On Kaprekar's Junction Numbers

Max A. Alekseyev
Department of Mathematics & Computational Biology Institute
George Washington University, Ashburn, VA 20147, USA
Email: maxal@gwu.edu

Donovan Johnson[1]
Castle Rock, CO, USA

and

N. J. A. Sloane
Mathematics Department
Rutgers University, Piscataway, NJ 08854, USA
Email: njasloane@gmail.com

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Abstract

A base b junction number u has the property that there are at least two ways to write it as $u = v + s(v)$, where $s(v)$ is the sum of the digits in the expansion of the number v in base b. For the base 10 case, Kaprekar in the 1950's and 1960's studied the problem of finding $K(n)$, the smallest u such that the equation $u = v + s(v)$ has exactly n solutions. He gave the values $K(2) = 101$, $K(3) = 10^{13} + 1$, and conjectured that $K(4) = 10^{24} + 102$. In 1966 Narasinga Rao gave the upper bound $10^{1111111111124} + 102$ for $K(5)$, as well as upper bounds for $K(6)$, $K(7)$, $K(8)$, and $K(16)$. We will present a set of recurrences which determine $K(n)$ for any base b, and in particular we will show that these conjectured values of $K(n)$ are correct. The key to our approach is an apparently new recurrence for $F(u)$, the number of solutions to $u = v + s(v)$. We give tables of $K(n)$ for bases $b \le 10$. Rather surprisingly, the solution to the base 2 problem is determined by the classical Thue-Morse sequence.

1 Introduction

For a fixed base $b \ge 2$, let $s(v)$ denote the sum of the digits in the base b expansion of $v \in \mathbb{N} = \{0, 1, 2, \ldots\}$, and let $f(v) = v + s(v)$. Sequences that arise by iterating f have a long history [1, 13] (the latter reference has an extensive bibliography). In the 1950's and 1960's, Dattaraya Ramchandra Kaprekar [3, 4, 5, 6, 7, 8] studied the inverse mapping to f in the base 10 case. Let $\mathrm{Gen}(u) = \{v \in \mathbb{N} \mid f(v) = u\}$ and $F(u) = |\,\mathrm{Gen}(u)|$. Kaprekar called the elements of $\mathrm{Gen}(u)$ the *generators* of u, and defined a *self-number* to be any number u for which $F(u) = 0$. The first few self-numbers (in base 10) are

$$1, 3, 5, 7, 9, 20, 31, 42, 53, 64, 75, 86, 97, 108, 110, 121, 132, 143, 154, 165, 176, \ldots \quad (1)$$

[1] We deeply regret that Donovan Johnson passed away in March, 2014.

(A003052).[2] Self-numbers are also known as *Colombian numbers*, after a problem proposed by Recamán [11].

Kaprekar called numbers with at least two generators *junction numbers*. The smallest junction number (again in base 10) is 101, which has generators 91 and 100, and the first few junction numbers are

$$101, 103, 105, 107, 109, 111, 113, 115, 117, 202, 204, 206, 208, 210, 212, 214, \ldots \quad (2)$$

(A230094).

Kaprekar was particularly interested in finding what we will call $K(n)$, the smallest number with n generators, which is the subject of the present paper. We will show that the sequence $K(n)$, for $n = 1, 2, 3, \ldots$, begins

$$\begin{aligned} &0,\ 101,\ 10^{13}+1,\ 10^{24}+102,\ 10^{1111111111124}+102,\ 10^{2222222222224}+10^{13}+2, \\ &\quad 10^{(10^{24}+10^{13}+115)/9}+10^{13}+2,\ 10^{(2\cdot 10^{24}+214)/9}+10^{24}+103, \\ &\quad 10^{(10^{1111111111124}+10^{24}+214)/9}+10^{24}+103,\ \ldots \end{aligned} \quad (3)$$

(A006064).[3] It is easy to check by hand that $K(2) = 101$, and with today's computers it is easy to verify $K(3) = 10^{13}+1$ by direct search. As to what was known by Kaprekar and his colleagues nearly fifty years ago, the various accounts given by Kaprekar [6], Narasinga Rao [9], and Gardner [2] do not quite agree, but it seems that Kaprekar believed that he had proved that $K(3) = 10^{13} + 1$, that the putative value $10^{24} + 102$ for $K(4)$ was discovered independently by Kaprekar and Professor Gunjikar in 1961, and that Kaprekar was convinced that it was the true value of $K(4)$ and not just an upper bound. Gardner [2] reports in 1975 that Kaprekar told him that he had also found what he conjectured to be the values of $K(5)$ and $K(6)$. Kaprekar's work on this problem is also discussed by Schorn [12] and Trotter [14].

However, Narasinga Rao, writing in 1963 [9] (although not published until 1966), states things slightly differently. He gives a recipe for finding junction numbers with a specified number of generators, improving on an earlier recipe of Kaprekar's, and gives Kaprekar's value of $K(3) = 10^{13} + 1$. He then *conjectures* that $K(4) = 10^{24} + 102$, and gives as a candidate for $K(5)$ the value $10^{1111111111124}+102$, remarking that no much smaller value is likely to exist. (Narasinga Rao's recipe does not necessarily produce the smallest junction number with a given number of generators.) He also gives upper bounds for $K(6)$, $K(7)$, $K(8)$, and $K(16)$. Remarkably enough, all of Narasinga Rao's upper bounds turn out to be the true values for these $K(n)$. We return to the base 10 case in Section 8.

The main goal of this paper is to present a set of recurrences which generate the sequence $\{K(n)\}$ for any base b (see Section 6 and in particular Theorem 15). These recurrences depend upon an apparently new recurrence for $F(u)$, discussed in Section 3.

Because the values of $K(n)$ for small b and n are both easy to determine and somewhat exceptional, we start by discussing the cases $n \leq 3$ in Section 4 and bases $b = 2$ and $b = 3$

[2]Throughout this article, six-digit numbers prefixed by A refer to entries in the OEIS [10].

[3]The reader may detect a pattern in these numbers, but should be warned that it breaks down after a while. See Tables 2 and 11.

in Section 5. Bases 2 and 3 are also exceptional since for them the recurrences for $K(n)$ are quite simple and can be obtained without the machinery developed in Section 6. Tables 1 and 2 in Section 4 collect the numerical values of $K(n)$ for $n \le 7$ and bases $b \le 10$.

Section 6 gives the recurrences for a general base b. For each b, a by-product of the recurrences is an infinite sequence $\{\tau(n), n \ge 1\}$ of integer values in the range $[0, b-2]$ if b is even, or $[0, \frac{b-3}{2}]$ if b is odd. Sections 7 and 8 apply the results of Section 6 to bases 4, 5, 7, and 10. For base 5, it turns out that $\{\tau(n)\}$ is essentially the classical Thue-Morse sequence A010060, and so, for any base, we refer to $\{\tau(n)\}$ as a "generalized Thue-Morse sequence". For example, in bases 4 and 7 we obtain the ternary sequence shown in (71), and in base 10 the sequence (73). It would be nice to know more about these generalized Thue-Morse sequences.

Section 9 (and in particular Conjecture 20) discusses the rate of growth of $K(n)$ as a function of n. We begin by applying the recurrence for $F(u)$ from Section 3 to establish some general bounds on $K(n)$. Then we consider the representation of $K(n)$ as a tower of exponentials. For bases $b \ne 3$, it appears that $K(n)$ is a tower

$$K(n) \;=\; b^{b^{\cdot^{\cdot^{\cdot^{b^{\omega(n)}}}}}}, \tag{4}$$

with $0 < \omega(n) \le 1$, of height $\lceil \log_2(n) \rceil + \lambda$, where $\lambda = 3$ if $b = 2$, $\lambda = 2$ if $b \ge 4$ is even, and $\lambda = 1$ if $b \ge 5$ is odd (base 3 is slightly exceptional).

Notation. We will always work in a fixed base $b \ge 2$. If the base b expansion of $v \in \mathbb{N} = \{0, 1, 2, \ldots\}$ is

$$v \;=\; \sum_{i=0}^{k-1} v_i b^i \tag{5}$$

(where $0 \le v_i < b$, $k = \lceil \log_b(v+1) \rceil$, and $v_{k-1} \ne 0$ unless $v = 0$), we refer to the v_i as "digits", even if $b \ne 10$, and say that v has "length" k. We will also write v as

$$v \;=\; [v_{k-1}, v_{n-2}, \ldots, v_1, v_0]_b\,, \tag{6}$$

where we omit the commas between the digits if there is no possibility of confusion. In this notation, $s(v) = \sum_{i=0}^{k-1} v_i$ and $f(v) = v + s(v) = \sum_{i=0}^{k-1} v_i(b^i + 1)$.

