Pou82] and call them the Entringer numbers. They satisfy the recurrence [Ent66, first lemma]

$$E_{0,0} = 1, \quad E_{n,0} = 0 \ (n \geq 1), \quad E_{n+1,k+1} = E_{n+1,k} + E_{n,n-k} \quad (n \geq k \geq 0)\ . \tag{1.1}$$

If these numbers are displayed in a triangular array with rows written alternately right to left and left to right, in boustrophedon (or "ox-plowing") manner (sequence A8280 in [OEIS]):

$$\begin{array}{ccccccccc}
 & & & & E_{00} & & & & \\
 & & & E_{10} & \rightarrow & E_{11} & & & \\
 & & E_{22} & \leftarrow & E_{21} & \leftarrow & E_{20} & & \\
 & E_{30} & \rightarrow & E_{31} & \rightarrow & E_{32} & \rightarrow & E_{33} & \\
E_{44} & \leftarrow & E_{43} & \leftarrow & E_{42} & \leftarrow & E_{41} & \leftarrow & E_{40} \\
 & & & & \cdots & & & &
\end{array}
\quad = \quad
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 0 & \rightarrow & 1 & & & \\
 & & 1 & \leftarrow & 1 & \leftarrow & 0 & & \\
 & 0 & \rightarrow & 1 & \rightarrow & 2 & \rightarrow & 2 & \\
5 & \leftarrow & 5 & \leftarrow & 4 & \leftarrow & 2 & \leftarrow & 0 \\
 & & & & \cdots & & & &
\end{array}
\tag{1.2}$$

then the entries are filled in by the rule that each row (after the zero-th) begins with a 0 and every subsequent entry is the sum of the previous entry in the same row and the entry above it in the previous row.

The earliest reference we have seen for this elegant observation is Arnold [Arn91], who refers to (1.2) as the Euler-Bernoulli triangle, but it may well be of much older origin. Dumont [Dum95] refers to (1.2) as the Seidel-Entringer-Arnold triangle, referring to Seidel [Sei77].

The numbers $E_n := E_{n,n}$ (sequence A111) appearing at the ends of the rows in (1.2) give the total number of permutations of $\{1, 2, \ldots, n\}$ that alternately fall and rise, i.e. the number of "down-up permutations" of n things. The history of these numbers goes back to André [And79], [And81], [Com74], [Sch61]. They have exponential generating function (e.g.f.)

$$\mathcal{E}(x) = \sum_{n=0}^{\infty} E_n \frac{x^n}{n!} = \sec x + \tan x\ . \tag{1.3}$$

Conway and Guy [CG96] call (1.2) the zig-zag triangle and the E_n the zig-zag permutation numbers. The Entringer numbers have also been shown to enumerate several classes of rooted planar trees as well as other mathematical objects [Arn91], [Arn92], [Kem33], [KPP94], [Pou82].

Guy [Guy95] observed that if the entries at the beginnings of the rows in (1.2) are changed from $1, 0, 0, 0, \ldots$ to say $1, 1, 1, 1, 1, \ldots$, or $1, 2, 4, 8, 16, \ldots$, etc., then the numbers at the ends of the rows form interesting-looking sequences not to be found in [SP95]. (Of course now they are in [OEIS].) Using $1,1,1,\ldots$ for example the triangle becomes

$$\begin{array}{ccccccccccc} &&&&& 1 &&&&& \\ &&&& 1 && 2 &&&& \\ &&& 4 && 3 && 1 &&& \\ && 1 && 5 && 8 && 9 && \\ & 24 && 23 && 18 && 10 && 1 & \\ 1 && 25 && 48 && 66 && 76 && 77 \\ &&&&& \ldots &&&&& \end{array} \tag{1.4}$$

yielding the sequence (A667)

$$1,2,4,9,24,77,294,1309,\ldots\ . \tag{1.5}$$

Guy asked if anything could be said about generating functions or combinatorial interpretations for these sequences. The purpose of this note is to answer his question.