As already mentioned, for $u \in \mathbb{N}$, we let $\mathrm{Gen}(u) = \{v \in \mathbb{N} \mid f(v) = u\}$, $F(u) = |\,\mathrm{Gen}(u)\}|$, and for $u < 0$ we set $\mathrm{Gen}(u) = \emptyset$ (the empty set) and $F(u) = 0$. For $n \ge 1$, $K(n)$ is defined to be the smallest $u \in \mathbb{N}$ such that $F(u) = n$. For $n \ge 2$ it will turn out that the leading base b digit of $K(n)$ is 1 (see Theorem 6), and we define $E(n)$ by $K(n) = b^{E(n)} +$ terms of smaller order (in other words, $E(n) + 1$ is the length of $K(n)$). In Tables 1 and 2 we write $E_b(n)$ to indicate the value of b. For use in proving the main theorem, in Section 4 we also define $K_i(n)$ to be the smallest $u \in \mathbb{N}$ such that $F(u) = n$ and $u \equiv i \pmod{b-1}$. Of course, the functions s, f, F, Gen, K, $K_i(n)$ all depend on the value of b, but to indicate this with subscripts would have made the equations unnecessarily complicated, so we hope the value of b will always be clear from the context.

2 Preliminary results

We begin with some elementary lemmas.

Lemma 1. *(i) If b is odd, then $f(v)$ is even for any $v \in \mathbb{N}$, and so $F(u) = 0$ if u is odd.*
(ii) If b is even, then $F(u) = 0$ if $u < b$ is odd.

Proof. (i) If b is odd and v is given by (5) then $f(v) = \sum_i v_i(b^i + 1)$ is even.
(ii) Numbers below b have at most one generator, v (say), for which $f(v) = 2v$ is even. □

The next lemma says that if $u = f(v)$, then v is smaller than u, but not too much smaller. This is useful when making computer searches.

Lemma 2. *If $u \geq 2, v \in \mathbb{N}$ satisfy $v + s(v) = u$, then*

$$u - (b-1)\lceil \log_b(u) \rceil \leq v \leq u - 1 \,. \tag{7}$$

Proof. Since $u \geq 2$, $v \geq 1$, $s(v) \geq 1$, and so $v \leq u - 1$. Since the length of v is $k = \lceil \log_b(v+1) \rceil \leq \lceil \log_b(u) \rceil$, we have $s(v) \leq (b-1)\lceil \log_b(u) \rceil$. □

The third lemma is a generalization of the observation that the Hamming weight of $2^m - 1 - v$ is equal to $m -$ Hamming weight(v). We omit the proof.

Lemma 3. *If $m \geq 0$, $1 \leq c \leq b - 1$, and $0 \leq v \leq cb^m - 1$, then*

$$s(cb^m - 1 - v) \;=\; (b-1)m + c - 1 - s(v) \,. \tag{8}$$

For example, in base 10, $s(281) = s(3 \cdot 10^2 - 1 - 18) = 9 \cdot 2 + 2 - 9 = 11$.

Lemma 4. *Let $n \geq 2$ be an integer. (i) If $\{a(i), i \geq 1\}$ is a sequence of nonnegative numbers such that $a(m+1) \geq 2a(m)$ for all $m = 1, 2, \ldots, n-2$, then*

$$\min_{1 \leq i \leq n-1} \{a(i) + a(n-i)\} \;=\; a\left(\left\lceil \frac{n}{2} \right\rceil \right) + a\left(\left\lfloor \frac{n}{2} \right\rfloor \right) \quad \text{for } n \geq 2 \,. \tag{9}$$

(ii) If $\{a(i), i \geq 1\}$ and $\{b(i), i \geq 1\}$ are a pair of sequences of nonnegative numbers such that

$$a(m+1) \geq a(m) + b(m) \quad \text{and} \quad b(m+1) \geq a(m) + b(m) \tag{10}$$

for all $m = 1, 2, \ldots, n-2$, then

$$\min_{1 \leq i \leq n-1} \{a(i) + b(n-i)\} \;=\; \text{either } a\left(\left\lceil \frac{n}{2} \right\rceil \right) + b\left(\left\lfloor \frac{n}{2} \right\rfloor \right) \text{ or } a\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + b\left(\left\lceil \frac{n}{2} \right\rceil \right). \tag{11}$$

Proof. (i) Suppose $n = 2t$. Then $2a(t) \leq a(t-1) + 2a(t) \leq a(t-1) + a(t+1) \leq 2a(t+1) \leq a(t+2) \leq a(t-2) + a(t+2) \leq \cdots \leq a(1) + a(n-1)$. If $n = 2t+1$ is odd, then $a(t) + a(t+1) \leq 2a(t+1) \leq a(t+2) \leq a(t-1) + a(t+2) \leq \cdots \leq a(1) + a(n-1)$. Part (ii) is proved similarly. □

We call a sequence $\{a(i), i \geq 1\}$ satisfying the hypothesis of part (i) for all $n \geq 2$ a *sequence of exponential type*, and we say that sequences $\{a(i), i \geq 1\}$ and $\{b(i), i \geq 1\}$ satisfying the hypotheses of part (ii) for all $n \geq 2$ form a *pair of sequences of exponential type*.

3 The recurrence for $F(v)$

The following result, in particular the recurrence (15), is the key to the whole paper.

Theorem 5. *We have* $\mathrm{Gen}(0) = \{0\}$, $F(0) = 1$, $\mathrm{Gen}(1) = \emptyset$, $F(1) = 0$. *For* $u \geq 2$, *write* u *uniquely as*

$$u \;=\; cb^m + k + c\,, \tag{12}$$

where $m \geq 0$, $1 \leq c \leq b-1$, *and*

$$\begin{cases} 0 \leq k \leq b^m, & \text{if } c < b-1; \\ 0 \leq k \leq b^m - b + 1, & \text{if } c = b-1. \end{cases} \tag{13}$$

Then

$$\mathrm{Gen}(u) \;=\; \{cb^m + v \mid v \in \mathrm{Gen}(k)\} \;\cup\; \{cb^m - 1 - v \mid v \in \mathrm{Gen}((b-1)m - k - 2)\} \tag{14}$$

and

$$F(u) \;=\; F(k) \;+\; F((b-1)m - k - 2)\,. \tag{15}$$

Remarks. (i) Since in (12) we require that $c \geq 1$, a number u of the form b^r is represented by taking $c = b-1$, $m = r-1$, and $k = b^{r-1} - b + 1$. (ii) Note that (15) does not mention c.

Proof. The first assertion is clear (there is no $v \in \mathbb{N}$ such that $f(v) = 1$), and (15) follows at once from (14). To prove (14), let $u \;=\; cb^m + k + c$, with $m \geq 0$, $1 \leq c \leq b-1$, and k satisfying (13). We will show that (i) any element of the right-hand side of (14) is a generator of u, and (ii) every generator of u is an element of the right-hand side of (14).

(i) Suppose $v \in \mathrm{Gen}(k)$. Since $k \leq b^m$, $v < b^m - 1$ (by (7)), we have $s(cb^m + v) = c + s(v)$, and $f(cb^m + v) = cb^m + v + c + s(v) = cb^m + k + c = u$. On the other hand, suppose $v \in \mathrm{Gen}((b-1)m - k - 2)$. Let $w = cb^m - 1 - v$. By Lemma 3, $s(w) = (b-1)m + c - 1 - s(v)$ (the condition $v \leq cb^m - 2$ follows from $v < (b-1)m - k - 2$). Then $f(w) = w + s(w) = cb^m - 1 - v + (b-1)m + c - 1 - s(v) = cb^m + k + c = u$.

(ii) Suppose w is a generator for $u = cb^m + c + k$. Clearly, $u \leq b^{m+1}$, and $u = b^{m+1}$ only when $c = b-1$ and $k = b^m - b + 1$. Trivially, either $w \geq cb^m$ or $w < cb^m$.

First, suppose $w \geq cb^m$ and write it as $w = cb^m + v$. If $v < b^m$ then $s(w) = c + s(v)$, and $w + s(w) = u$ implies $v + s(v) = k$ and $v \in \mathrm{Gen}(k)$. If $v \geq b^m$ then if $c = b-1$, $w \geq b^{m+1} \geq u$, contradicting (7). So $c \leq b-2$ and $w = (c+1)b^m + \mu$, where $\mu = v - b^m \geq 0$, which implies $u \geq (c+1)b^m$, that is, $u = (c+1)b^m + \lambda$, where $\lambda = c + k - b^m \leq c$. But $w + s(w) = u$ implies $\mu + s(\mu) + c + 1 \;=\; \lambda$, a contradiction. So $v \geq b^m$ cannot happen.

Second, suppose $w < cb^m$, that is, $w = cb^m - 1 - v$ with $0 \leq v \leq cb^m - 2$. By Lemma 3, $s(w) = (b-1)m + c - 1 - s(v)$, and $w + s(w) = u$ implies $v + s(v) = (b-1)m - k - 2$ and $v \in \mathrm{Gen}((b-1)m - k - 2)$. ∎

For example, in base 10, if $u = 10^{13}+1$, we have $c = 1$, $m = 13$, $k = 0$, so $F(10^{13}+1) = F(0) + F(115)$. Now $115 = 10^2 + 1 + 14$, so $F(115) = F(14) + F(2) = 1 + 1$, and therefore $F(10^{13} + 1) = 3$. In Section 4 we will confirm Kaprekar's result that there is no smaller number with three inverses.

When $u = K(n)$, the smallest number with n generators in base b, we can make a stronger assertion than (12). (Note that $K(1) = 0$ for any base.)

Theorem 6. *Let $b \ge 2$ and $n \ge 2$.*

(i) We have

$$K(n) \;=\; b^{E(n)} + k + 1\,, \tag{16}$$

where the exponent $E(n)$ is at least 1 and $0 \le k \le (b-1)E(n) - 2$.

(ii) If b is odd then $K(n)$ and k are both even.

Proof. (i) Let $u = K(n)$, and write $u = cb^m + k + c$ as in Theorem 5, with $m \ge 0$, $1 \le c \le b-1$, and k as in (13). We consider three cases: $c = 1$ and $k \le (b-1)m-2$; $c = 1$ and $k > (b-1)m - 2$; and $c > 1$. In the first case, nothing needs to be done (it is easy to see that m cannot be zero), and we set $E(n) = m$. In the second case we apply (15), obtaining $F(u) = n = F(k) + F((b-1)m - k - 2)$. Since the argument of the last term is negative, we have $F(u) = F(k)$ and thus $k < u$ also has n generators, a contradiction. In the third case, we let $u' = b^k + k + 1$ and notice that by (15) $F(u') = F(u) = n$ while $u' < u$, a contradiction.

(ii) If b is odd, $K(n)$ is even by Lemma 1, and therefore k is even. □

4 Values of $K(n)$ for bases $b \le 10$ and $n \le 7$

Tables 1 and 2 show the values of $K(n)$ for $n \le 7$ and bases $2 \le b \le 10$. The small entries in these two tables, that is, entries below about 10^{10}, were obtained by computer, either by C or Maple programs based on Lemma 4, or by using the PARI/GP program given in the Appendix. The values of $K(2)$ and $K(3)$ for any base b will be derived in this section, and the values of $K(n)$ for any n and bases 2 and 3 in the next section.. The values of $K(n)$ in Tables 1 and 2 for $n \ge 4$ and bases $b \ge 4$ are included here for convenience, but they will not be officially established until we have the main theorem of Section 6. (The values of $K(2)$ and $K(3)$ could also be obtained from the recurrences of Section 6, but it seems more informative to calculate them directly.)

Theorem 7.

$$K(2) \;=\; \begin{cases} b^2+1, & \text{if } b \text{ is even,} \\ b+1, & \text{if } b \text{ is odd.} \end{cases} \tag{17}$$

Proof. Suppose b is even. The number $b^2 + 1 = [101]_b$ has the two generators $b^2 = [100]_b$ and $b^2 - b + 1 = [b-1, 1]_b$, so $K(2) \le b^2 + 1$. However, it is easy to check by hand that the values of $f(v)$ for $0 \le v \le b^2$ are all distinct, so $K(2) = b^2 + 1$. The case when b is odd is even easier to verify and we omit the details. □

b	2	3	4	5	6
$K(1)$	0	0	0	0	0
$K(2)$	5	4	17	6	37
$K(3)$	129	28	4^7+1	26	6^9+1
$K(4)$	4102	248	$4^{12}+18$	632	$6^{16}+38$
$K(5)$	$2^{136}+6$	$3^{17}+5$	$4^{5468}+18$	5^9+9	$6^{(6^9+44)/5}+38$
$K(6)$	$2^{260}+130$	$3^{29}+29$	$4^{10924}+4^7+2$	$5^{15}+27$	$6^{(2\cdot 6^9+8)/5}+6^9+2$
$K(7)$	$2^{4233}+130$	$3^{139}+29$	$4^{E_4(7)}+4^7+21$, $E_4(7)=(4^{12}+4^7+40)/3$	$5^{165}+27$	$6^{E_6(7)}+6^9+2$, $E_6(7)=(6^{16}+6^9+43)/5$

Table 1: Values of $K(1),\ldots,K(7)$ for bases $b=2,\ldots,6$ (the columns are A230303, A230640, A230638, A230867, A238840).

b	7	8	9	10
$K(1)$	0	0	0	0
$K(2)$	8	65	10	101
$K(3)$	50	$8^{11}+1$	82	$10^{13}+1$
$K(4)$	352	$8^{20}+66$	740	$10^{24}+102$
$K(5)$	$7^{10}+9$	$8^{E_8(5)}+66$, $E_8(5)=(8^{11}+76)/7$	$9^{12}+11$	$10^{E_{10}(5)}+102$, $E_{10}(5)=(10^{13}+116)/9$
$K(6)$	$7^{17}+51$	$8^{E_8(6)}+8^{11}+2$, $E_8(6)=(2\cdot 8^{11}+12)/7$	$9^{21}+83$	$10^{E_{10}(6)}+10^{13}+2$, $E_{10}(6)=2(10^{13}+8)/9$
$K(7)$	$7^{67}+51$	$8^{E_8(7)}+8^{11}+2$, $E_8(8)=(8^{20}+8^{11}+75)/7$	$9^{103}+83$	$10^{E_{10}(7)}+10^{13}+2$, $E_{10}(7)=(10^{24}+10^{13}+115)/9$

Table 2: Values of $K(1),\ldots,K(7)$ for bases $b=7,\ldots,10$ (the columns are A238841, A238842, A238843, A006064).

Theorem 8.

$$K(3) \;=\; \begin{cases} 129, & \textit{if } b=2, \\ 28, & \textit{if } b=3, \\ b^{b+3}+1, & \textit{if } b\geq 4 \textit{ is even}, \\ b^2+1, & \textit{if } b\geq 5 \textit{ is odd}. \end{cases} \tag{18}$$

Proof. For $b=2,3,4$ we refer to Table 1. Suppose first that $b\geq 6$ is even. Certainly $b^{b+3}+1$ has three generators, namely $b^{b+3}-1-[1,0,b-3]_b$, $b^{b+3}-1-[b-1,b-2]_b$, and b^{b+3} (this is easily checked using Lemma 3). So $K(3)\leq b^{b+3}+1$. If $u=K(3)<b^{b+3}+1$, then by Theorem 6 we would have $u=b^m+k+1$ with $m\leq b+2$ and $0\leq k\leq (b-1)m-2$. By (15) we have $3=F(u)=F(k)+F((b-1)m-k-2)$. So either $F(k)=1$ and $F((b-1)m-k-2)=2$, or $F(k)=2$ and $F((b-1)m-k-2)=1$. We discuss only the first possibility, the second one being similar. From $k\geq K(1)=0$ and $(b-1)m-k-2\geq K(2)=b^2+1$ we find that m must equal $b+2$, and $0\leq k\leq b-5$. By Lemma 1, k is even. Write $\lambda=b-5-k$, where

λ is odd and $0 \le \lambda \le b-5$. But now $F(b^2+\lambda+1) = 2 = F(\lambda) + F(\lambda-4)$ implies (again by Lemma 1) that λ is even, a contradiction. This completes the proof of $K(3) = b^{b+3}+1$ for even $b \ge 6$.

Second, suppose $b \ge 5$ is odd. Certainly b^2+1 has three generators, $b^2-1-[1,(b-5)/2]_b$, $b^2-1-[b-2]_b$, and b^2, and we can easily check that at most two of the values $f(v)$ for $0 \le v \le b^2$ can coincide. □

Theorem 8 confirms Kaprekar's result that $K(3) = 10^{13}+1$ in base 10.

When we come to study the case of a general base b in §6, we will need to know the values of a refined version of $K(n)$. We define $K_i(n)$ to be the smallest number $v \equiv i \pmod{b-1}$ for which $F(v) = n$, where i is a residue class modulo $b-1$. If b is odd, by Lemma 1 we need only consider even values of i, so we can say more precisely that $i \in \mathcal{I}$, where $\mathcal{I}$ is a subset of the residue classes modulo $b-1$ given by

$$\mathcal{I} = \begin{cases} \{0,1,2,3,\ldots,b-2\}, & \text{if } b \text{ is even,} \\ \{0,2,4,6,\ldots,b-3\}, & \text{if } b \text{ is odd.} \end{cases}$$

Then we have:

$$K(n) \;=\; \min_{i \in \mathcal{I}} K_i(n)\,. \tag{19}$$

There is an analog of Theorem 6 for $K_i(n)$.

Theorem 9. *For any $b \ge 2$, $i \in \mathcal{I}$, and $n \ge 2$, we have*

$$K_i(n) \;=\; c b^{E(n)} + k + c\,, \tag{20}$$

where $E(n)$ is as in (16), for some integers c and k satisfying $1 \le c \le b-1$ and $0 \le k \le (b-1)E(n)-2$.

Proof. Theorem 6 states that for some k', we have $F(b^{E(n)}+k'+1) = n$. Theorem 5 further implies that for any c' and the same k',

$$F(c'b^{E(n)}+k'+c') = F(k') + F((b-1)E(n)-k'-2) = F(b^{E(n)}+k'+1) = n\,.$$

Let us show that there exists a value $c' = c_i$ such that $c_i b^{E(n)} + k' + c_i \equiv i \pmod{b-1}$. Since $c_i b^{E(n)} + k' + c_i \equiv k' + 2c_i \pmod{b-1}$, we want $k'+2c_i \equiv i \pmod{b-1}$. For even b, this congruence is trivially solvable for c_i in the interval $1 \le c_i \le b-1$. For odd b, $i \in \mathcal{I}$ is even and so is k' (by Theorem 6), so the congruence is again solvable for c_i.