2. The Boustrophedon transform

Given a sequence[¶] $\mathbf{a} = (a_1, a_1, a_2, \ldots)$ we define its boustrophedon transform to be the sequence $\mathbf{b} = (b_0, b_1, b_2, \ldots)$ produced by the triangle

$$\begin{array}{ccccccccc} &&&& a_0 = b_0 &&&& \\ &&& a_1 & \rightarrow & b_1 = a_0 + a_1 &&& \\ & b_2 = a_1 + a_2 + b_1 & \leftarrow & a_2 + b_1 && \leftarrow & a_2 && \\ a_3 & \rightarrow & a_3 + b_2 & \rightarrow & a_2 + a_3 + b_1 + b_2 & \rightarrow & b_3 = 2a_2 + a_3 + b_1 + b_2 && \\ &&&& \ldots &&&& \end{array} \tag{2.1}$$

when it is filled in using the rule described in Section 1. Formally, the entries $T_{n,k}$ $(n \ge k \ge 0)$ in the triangle are defined by

$$\begin{aligned} T_{n,0} &= a_n \quad (n \ge 0)\ , \\ T_{n+1,k+1} &= T_{n+1,k} + T_{n,n-k} \quad (n \ge k \ge 0)\ , \end{aligned} \tag{2.2}$$

and then

$$b_n = T_{n,n} \quad (n \ge 0)\ .$$

Although many operations on sequences have been studied in the past (see [BS95] and the references therein), this transformation appears to have been overlooked.

[¶]In this paper we consider only integer-valued sequences, although the transformation can be applied to sequences over any ring.

Theorem 1. *The boustrophedon transform* $\mathbf{b}$ *of a sequence* $\mathbf{a}$ *is given by*

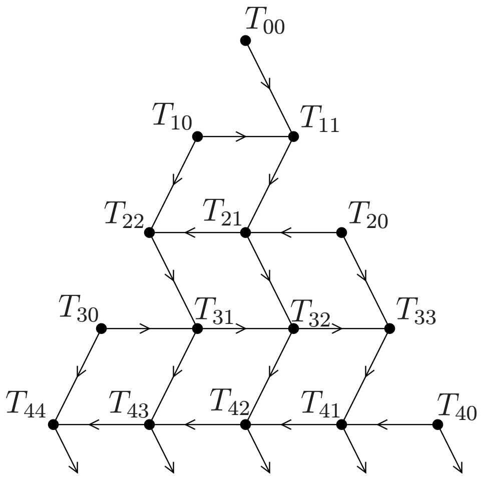
$$b_n = \sum_{k=0}^{n} \binom{n}{k} a_k E_{n-k}, \qquad (n \geq 0) \ , \tag{2.3}$$

$$a_n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} b_k E_{n-k}, \qquad (n \geq 0) \ , \tag{2.4}$$

and the e.g.f.'s of $\mathbf{b}$ *and* $\mathbf{a}$ *are related by*

$$\mathcal{B}(x) = (\sec x + \tan x) \ \mathcal{A}(x) \ . \tag{2.5}$$

Proof. We redraw (2.1) as a directed graph Γ whose nodes are labeled by the numbers $T_{n,k}$ (see Fig. 1). Let $\pi(n,k,i)$ denote the number of paths in Γ from the node labeled $T_{i,0}$ to the



(2.6)

Figure 1: Directed graph Γ underlying the boustrophedon transform.

node labeled $T_{n,k}$. It follows from the rule for constructing the triangle that the numbers $T_{n,k}$ are given by

$$T_{n,k} = \sum_{i=0}^{n} \pi(n,k,i) a_i \ . \tag{2.7}$$

From Section 1 we know that the boustrophedon transform of the sequence $1, 0, 0, 0, \ldots$ is E_0, E_1, E_2, E_3, $\ldots$, and so (from (2.7))

$$E_n = \pi(n,n,0) \quad (n \geq 0) \ . \tag{2.8}$$

We will give a direct proof of this (although of course it is known result, cf. [Arn92]), in order to establish a bijection between paths in Γ and up-down permutations.

Proposition 1. $\pi(n,n,0)$ *is equal to* E_n, *(A1111), the number of down-up permutations of* $\{1, 2, \ldots, n\}$.

Proof. Let P be a path in Γ from the top node to the node labeled $T_{n,n}$. (Fig. 2 shows an example for $n = 5$.) Let $T_{i,f(i)}$ be the label of the node where P arrives at level i ($1 \leq f(i) \leq$

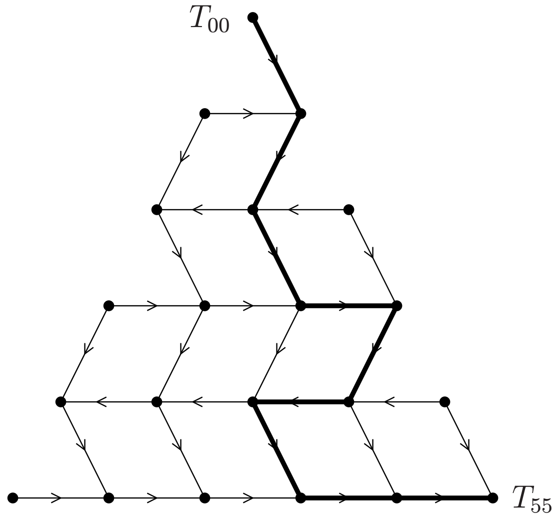


Figure 2: A path from $T_{0,0}$ to $T_{5,5}$.

$i \leq n$). We construct a box diagram to represent P by the following procedure (see Fig. 3). The bottom row contains n boxes labeled $1, \ldots, n$ from left to right (if n is even) or from right to left (if n is odd). The box labeled $f(n)$ is starred. We now repeatedly place a row of boxes above the empty boxes, putting a star in the $f(i)$-th box, always counting from the left if i is even or from the right if i is odd, for $i = n-1, n-2, \ldots, 1$.

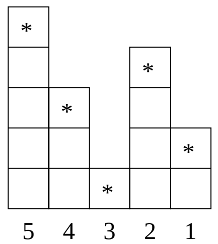


Figure 3: Box diagram corresponding to path P in Fig. 2.

We convert the box diagram into a permutation of $\{1, \ldots, n\}$ by reading the rows from the bottom up and recording the number at the foot of the column containing the star. (The permutation corresponding to the above example is $(3, 1, 4, 2, 5)$.) We omit the easy verification that this process defines a bijection between paths and down-up permutations. ■

Proposition 2.

$$\pi(n,n,k) = \binom{n}{k} E_{n-k}, \quad \textit{for} \;\; 0 \le k \le n \ .$$

Sketch of proof. Consider a path from the node labeled $T_{k,0}$ to the node labeled $T_{n,n}$, such as the path from $T_{4,0}$ to $T_{9,9}$ shown in Fig. 4. The procedure used in the proof of Proposition 1