By the definition of $K_i(n)$, we have

$$b^{E(n)} + k' + 1 = K(n) \le K_i(n) \le c_i b^{E(n)} + k' + c_i,$$

implying that $K_i(n)$ has the form (20) if we take $c = c_i$. □

For bases $b = 2$ and 3, we have $\mathcal{I} = \{0\}$, so there is only one $K_i(n)$, which is $K_0(n) = K(n)$. For bases $b \ge 4$ and $n \le 3$, we can give $K_i(n)$ explicitly. In the following three theorems, the subscripts i in $K_i(n)$ are elements of $\mathcal{I}$, and in particular are to be read modulo $b-1$. We omit the proofs, which are similar to those of Theorems 7 and 8. Table 3 illustrates these theorems.

Theorem 10. *For bases $b \geq 4$,*

$$K_{2\lambda}(1) = 2\lambda \tag{21}$$

for $0 \leq \lambda \leq b-2$ (if b even) or $0 \leq \lambda \leq \frac{b-3}{2}$ (if b odd).

Theorem 11. *For even bases $b \geq 4$,*

$$K_{2+2\lambda}(2) = b^2 + 1 + 2\lambda \text{ for } 0 \leq \lambda \leq b-2; \tag{22}$$

for odd bases $b \geq 5$,

$$K_{2+2\lambda}(2) = b + 1 + 2\lambda \text{ for } 0 \leq \lambda \leq \frac{b-3}{2}. \tag{23}$$

Theorem 12. *For even bases $b \geq 4$,*

$$K_0(3) = b^{b+3} + b^2 + 2b - 4, \tag{24}$$

$$K_{2+2\lambda}(3) = b^{b+3} + 1 + 2\lambda \text{ for } 0 \leq \lambda \leq b-3; \tag{25}$$

for odd bases $b \geq 5$,

$$K_0(3) = b^2 + 2b - 3, \tag{26}$$

$$K_{2+2\lambda}(3) = b^2 + 1 + 2\lambda \text{ for } 0 \leq \lambda \leq \frac{b-5}{2}. \tag{27}$$

For bases $b \geq 4$ the minimal values of $K_i(2)$ and $K_i(3)$ occur when $\lambda = 0$, that is, when $i = 2$, and (via (19)) confirm the values of $K(2)$ and $K(3)$ given in Theorems 7 and 8.

$b=6$				$b=9$			
$i \setminus n$	1	2	3	$i \setminus n$	1	2	3
0	0	45	6^9+44	0	0	16	96
1	6	41	6^9+5	2	2	10	82
2	2	37	6^9+1	4	4	12	84
3	8	43	6^9+7	6	6	14	86
4	4	39	6^9+3				

Table 3: Values of $K_i(n)$ ($n \leq 3$) for bases $b = 6$ (left) and $b = 9$ (right), illustrating Theorems 10-12.

5 $K(n)$ for bases 2 and 3

We first discuss the base 2 case for general n. The initial values of $f(u)$ and $F(u)$ are shown in Table 4. We see that the smallest numbers with 1 and 2 generators are $K(1) = 0$ and $K(2) = 5$, respectively. Direct search by computer gives $K(3) = 129$ and $K(4) = 4102$, as we have already seen in Table 1 (although $K(5) = 2^{136} + 6$ is out of reach). The general solution is given by the following pair of recurrences.

u	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
$f(u)$	0	2	3	5	5	7	8	10	9	11	12	14	14	16	17	19	17	19	20	22	22
$F(u)$	1	0	1	1	0	2	0	1	1	1	1	1	1	0	2	0	1	2	0	2	1

Table 4: Values of $f(u)$ and $F(u)$ in base 2 (A092391, A228085)

Theorem 13. *For base $b = 2$, for $n \geq 2$,*

$$K(n) \;=\; 2^{E(n)} + K\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1\,, \tag{28}$$

where

$$E(n) \;=\; K\left(\left\lceil \frac{n}{2} \right\rceil \right) + K\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 2\,. \tag{29}$$

Also

$$K(n) \;>\; 2K(n-1)\,. \tag{30}$$

Proof. The proof is by induction on n. The results are true for $n \leq 3$, so we assume $n \geq 4$. As in Theorem 6, let $K(n) = u = 2^m + k + 1$, where $m = E(n)$ and $0 \leq k \leq m - 2$. By (15), $F(u) = n = F(k) + F(m-k-2)$. Let $x = F(k)$, $y = F(m-k-2)$ so that $x + y = n$. Then $k \geq K(x)$, $m - k - 2 \geq K(y)$, and thus

$$m \geq K(x) + K(y) + 2\,. \tag{31}$$

We know from (30) that $\{K(n)\}$ is of exponential type, so by Lemma 4 the right-hand side of (31) is minimized when either $x = \lceil n/2 \rceil$, $y = \lfloor n/2 \rfloor$ or $x = \lfloor n/2 \rfloor$, $y = \lceil n/2 \rceil$ (if n is even there is no difference). It follows that the value of $E(n) = m$ is given by (29). The minimal value of k is the smaller of $K(\lceil n/2 \rceil)$ and $K(\lfloor n/2 \rfloor)$, which is $K(\lfloor n/2 \rfloor)$. This proves (28). The proof of (30) is now a routine calculation; we omit the details. □

Remark. The proof also shows that

$$\mathrm{Gen}(K(n)) \;=\; \{2^{E(n)} + v \mid v \in \mathrm{Gen}\left(K\left(\left\lfloor \frac{n}{2} \right\rfloor \right)\right)\} \,\cup\, \{2^{E(n)} - 1 - v \mid v \in \mathrm{Gen}\left(K\left(\left\lceil \frac{n}{2} \right\rceil \right)\right)\}\,. \tag{32}$$

Table 5 extends the $b = 2$ column of Table 1 to $n = 16$. (The first 100 terms of $E(n)$ and $K(n)$ are given in the entries A230302 and A230303 in [10]).

There is a similar pair of recurrences in the base 3 case.

Theorem 14. *For base $b = 3$, for $n \geq 2$,*

$$K(n) \;=\; 3^{E(n)} + K\left(\left\lfloor \frac{n}{2} \right\rfloor \right) + 1\,, \tag{33}$$

where

$$E(n) \;=\; \frac{K(\lceil \frac{n}{2} \rceil) + K(\lfloor \frac{n}{2} \rfloor) + 2}{2}\,. \tag{34}$$

Also

$$K(n) \;>\; 3K(n-1)\,. \tag{35}$$

n	$E(n)$	$K(n)$
8	8206	$2^{8206} + 4103$
9	$2^{136} + 4110$	$2^{E(9)} + 4103$
10	$2^{137} + 14$	$2^{E(10)} + 2^{136} + 7$
11	$2^{260} + 2^{136} + 138$	$2^{E(11)} + 2^{136} + 7$
12	$2^{261} + 262$	$2^{E(12)} + 2^{260} + 131$
13	$2^{4233} + 2^{260} + 262$	$2^{E(13)} + 2^{260} + 131$
14	$2^{4234} + 262$	$2^{E(14)} + 2^{4233} + 131$
15	$2^{8206} + 2^{4233} + 4235$	$2^{E(15)} + 2^{4233} + 131$
16	$2^{8207} + 8208$	$2^{E(16)} + 2^{8206} + 4104$

Table 5: Base 2: $E(n)$ and $K(n)$ for $n = 8, \ldots, 16$, extending Table 1.

Proof. The proof is similar to that of Theorem 13, except at one step. Again we use induction on $n \geq 4$. Suppose $K(n) = u = 3^m + k + 1$, where $0 \leq k \leq 2m - 2$. Then $F(u) = n = F(k) + F(2m - k - 2) = x + y$, say, with $x + y = n$. Then $k \geq K(x)$, $2m - k - 2 \geq K(y)$, so

$$2m \geq K(x) + K(y) + 2\,. \tag{36}$$

The difference from (31) in the base 2 case lies in the presence of the factor of 2 (in general it will be $b - 1$) on the left-hand side of this inequality. So now we must minimize the sum $K(x) + K(n - x)$ subject to the additional requirement that the sum is even. Here that does not cause any difficulty, because all values of K are even (by Lemma 1). We complete the proof as in the base 2 case, by taking $x = \lfloor n/2 \rfloor$, $y = \lceil n/2 \rceil$. □

The first seven terms of $E(n)$ and $K(n)$ are shown in Table 1; the first 100 terms may be found in A230639 and A230640.

6 $K(n)$ for a general base b

In this section we give a set of recurrences that determine $K(n)$ for a general base $b \geq 2$. The divisibility requirement that we encountered in (36) for the base 3 case makes the recurrences considerably more complicated in the general case.

We know from Theorem 6 that $K(n)$ has the form

$$K(n) = u = b^m + k + 1, \tag{37}$$

where $m = E(n)$ and $0 \leq k \leq (b - 1)m - 2$. We will find the minimum value of u by first minimizing m and then minimizing k. By (15),

$$F(u) = n = F(k) + F((b - 1)m - k - 2) = x + y\,,$$

where $x = F(k)$ and $y = F((b - 1)m - k - 2)$. As in the proofs of Theorems 13 and 14, this implies $k \geq K(x)$, $(b - 1)m - k - 2 \geq K(y)$, and therefore

$$(b - 1)m - 2 \geq K(x) + K(y)\,. \tag{38}$$

Since in general $K(x)+K(y)+2$ will not be a multiple of $b-1$, the implied lower bound on m cannot always be attained. We therefore refine the inequality (38) using the functions $K_i(n)$ introduced in §4 and replace (38) with an inequality where the implied lower bound on m *can* always be attained. If $k \equiv i \pmod{b-1}$ for $i \in \mathcal{I}$, then $(b-1)m-k-2 \equiv -i-2 \pmod{b-1}$ and so $k \geq K_i(x)$, $(b-1)m-k-2 \geq K_{-i-2}(y)$, and

$$(b-1)m - 2 \geq K_i(x) + K_{-i-2}(y). \tag{39}$$

Now, in contrast to (38), the right-hand side is congruent to $-2 \pmod{b-1}$ and so we obtain an integral lower bound on m (for some i and $x+y=n$). Namely, (39) implies

$$(b-1)m \geq 2 + \min_{i\in\mathcal{I},\, 0<x<n} \left\{K_i(x) + K_{-i-2}(n-x)\right\}. \tag{40}$$

We will prove by induction that for any $i \in \mathcal{I}$, the sequences $\{K_i(n), n \geq 1\}$ and $\{K_{-i-2}(n), n \geq 1\}$ form a pair of sequences of exponential type. By Lemma 4, we may replace (40) with

$$(b-1)m \geq 2 + \min_{i\in\mathcal{I}} K_i'(n)\,, \tag{41}$$

where

$$K_i'(n) \;=\; \min\; \left\{K_i\left(\left\lceil \frac{n}{2}\right\rceil\right) + K_{-i-2}\left(\left\lfloor \frac{n}{2}\right\rfloor\right),\; K_i\left(\left\lfloor \frac{n}{2}\right\rfloor\right) + K_{-i-2}\left(\left\lceil \frac{n}{2}\right\rceil\right)\right\}. \tag{42}$$

From (41) we trivially obtain

$$E(n) \;=\; m \geq \frac{\min_{i\in\mathcal{I}} K_i'(n) + 2}{b-1}\,. \tag{43}$$

As we will show below, this lower bound is achievable and the right hand side of (43) is thus the exact value of $E(n)$.

To state the main theorem of the paper, we need to introduce the following intermediate quantities. For $n \geq 2$ and $i \in \mathcal{I}$, we define

$$c_{i,n} \;=\; \text{minimum positive integer } c \text{ such that } K_{i-2c}'(n) = \min_{j\in\mathcal{I}} K_j'(n)\,, \tag{44}$$

with $1 \leq c_{i,n} \leq b-1$, and

$$h_{i,n} \;=\; \begin{cases} \left\lceil \frac{n}{2}\right\rceil, & \text{if } K_{i-2c_{i,n}}\left(\left\lceil \frac{n}{2}\right\rceil\right) + K_{2c_{i,n}-i-2}\left(\left\lfloor \frac{n}{2}\right\rfloor\right) < K_{i-2c_{i,n}}\left(\left\lfloor \frac{n}{2}\right\rfloor\right) + K_{2c_{i,n}-i-2}\left(\left\lceil \frac{n}{2}\right\rceil\right), \\ \left\lfloor \frac{n}{2}\right\rfloor, & \text{otherwise.} \end{cases} \tag{45}$$

Theorem 15. *The $K_i(n)$ $(i \in \mathcal{I})$ satisfy the following recurrences. For all $n \geq 2$ and all $i \in \mathcal{I}$ we have:*

$$E(n) \;=\; \frac{\min_{j\in\mathcal{I}} K_j'(n) + 2}{b-1}\,, \tag{46}$$

$$K_i(n) \;=\; c_{i,n} b^{E(n)} + K_{i-2c_{i,n}}(h_{i,n}) + c_{i,n}\,, \tag{47}$$

$$K(n) \;=\; \min_{i\in\mathcal{I}} K_i(n)\,, \tag{48}$$

$$b^{E(n)} \ < \ K(n) \ \le \ K_i(n) \ < \ b^{E(n)+1}\,, \tag{49}$$

$$E(n) \ \ge \ E(n-1)+2, \qquad \textit{for } n \ge 5\,, \tag{50}$$

and for any p*,* $0 < p < n$*,*

$$K_i(p) + K_{-i-2}(n-p) \ge K_i'(n)\,. \tag{51}$$

Proof. We will prove all five statements (46), (47), (49), (50), (51) together by induction on n. (Eq. (48) is an immediate consequence of the definition of $K_i(n)$, as we already saw in (19). It is included in the theorem for completeness.)

We divide the proof into several parts, and we write $(46)_i$, $(47)_i$, ... to refer to the statements (46), (47), etc. for $n = i$.

(I) $(46)_2, (47)_2, (49)_2, (51)_2$ are true.

Proof. This is easily verified using the values of $K(n)$ and $K_i(n)$ from Theorems 7, 10, 11, as well as the facts that $c_{i,2} = 1$ and $h_{i,2} = 1$ for all i.

(II) $(46)_n$ follows from $(50)_{n-1}$ and $(51)_{n-1}$.

Proof. Let $\hat{E}(n)$ denote the right-hand side of $(46)_n$, so that $(46)_n$ takes the form $E(n) = \hat{E}(n)$. To prove this, we first use Theorem 9 to write $K_i(n) = cb^{E(n)} + k + c$ for some integers c and k satisfying $1 \le c \le b-1$ and $0 \le k \le (b-1)E(n) - 2$. Then (15) implies $n = F(K_i(n)) = F(k) + F((b-1)E(n) - k - 2)$.

Since $K_i(n) \equiv i \pmod{b-1}$, we have $k \equiv i - 2c \pmod{b-1}$. Let $x = F(k)$. Then $k \ge K_{i-2c}(x)$ and $(b-1)E(n) - k - 2 \ge K_{2c-i-2}(n-x)$, thus

$$E(n) \ \overset{\text{(i)}}{\ge} \ \frac{K_{i-2c}(x) + K_{2c-i-2}(n-x) + 2}{b-1} \ \overset{\text{(ii)}}{\ge} \ \frac{K'_{i-2c}(n) + 2}{b-1} \ \overset{\text{(iii)}}{\ge} \ \hat{E}(n), \tag{52}$$

where inequality (ii) follows from $(51)_n$ by induction and Lemma 4.

MAX, isn't there a problem here? We seem to need $(51)_n$ to prove (ii), but I thought we would only use $(51)_{n-1}$??? We also need to check that $0 < x < n$, right? Trouble?

Conversely, we now prove that $E(n) \le \hat{E}(n)$. Let $\hat{K}_i(n) = c_{i,n} b^{\hat{E}(n)} + K_{i-2c_{i,n}}(h_{i,n}) + c_{i,n}$. We claim that $\hat{K}_i(n) \equiv i \pmod{b-1}$ and $F(\hat{K}_i(n)) = n$. The first property follows instantly since $\hat{K}_i(n) \equiv c_{i,n} + (i - 2c_{i,n}) + c_{i,n} \equiv i \pmod{b-1}$. From (47), using (15) [MAX, in case you happen to be reading this, I don't think we need (47) here, and in fact we should not be using it because it hasn't been proved yet!]

I hope we don't really need (47)!

and the definition of $\hat{E}(n)$, we have

$$F(\hat{K}_i(n)) \ = \ F(K_{i-2c_{i,n}}(h_{i,n})) + F(\min_{j \in \mathcal{I}} K_j'(n) - K_{i-2c_{i,n}}(h_{i,n}))\,. \tag{53}$$

From (44), (42), and (45), it follows that

$$\begin{aligned}
&\min_{j\in\mathcal{I}} K_j'(n) - K_{i-2c_{i,n}}(h_{i,n}) \\
&\quad = K'_{i-2c_{i,n}}(n) - K_{i-2c_{i,n}}(h_{i,n}) \\
&\quad = \min\left\{ K_{i-2c_{i,n}}\left(\left\lceil \frac{n}{2} \right\rceil\right) + K_{2c_{i,n}-2-i}\left(\left\lfloor \frac{n}{2} \right\rfloor\right),\ K_{i-2c_{i,n}}\left(\left\lfloor \frac{n}{2} \right\rfloor\right) + K_{2c_{i,n}-2-i}\left(\left\lceil \frac{n}{2} \right\rceil\right) \right\} \\
&\qquad - K_{i-2c_{i,n}}(h_{i,n}) \\
&\quad = K_{2c_{i,n}-2-i}(n - h_{i,n})\,.
\end{aligned} \tag{54}$$

Then (53) gives

$$F(\hat{K}_i(n)) = F(K_{i-2c_{i,n}}(h_{i,n})) + F(K_{2c_{i,n}-2-i}(n-h_{i,n})) = h_{i,n} + n - h_{i,n} = n,$$

as required.