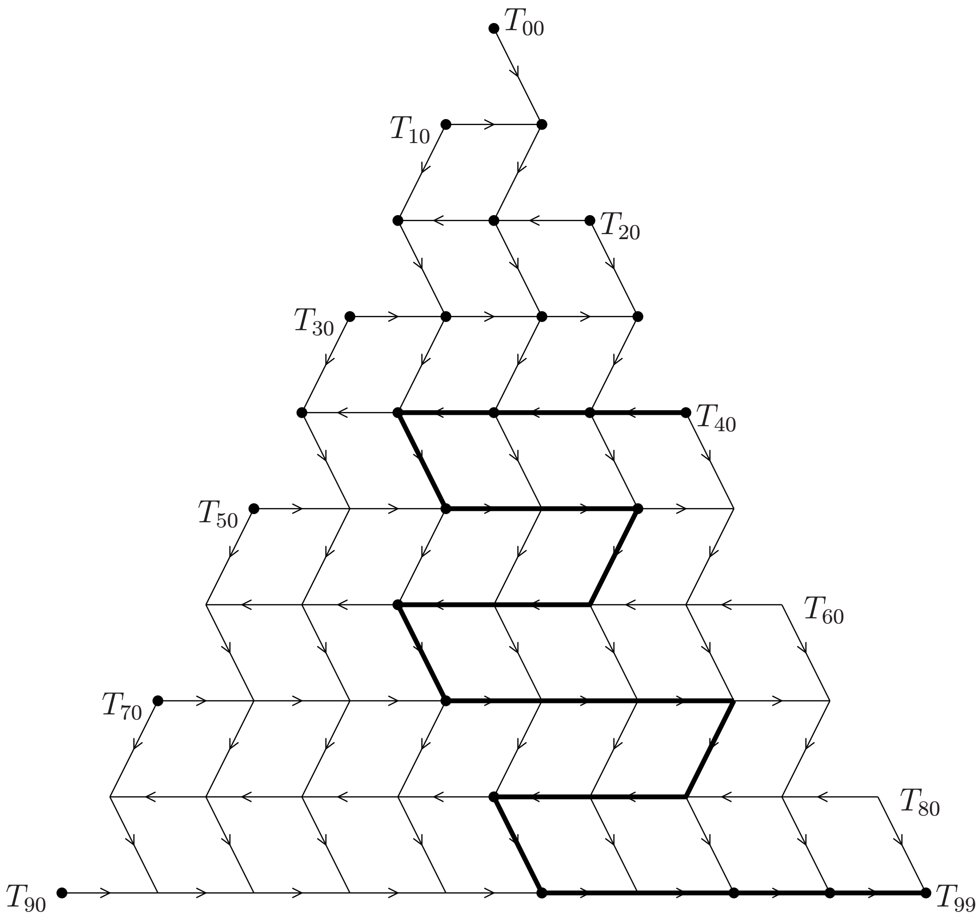


Figure 4: A path from $T_{4,0}$ to $T_{9,9}$.

converts this into a box diagram, which for this example is shown in Fig. 5. The columns that do not contain stars identify one of the $\binom{n}{k}$ k-subsets of $\{1,\ldots,n\}$, while the starred columns themselves form a box diagram (in this case it is that shown in Fig. 3) that identifies a down-up permutation of $\{1,\ldots,n-k\}$. ∎

From Proposition 2 and (2.7) we obtain

$$b_n = T_{n,n} = \sum_{k=0}^{n} \binom{n}{k} E_{n-k} a_k \ ,$$

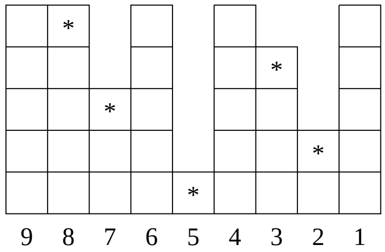


Figure 5: Box diagram for path shown in Fig. 4.

which establishes (2.3). Equations (2.5) and (2.4) now follow immediately. This completes the proof of the theorem. ■

Remark. With only a little more effort we can determine all the "boustrophedon numbers" $\pi(n, k, i)$. Note that $\pi(n, 0, i) = 0$ for $n \geq 1$, $0 \leq i \leq n-1$, and $\pi(n, 0, n) = 1$.

Proposition 3. *For $n \geq 1$, $0 \leq k \leq n-1$,*

$$\pi(n, k, 0) = E_{n,k} = \sum_{r=0}^{[(k-1)/2]} (-1)^r \binom{k}{2r+1} E_{n-2r-1} \ .$$

Proof. $\pi(n, k, 0) = E_{n,k}$ follows from (2.7) and the definition of $E_{n,k}$ (see (1.2)), and the formula for $E_{n,k}$ is given in [Ent66]. ■

Remark. If the path is extended to reach the node labeled $T_{n+1,n+1}$, the corresponding box diagram has the same format as those arising in Proposition 1, except that the star in the last row is constrained to appear in the box labeled $k+1$.

Proposition 4. *For $n \geq 2$, $0 < k < n$, $0 < i \leq n$,*

$$\pi(n, k, i) = \sum_{s=0}^{\min\{k, n-i\}} \binom{k}{s} \binom{n-k}{n-i-s} \pi(n-i, s, 0) \ . \tag{2.9}$$

Sketch of proof. Consider a path P from $T_{i,0}$ to $T_{n,k}$, and complete it to a path Q from $T_{i,0}$ to $T_{n+1,n+1}$ by extending P by a downward sloping edge and a series of horizontal edges, as illustrated in Fig. 6. We form the box diagram for Q, as in Proposition 2 (see Fig. 7a). After deleting all the unstarred columns we obtain the box diagram for a path of type $\pi(n-i, s, 0)$, for some s (Fig. 7b).