From the properties $\hat{K}_i(n) \equiv i \pmod{b-1}$ and $F(\hat{K}_i(n)) = n$, it follows that $K_i(n) \leq \hat{K}_i(n)$. Then Theorem 9 implies that $E(n) \leq \hat{E}(n)$, which together with (52) establishes $(46)_n$.

(III) $(47)_n$ follows from $(46)_n$ and $(49)_{n-1}$.

Proof. Since we now know that $E(n) = \hat{E}(n)$, all the inequalities (i), (ii), and (iii) in (52) are, in fact, equalities. The equality (i) implies that $k = K_{i-2c}(x)$. The equality (ii) implies that $x \leq \lceil \frac{n}{2} \rceil$, and thus by $(49)_{n-1}$

MAX This used to cite S4, (50), but I think S3, (49), is what we need here

$$k \leq K_{i-2c}\left(\lceil \tfrac{n}{2} \rceil\right) < b^{E(\lceil \frac{n}{2} \rceil)+1} \leq b^{E(n)}.$$

It follows that contribution of k into $K_i(n)$ is less than that of c, which together with the equality (iii) implies $c = c_{i,n}$. The equality (ii) now further implies that $k = K_{i-2c_{i,n}}(h_{i,n})$. This completes the proof of $(47)_n$.

(IV) $(49)_n$ follows from $(47)_n$ and $(49)_{n-1}$.

Proof. For $(49)_n$, the only inequality that is not obvious is $K_i(n) < b^{E(n)+1}$. This is easily verified for $n \leq 3$. Suppose then that $n \geq 4$. From $(47)_n$ and the induction hypothesis,

$$K_i(n) \ < \ (b-1)b^{E(n)} + b^{E(\lceil n/2 \rceil)+1} + b - 1\,.$$

We claim that the last expression is less than $b^{E(n)+1}$, or in other words that

$$b^{E(\lceil n/2 \rceil)+1} + b - 1 \ < \ b^{E(n)}\,.$$

MAX This used to cite S4, (50), but I think S3, (49), is what we need here

But this follows from $(49)_{n-1}$ and the induction hypothesis. We omit the details.

(V) $(50)_n$ follows from $(49)_{n-1}$ and $(50)_{n-1}$.

Proof. To prove $(50)_n$, we consider the cases $n = 2t$ and $n = 2t-1$ separately. If $n = 2t$, we have

$$E(2t) \ = \ \frac{\min_i\{K_i(t) + K_{-i-2}(t)\} + 2}{b-1} \ \geq \ \frac{2K(t) + 2}{b-1} \qquad (55)$$

and

$$E(2t-1) \ = \ \frac{\min_j \min\{K_j(t) + K_{-j-2}(t-1), K_j(t-1) + K_{-j-2}(t)\} + 2}{b-1}\,. \qquad (56)$$

We obtain an upper bound on the right-hand side of (56) if we choose any particular value of j, so let us choose j so that the expression in brackets becomes $K(t) + K_\ell(t-1)$ for some $\ell \in \mathcal{I}$. Then $(50)_n$ will follow if we show that

$$\frac{2K(t)+2}{b-1} \ > \ \frac{K(t) + K_\ell(t-1) + 2}{b-1} + 2\,,$$

i.e., if

$$K(t) \ > \ K_\ell(t-1) + 2(b-1)\,.$$

But this follows from $(49)_{n-1}$, $(50)_{n-1}$, and the induction hypothesis; again we omit the details. There is a similar argument in the case when $n = 2t-1$.

(VI) $(51)_n$ follows from $(49)_n$ and $(50)_n$.

Proof. For final step, to establish $(51)_n$, we will prove that for any $i \in \mathcal{I}$, the sequences $\{K_i(n), n \geq 1\}$ and $\{K_{-i-2}(n), n \geq 1\}$ form a pair of sequences of exponential type. For this, we must show that

$$K_i(n) \;>\; K_i(n-1) + K_{-i-2}(n-1)\,. \tag{57}$$

This can be checked directly for $n \leq 4$. For $n \geq 5$, $K_i(n) > b^{E(n-1)+2}$ by $(49)_n$, $(50)_n$, while the right-hand side of (57) is at most $2b^{E(n-1)+1} \leq b^{E(n-1)+2}$. This complete the proof of the theorem. □

Examples. We illustrate Theorem 15 in the case of even $b \geq 4$ and $n = 2, 3, 4$.

For $n = 2$, we find that $K'_i(2) = 2b-4$ for all i, $E(2) = (2b-2)/(b-1) = 2$, and all $c_{i,2} = 1$, all $h_{i,2} = 1$. From this we obtain the values of $K_i(2)$ that we saw in Theorem 11.

For $n = 3$ we find that $K'_i(3) = b^2+2b-5$ for all i, $E(3) = (b^2+2b-3)/(b-1) = b+3$, all $c_{i,3} = 1$, and $h_{0,3} = 2$, $h_{1,3} = h_{2,3} = h_{3,3} = \cdots = 1$. From this we obtain the values of $K_i(3)$ that we saw in Theorem 12.

For $n = 4$ we find that $K'_0(4) = 2b^2+2b-8$, $K'_i(4) = 2b^2+2b-6$ for $i \geq 1$, $E(4) = (2b^2+2b-4)/(b-1) = 2b+4$, $c_{0,4} = 2$, $c_{1,4} = 1$, $c_{2,4} = 3$, $c_{i,4} = 1$ for $3 \leq i \leq b-2$, and all $h_{i,4} = 2$. Then

$$\begin{aligned} &K_0(4) = 2b^{2b+4} + b^2 + 2b - 5, K_1(4) = b^{2b+4} + b^2 + b - 2, K_2(4) = 3b^{2b+4} + b^2 + 2b - 4, \\ &K_3(4) = b^{2b+4} + b^2 + b, K_4(4) = b^{2b+4} + b^2 + 2, \ldots\,, \end{aligned} \tag{58}$$

and

$$K(4) = K_4(4) = b^{2b+4} + b^2 + 2\,. \tag{59}$$

This confirms Kaprekar's conjecture of $10^{24} + 102$ in base 10.

Table 6 summarizes the results from the recurrence for $n \leq 7$ and even bases $b \geq 4$ and odd bases $b \geq 7$. (For smaller values of b see Tables 1, 2, 5.)

	even $b \geq 4$		odd $b \geq 7$	
n	$E(n)$	$K(n)$	$E(n)$	$K(n)$
1	–	0	–	0
2	2	b^2+1	1	$b+1$
3	$b+3$	$b^{b+3}+1$	2	b^2+1
4	$2b+4$	$b^{2b+4}+b^2+2$	3	b^3+b+2
5	$\frac{b^{b+3}+b^2+2b-4}{b-1}$	$b^{E(5)}+b^2+2$	$b+3$	$b^{b+3}+b+2$
6	$\frac{2b^{b+3}+2b-4}{b-1}$	$b^{E(6)}+b^{b+3}+2$	$2b+3$	$b^{2b+3}+b^2+2$
7	$\frac{b^{2b+4}+b^{b+3}+b^2+2b-5}{b-1}$	$b^{E(7)}+b^{b+3}+2$	b^2+2b+4	$b^{E(7)}+b^2+2$

Table 6: Values of $E(n)$ and $K(n)$ for $n \leq 7$ and even bases $b \geq 4$, odd bases $b \geq 7$.

Remarks on Theorem 15. (1) Since, by definition, $K_i(n) \equiv i \pmod{b-1}$, we have $K_i(n) \neq K_j(n)$ for $i \neq j$, $i, j \in \mathcal{I}$. It follows that the choice of $i \in \mathcal{I}$ in (19) is unique, and so we may define a "generalized Thue-Morse sequence" $\{\tau(n), n \geq 1\}$ for base b by:

$$K(n) = \begin{cases} K_{\tau(n)}(n), & \text{if } b \text{ is even,} \\ K_{2\tau(n)}(n), & \text{if } b \text{ is odd,} \end{cases} \tag{60}$$

where $0 \leq \tau(n) \leq b-2$ if b is even, and $0 \leq \tau(n) \leq \frac{b-3}{2}$ if b is odd. For examples see Table 8, (71), and (73).

(2) The $K'_i(n)$ are not all distinct. It follows at once from (42) that:
(i) if b is even, the distinct $K'_i(n)$ are

$$K'_i(n), \text{ for } 0 \leq i \leq (b-4)/2, \text{ and } K'_{b-2}(n)\,, \tag{61}$$

where the remaining values are given by $K'_i(n) = K'_{b-i-3}(n)$;
(ii) if $b \equiv -1 \pmod 4$, the distinct $K'_i(n)$ are

$$K'_{2i}(n), \text{ for } 0 \leq i \leq (b-7)/4, \text{ and } K'_{(b-3)/2}(n)\,, \tag{62}$$

where the remaining values are given by $K'_{2i}(n) = K'_{b-2i-3}(n)$; and
(iii) if $b \equiv 1 \pmod 4$, the distinct $K'_i(n)$ are

$$K'_{2i}(n), \text{ for } 0 \leq i \leq (b-5)/4\,, \tag{63}$$

where the remaining values are given by $K'_{2i}(n) = K'_{b-2i-3}(n)$.

(3) If n is even, no minimization is needed in (42), since the two terms inside the braces are the same. Also, again if n is even, $h_{i,n} = n/2$ for all $i \in \mathcal{I}$.

(4) Two further properties are worth mentioning. It follows from (46) and (44) that

$$E(n) \;=\; \frac{K'_{i-2c_{i,n}}(n) + 2}{b-1} \tag{64}$$

holds for any $i \in \mathcal{I}$, and, from (60),

$$c_{\tau(n),n} = 1 \text{ if } \; b \text{ is even}, \quad c_{2\tau(n),n} = 1 \text{ if } \; b \text{ is odd}\,. \tag{65}$$

Max, I can't get the next remark - it would be Remark (5) - to make sense. Maybe (55) is actually true? Maybe we should just delete this whole Remark (5)?

(5) NOT FINISHED, NEEDS WORK! We initially thought that for odd n, the minimization in (42) would be determined by which of $K_i(\lceil n/2 \rceil)$ and $K_{-i-2}(\lceil n/2 \rceil)$ was the smaller (since by (49) and (50), $K_i(\lceil n/2 \rceil)$ is much larger than $K_i(\lfloor n/2 \rfloor)$), which would imply that

$$\min_{i \in \mathcal{I}} K'_i(n) \;\stackrel{?}{=}\; K(\lceil n/2 \rceil) + K_\ell(\lfloor n/2 \rfloor)\,, \tag{66}$$

for some $\ell \in \mathcal{I}$, with a corresponding simplification in (46). To prevent others from falling into this trap, we mention that (66) is false. For base 10, for example, $\min_{i \in \mathcal{I}} K'_i(3) = K_8(2) + K_8(1) = 107 + 8 = 115$, whereas $K(2) = K_2(2) = 101$. NOT SURE ABOUT THIS! We can write $\min_{i \in \mathcal{I}} K'_i(3) = K_2(2) + K_5(1) = 101 + 4 = 115$ which IS of the form (66)!

The next two sections give further information about bases 4, 5, 7, and 10.

n	$K_0'(n)$	$E(n)$	$h_{0,n}$	$h_{2,n}$	$K_0(n)$		$K_2(n)$
1	–	–	–	–	$\mathbf{0}$	$\searrow$	2
2	2	1	1	1	$5+3$	$\searrow$	$\mathbf{5+1}$
3	6	2	2	1	5^2+7	$\swarrow$	$\mathbf{5^2+1}$
4	14	4	2	2	$\mathbf{5^4+7}$	$\searrow$	5^4+9
5	34	9	3	2	5^9+27	$\swarrow$	$\mathbf{5^9+9}$
6	58	15	3	3	$\mathbf{5^{15}+27}$	$\swarrow$	$5^{15}+33$
7	658	165	3	4	$\mathbf{5^{165}+27}$	$\searrow$	$5^{165}+633$
8	1266	317	4	4	$5^{317}+635$	$\searrow$	$\mathbf{5^{317}+633}$
9	5^9+5^4+16	488442	5	4	$5^{E(9)}+5^9+10$	$\swarrow$	$\mathbf{5^{E(9)}+5^4+8}$
10	$2\cdot 5^9+36$	976572	5	5	$\mathbf{5^{E(10)}+5^9+10}$	$\swarrow$	$5^{E(10)}+5^9+28$

Table 7: Base 5: $K(n)$ is shown in bold font. The meaning of the arrows is explained in the text.

7 $K(n)$ for bases 4, 5, and 7

We discuss base 5 first, since this turns out to be simpler than bases 4 or 7.

For $b=5$, the index set is $\mathcal{I}=\{0,2\}$. From (61), there is only one $K_i'(n)$ to consider, namely

$$K_0'(n) \;=\; \min\left\{K_0\left(\left\lceil\frac{n}{2}\right\rceil\right)+K_2\left(\left\lfloor\frac{n}{2}\right\rfloor\right),\; K_0\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+K_2\left(\left\lceil\frac{n}{2}\right\rceil\right)\right\}. \tag{67}$$

Then $E(n)=(K_0'(n)+2)/4$, $c_{0,n}=c_{2,n}=1$ for all n,

$$h_{0,n} \;=\; \begin{cases}\left\lceil\frac{n}{2}\right\rceil, & \text{if } K_0\left(\left\lfloor\frac{n}{2}\right\rfloor\right)+K_2\left(\left\lceil\frac{n}{2}\right\rceil\right) < K_0\left(\left\lceil\frac{n}{2}\right\rceil\right)+K_2\left(\left\lfloor\frac{n}{2}\right\rfloor\right),\\ \left\lfloor\frac{n}{2}\right\rfloor, & \text{otherwise,}\end{cases} \tag{68}$$

and $h_{2,n}=n-h_{0,n}$. Also

$$K_0(n) \;=\; 5^{E(n)}+K_2(h_{0,n})+1, \quad K_2(n) \;=\; 5^{E(n)}+K_0(h_{2,n})+1\,, \tag{69}$$

and $K(n)=\min\{K_0(n),K_2(n)\}$. The initial values of these variables are shown in Table 7. The value of $K(n)$ is shown in bold font (the first 100 values of $E(n)$ and $K(n)$ can be found in A230868 and A230867). The symbol in the penultimate column of the table indicates which choice in (67) was made when calculating $K_0'(n)$ for n odd. An arrow $\searrow$ in row i indicates that $K_0'(2i+1)=K_0(i)+K_2(i+1)$, while an arrow $\swarrow$ indicates that $K_0'(2i+1)=K_0(i+1)+K_2(i)$.[4]

The values of the "generalized Thue-Morse sequence" $\tau(n)$ (see (60)) are shown in Table 8.

In this case $\tau(n)$ actually *is* the classical Thue-Morse sequence, except shifted by one step.[5] We prove this in the next theorem.

[4]The arrows are intended to suggest, for the four elements $K_0(i), K_2(i), K_0(i+1), K_2(i+1)$, whether it is better to pair up the North West and South East entries, or the North East and South West entries.

[5]The classical sequence is $\tau(n-1)$.

n	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
$\tau(n)$	0	1	1	0	1	0	0	1	1	0	0	1	0	1	1	0

Table 8: Initial values of $\tau(n)$ (this is essentially the Thue-Morse sequence A010060).

Theorem 16. *For $n \geq 1$, $\tau(n) = \tau(\lceil \frac{n}{2} \rceil)$ if n is odd, $\tau(n) = 1 - \tau(\frac{n}{2})$ if n is even.*

Proof. (Sketch.) The basis for the inductive proof are the following observations. (i) If n is even, then (69) implies that $K_0(n) < K_2(n)$ if and only if $K_2(n/2) < K_0(n/2)$, and hence $\tau(n) = 1 - \tau(n/2)$. (ii) Suppose on the other hand that $n = 2i + 1$ is odd. There are two possibilities. If

$$K_0(i) + K_2(i+1) \ < \ K_0(i+1) + K_2(i) \tag{70}$$

(the $\searrow$ case in Table 7), then $K_2(i+1) < K_0(i+1)$, $\tau(i+1) = 2$, $K(n) = K_2(n)$, and hence $\tau(n) = 1 = \tau(\lceil n/2 \rceil)$. If the inequality in (70) is reversed, we similarly find that $\tau(n) = 2 = \tau(\lceil n/2 \rceil)$. For the induction to work, we need to also show that the values of $E(n)$, $K_0(n)$, and $K_2(n)$ are considerably larger than the values of $E(n-1)$, $K_2(n-1)$, and $K_0(n-1)$, respectively, but this follows from (49) and (50). We omit the details of the proof. $\square$

The situation is more complicated in base 4. Now the index set is $\mathcal{I} = \{0, 1, 2\}$ (modulo 3), $K(n)$ is the minimum of the three terms $K_0(n)$, $K_1(n)$, $K_2(n)$, and so is specified by a ternary sequence $\tau(n)$. The first 100 values of $E(n)$ and $K(n)$ can be found in A230637 and A230638, and the first 100 terms of $\tau(n)$ are:

$$\begin{aligned}&0,2,2,1,1,1,2,0,2,0,2,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,2,0,2,0,2,0,2,0,2,0,2,0,2,0,2,\\&0,2,0,2,0,2,0,2,0,2,0,0,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\\&1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,\ldots\end{aligned} \tag{71}$$

(A239110). The same sequence $\tau(n)$ also arises from the base 7 case, where again the index set contains three elements. We would like to understand this sequence better!