In the box diagram for Q itself, the star in the last row divides the remaining stars into two sets of sizes s (to the right) and $n-i-s$ (to the left), and the binomial coefficients in (2.9) count the ways in which the corresponding columns can be selected. ■

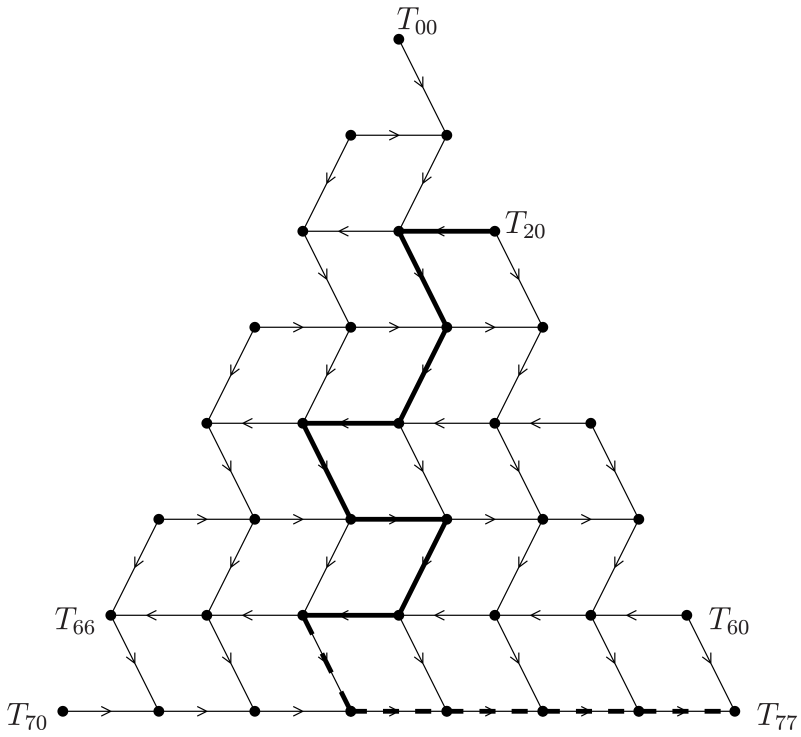


Figure 6: Path from $T_{2,0}$ to $T_{6,4}$ (solid line) and its continuation to $T_{7,7}$ (broken line).

Propositions 1–4 together express all the boustrophedon numbers in terms of the E_n's, and via (2.7) give an explicit formula for every entry in the triangle (2.1).

3. Combinatorial interpretations and examples

Equation (2.3) yields many possible combinatorial interpretations for the numbers b_n. For example, if a_n is the number of arrangements of n labeled objects so that they have some property Q, then b_n is the number of ways of dividing n objects into two groups so that the first group has property Q and the second forms a down-up sequence. Since E_n is also the number of ordered binary trees on n nodes (cf. [Pou82], [KPP94]), other interpretations for the b_n can be given in terms of graphs.

Example 1. We can see now that (1.5) has e.g.f. $e^x(\sec x + \tan x)$, and that the n-th term of this sequence gives the number of ways that we can form a down-up sequence of some length $\ell \geq 0$ from $\{1, \ldots, n\}$. E.g. for $n = 3$ there are 9 possibilities: $\phi, 1, 2, 3, 21, 31, 32, 213, 312$.

Example 2. The boustrophedon transform of the Bell numbers (cf. [SP95], Fig. M4981; sequence A110); produces the sequence $1, 2, 5, 16, 60, 258, \ldots$, (A764) whose n-th term gives the number of ways to take blocks labeled $1, \ldots, n$ and to partition some of them into heaps and to arrange the rest so they form a down-up sequence.

Example 3. The boustrophedon transform of the E_n sequence shifted one place to the left is the same sequence shifted two places to the left:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & \rightarrow & 2 & & & \\
 & & 5 & \leftarrow & 4 & \leftarrow & 2 & & \\
 & 5 & \rightarrow & 10 & \rightarrow & 14 & \rightarrow & 16 & \\
61 & \leftarrow & 56 & \leftarrow & 46 & \leftarrow & 32 & \leftarrow & 16 \\
 & & & & \cdots & & & &
\end{array}$$

In view of Theorem 1, this means the e.g.f. $\mathcal{E}(x)$ satisfies

$$\mathcal{E}(x)\mathcal{E}'(x) = \mathcal{E}''(x) \ .$$

The initial conditions $E_0 = E_1 = 1$ then give $\mathcal{E}(x) = \sec x + \tan x$ as the solution.

Example 4. The sequence $1, 0, 1, 1, 2, 6, 17, 62, 259, 1230, \ldots$ (A661) is the lexicographically earliest sequence that begins with 1 and shifts *two* places left under the boustrophedon transform. (Examples 3 and 4 are both eigen-sequences for this transform, in the notation of [BS95].) We do not know of any combinatorial interpretation for these numbers.

Example 5. The double-ox transform. Generalizing some examples of Arnold ([Arn92], see also [Dum95]), we consider two oxen plowing separate fields with a messenger that takes the output at the end of one row and rushes it to be used by the other ox as input to the next row. For example, if the initial sequence (shown in italics in Fig. 8) is $1, 1, 1, \ldots$, this produces the output sequence (shown in bold) $1, 3, 9, 35, 177, 1123, \ldots$ (A834).

Less colorfully, let $\mathbf{a} = a_0, a_1, \ldots$ be the initial sequence, $\mathbf{m} = m_0, m_1, \ldots$ the middle (or messenger) sequence, and $\mathbf{b} = b_0, b_1, \ldots$ the transformed sequence. We define two triangles of numbers $\{L_{n,k}\}$ and $\{R_{n,k}\}$, with $0 \le k \le n$, by

$$\begin{aligned}
L_{2i,0} &= a_{2i}, \quad R_{2i+1,0} = a_{2i+1}, \\
L_{2i,2i} &= R_{2i,0} = m_{2i}, \quad L_{2i+1,0} = R_{2i+1,2i+1} = m_{2i+1} \\
L_{2i+1,2i+1} &= b_{2i+1}, \quad R_{2i,2i} = b_{2i},
\end{aligned}$$

and

$$L_{n+1,k+1} = L_{n+1,k} + L_{n,n-k}, \quad R_{n+1,k+1} = R_{n+1,k} + R_{n,n-k} .$$

We were happy to find that Theorem 1 leads to an equally simple description of this transformation. The proof is left to the reader.

Theorem 2. *The e.g.f.'s of* **a**, **m** *and* **b** *are related by*

$$\mathcal{M}(x) = \frac{1}{\cos x - \sin x} \mathcal{A}(x) ,$$

$$\mathcal{B}(x) = \frac{\cos x + \sin x}{\cos x - \sin x} \mathcal{A}(x) .$$

Acknowledgements

We thank Richard Guy for suggesting the problem, and J. H. Conway, I. M. Pak and S. Plouffe for helpful discussions. A referee suggests that since the boustrophedon and double-ox transforms are associated with the root systems A_n and C_n, respectively (cf. [Arn92]), it would be worthwhile studying analogues of these transforms for other root systems.

Postscript added June 10 2002

(1) David Callan points out that the proof of Proposition 1 is not quite right. He says:

> The bijection from paths to down-up permutations is not quite right. (It is possible for $f(n)$ to equal 1 and then the box diagram procedure as described would give a permutation beginning with 1—definitely not down-up.) One way to correct it would be the following. Label the n boxes in the bottom row $1, 2, 3, \dots, n$ from left to right in all cases (whether n is even or odd). Then place stars as described in the present proof, except counting from the right for the first (bottom-most) star, then alternately left and right for subsequent stars.

We agree with this comment, and thank him for the correction.

(2) Mike Atkinson drew our attention to two earlier references that we could have cited:

M.D. Atkinson: Zigzag permutations and comparisons of adjacent elements, *Information Processing Letters*, **21** (1985), 187–189.

M.D. Atkinson: Partial orders and comparison problems, in *Sixteenth Southeastern Conference on Combinatorics, Graph Theory and Computing, (Boca Raton, February 1985)*, Congressus Numerantium **47**, 77–88.

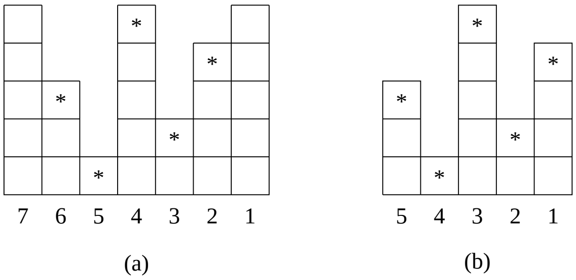


Figure 7: (a) Box diagram corresponding to path shown in Fig. 6. (b) Reduced box diagram, representing path of type $\pi(4,3,0)$.

		1					**1**		
	3		2			2		*1*	
1		4		6	6		8		**9**
35	34		30	24	24	18		10	*1*
1	36	70	100	124	124	148	166	176	**177**
		...					...		

Figure 8: The double-ox transform of $1,1,1,\ldots$ is $1,3,9,35,177,\ldots$.

References

[And79] D. André, Développements de sec x et de tang x, *C. R. Acad. Sci. Paris*, **88** (1879), 965–967.

[And81] D. André, Sur les permutations alternées, *J. Math. Pures Appl.*, **7** (1881), 167–184.

[Arn91] V. I. Arnold, Bernoulli-Euler updown numbers associated with function singularities, their combinatorics and arithmetics, *Duke Math. J.*, **63** (1991), 537–555.

[Arn92] V. I. Arnold, The calculus of snakes and the combinatorics of Bernoulli, Euler and Springer numbers of Coxeter groups, *Uspekhi Mat. Nauk*, **47**:**1** (1992), 3–45. English translation in *Russian Math. Surveys*, **47** (1992), 1–51.

[BS95] M. Bernstein and N. J. A. Sloane, Some canonical sequences of integers, *Lin. Alg. Appl.*, **226–228** (1995), 57–72.

[Com74] L. Comtet, *Advanced Combinatorics*, Reidel, Dordrecht, Holland, 1974, pp. 258 ff.

[CG96] J. H. Conway and R. K. Guy, *The Book of Numbers*, Springer-Verlag, NY, 1997.

[Dum95] D. Dumont, Further triangles of Seidel-Arnold type and continued fractions related to Euler and Springer numbers, *Advances Appl. Math.*, **16** (1995), 275–296.

[Ent66] R. C. Entringer, A combinatorial interpretation of the Euler and Bernoulli numbers, *Nieuw. Arch. Wisk.*, **14** (1966), 241–246.

[Guy95] R. K. Guy, personal communication, 1995.

[Kem33] A. Kempner, On the shape of polynomial curves, *Tôhoku Math. J.*, **37** (1933), 347–362.

[KPP94] A. G. Kuznetzov, I. M. Pak and A. E. Postnikov, Increasing trees and alternating permutations (in Russian), *Uspekhi Mat. Nauk*, **49**:**6** (1994), 79–110. English translation in *Russian Math Surveys*, **49:6** (1994), 79–114.

[Pou82] C. Poupard, De nouvelles significations enumeratives des nombres d'Entringer, *Discrete Math.*, **38** (1982), 265–271.

[Sch61] I. J. Schwatt, *An Introduction to the Operations with Series*, Chelsea, NY, 2nd ed., 1961, esp. Chap. XV.

[Sei77] L. Seidel, Über eine einfache Entstehungsweise der Bernoullischen Zahlen und einiger verwandten Reihen, *Sitzungsber. Münch. Akad.*, **4** (1877), 157–187.

[OEIS] N. J. A. Sloane, The On-Line Encyclopedia of Integer Sequences, published electronically at http://www.research.att.com/∼/njas/sequences/

[SP95] N. J. A. Sloane and S. Plouffe, *The Encyclopedia of Integer Sequences*, Academic Press, San Diego, 1995.