8 $K(n)$ for base 10

In base $b = 10$, the index set is $\mathcal{I} = \{0, 1, 2, \ldots, 8\}$ (modulo 9), and, from (61), there are five distinct $K'_i(n)$, namely $K'_0(n) = K'_7(n)$, $K'_1(n) = K'_6(n)$, $K'_2(n) = K'_5(n)$, $K'_3(n) = K'_4(n)$, and $K'_8(n)$. There are nine variables $c_{i,n}$, $h_{i,n}$, and $K_i(n)$, with $0 \leq i \leq 8$. The values of $K_i(n)$ for $n \leq 7$ are shown in Table 9. Then

$$K(n) \ = \ \min_{0 \leq i \leq 8} K_i(n) \ = \ 10^{E(n)} + \text{ terms of smaller order}\,. \tag{72}$$

We have already seen $E(n)$ and $K(n)$ for $n \leq 7$ in Table 1. Tables 10 and 11 extend these values to $n = 16$, going far enough that we can see – and confirm! – the values for $K(4), \ldots, K(8)$, and $K(16)$ found by Karmarkar and Narendra Rao more than fifty years

	$i=0$	$i=1$	$i=2$	$i=3$	$i=4$
$K_i(1)$	$\mathbf{0}$	10	2	12	4
$K_i(2)$	117	109	$\mathbf{101}$	111	103
$K_i(3)$	$10^{13}+116$	$10^{13}+9$	$\mathbf{10^{13}+1}$	$10^{13}+11$	$10^{13}+3$
$K_i(4)$	$2\cdot 10^{24}+115$	$10^{24}+108$	$3\cdot 10^{24}+116$	$10^{24}+110$	$\mathbf{10^{24}+102}$
$K_i(5)$	$2\cdot 10^{E(5)}+115$	$10^{E(5)}+108$	$3\cdot 10^{E(5)}+116$	$10^{E(5)}+108$	$\mathbf{10^{E(5)}+102}$

	$i=5$	$i=6$	$i=7$	$i=8$	
$K_i(1)$	14	6	16	8	
$K_i(2)$	113	105	115	107	
$K_i(3)$	$10^{13}+13$	$10^{13}+5$	$10^{13}+15$	$10^{13}+7$	
$K_i(4)$	$10^{24}+112$	$10^{24}+104$	$10^{24}+114$	$10^{24}+106$	
$K_i(5)$	$10^{E(5)}+112$	$10^{E(5)}+104$	$10^{E(5)}+114$	$10^{E(5)}+106$	

Table 9: Base 10: $K_i(n)$ for $n \le 5$; the value of $K(n)$ (A006064) is shown in bold font. In the $K_i(5)$ rows, $E(5) = (10^{13}+116)/9 = 1111111111124$ as in Table 1.

n	$E(n)$
8	$(2\cdot 10^{24}+214)/9$
9	$(10^{(10^{13}+116)/9}+10^{24}+214)/9$
10	$(2\cdot 10^{(10^{13}+116)/9}+214)/9$
11	$(10^{(2\cdot 10^{13}+16)/9}+10^{(10^{13}+116)/9}+10^{13}+114)/9$
12	$(2\cdot 10^{(2\cdot 10^{13}+16)/9}+2\cdot 10^{13}+14)/9$
13	$(10^{E(7)}+10^{(2\cdot 10^{13}+16)/9}+2\cdot 10^{13}+14)/9$
14	$(2\cdot 10^{E(7)}+2\cdot 10^{13}+14)/9$
15	$(10^{E(8)}+10^{E(7)}+E(7)-2)/9$
16	$(2\cdot 10^{E(8)}+E(8)-2/9$

Table 10: Base 10: $E(n)$ for $8 \le n \le 16$, extending Table 1; $E(7) = (10^{24}+10^{13}+115)/9$.

ago (see the discussion in the Introduction). The first 100 terms of these two sequences can be seen in entries A230857 and A006064 in [10].

The first 100 terms of the base 10 generalized Thue-Morse sequence $\tau(n)$ are as follows:

$$0,2,2,4,4,4,4,6,6,6,6,6,6,6,6,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,8,1,1,1,1,1,1,1,1,1,1,\\ 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,\\ 8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,8,3,\ldots \tag{73}$$

(A239896). If we write 6^8 to denote a run of eight successive 6's, etc., this can be rewritten as $0,2^2,4^4,6^8,8^{16},1^{31},(8,3)^{\ge 19}$. Again, we wish we understood this sequence better.

n	$K(n)$
8	$10^{E(8)} + 10^{24} + 103$
9	$10^{E(9)} + 10^{24} + 103$
10	$10^{E(10)} + 10^{(10^{13}+116)/9} + 103$
11	$10^{E(11)} + 10^{(10^{13}+116)/9} + 103$
12	$10^{E(12)} + 10^{(2\cdot 10^{13}+16)/9} + 10^{13} + 3$
13	$10^{E(13)} + 10^{(2\cdot 10^{13}+16)/9} + 10^{13} + 3$
14	$10^{E(14)} + 10^{(10^{24}+10^{13}+115)/9} + 10^{13} + 3$
15	$10^{E(15)} + 10^{(10^{24}+10^{13}+115)/9} + 10^{13} + 3$
16	$10^{E(16)} + 10^{(2\cdot 10^{24}+214)/9} + 10^{24} + 104$

Table 11: Base 10: $K(n)$ for $8 \le n \le 16$, extending Table 1.

9 Growth of $K(n)$

In this section we discuss the rate of growth of $K(n)$ for a fixed b. We begin with three general bounds. They are somewhat weak, but are helpful in building intuition.

Theorem 17. *For $b \ge 2$ and $n \ge 2$, there is an integer s in the range $1 \le s \le n-1$ such that*

$$K(n) \; > \; b^{\left\lceil \frac{K(s)+K(n-s)+2}{b-1} \right\rceil} . \tag{74}$$

Proof. Let $u = K(n) = b^m + k + 1$ as in Theorem 6, with $m = E(n) \ge 1$ and $0 \le k \le (b-1)E(n) - 2$. By (15), $F(u) = n = F(k) + F((b-1)m - k - 2)$, and let $s = F(k)$, $t = F((b-1)m - k - 2)$, with $s + t = n$. Therefore $k \ge K(s)$, $(b-1)m - k - 2 \ge K(t)$, so $m \ge (K(s) + K(t) + 2)/(b-1)$. Since m is an integer, the result follows. ☐

Theorem 18. *For $b \ge 2$ and $n \ge 2$,*

$$K(n) \; < \; b^{\frac{K(t)+(b-1)K(n-t)+2}{b-1}+1} , \tag{75}$$

for all t in the range $1 \le t \le n-1$.

Proof. For any t with $1 \le t \le n-1$, we construct a number

$$\theta(t) \; = \; b^{D(t)} + K(t) + 1 \tag{76}$$

which has n generators (so that $K(n) \le \theta(t)$), and show that $\theta(t)$ is less than the right-hand side of (75).

We begin by constructing $D(t)$. By Theorem 6, $K(n-t) = b^m + k + 1$ for some $m \ge 1$, $0 \le k \le (b-1)m - 2$. By (15), $n - t = F(b^m + k + 1) = F(cb^m + k + c)$, for all c with $1 \le c \le b-1$. We choose $c = c(t)$ so that

$$K(t) + (cb^m + k + c) + 2 \; \equiv \; 0 \pmod{b-1} . \tag{77}$$

(To see that this is possible, we rewrite (77) as

$$(b^m + 1)c \; \equiv \; -(K(t) + k + 2) \pmod{b-1} . \tag{78}$$

If b is even, then $b^m+1 \equiv 1 \pmod{b-1}$, and we can solve (78) for c. If b is odd, $K(t)+k+2$ is even by Lemma 1, and we can divide (78) through by 2 and again solve for c.) We now define

$$D(t) \;=\; \frac{K(t)+cb^m+k+c+2}{b-1}\,, \tag{79}$$

and define $\theta(t)$ by (76). By (15), $F(\theta(t)) = F(D(t)) + F((b-1)D(t) - K(t) - 2) = t + F(cb^m + k + c) = t + n - t = n$. Therefore $K(n) \le \theta(t)$.

The following two straightforward calculations complete the proof:

$$D(t) \;<\; \frac{K(t)+(b-1)K(n-t)+2}{b-1} \tag{80}$$

and

$$\theta(t) \;<\; b^{D(t)+1}\,. \tag{81}$$

We omit the details. □

The next theorem generalizes the inequalities in (30) and (35). It implies that, for any base b, $\{K(n), n \ge 1\}$ is a sequence of exponential type (cf. Lemma 4).

Theorem 19. *For $b \ge 2$ and $n \ge 1$,*

$$K(n+1) \;>\; bK(n)\,, \tag{82}$$

except that for odd $b \ge 5$, we only have

$$K(3) \;>\; (b-1)K(2)\,. \tag{83}$$

Proof. We fix b and use induction on n. From Tables 1 and6, we see that the claims are true for $n \le 6$. Note that (82) is simply not true for $n = 2$ and odd $b \ge 5$. For the induction step, suppose $n \ge 7$. We know from Theorem 17 that for some s with $1 \le s \le n$,

$$K(n+1) \;>\; b^{\frac{K(s)+K(n+1-s)+2}{b-1}}\,.$$

Without loss of generality we may assume $s \le n+1-s$, so $s \le \lfloor (n+1)/2 \rfloor$, $n-s \ge \lfloor n/2 \rfloor \ge 3$. By the induction hypothesis, $K(n+1-s) > bK(n-s)$. Therefore

$$\begin{aligned} K(n+1) \;&>\; b^{(K(s)+bK(n-s)+2)/(b-1)} \\ &\ge\; b \cdot b^{(K(s)+(b-1)K(n-s)+2)/(b-1)+1} \\ &\ge\; bK(n)\,, \end{aligned} \tag{84}$$

where we first used $K(n-s) \ge K(3) \ge 2(b-1)$ and then Theorem 18. □

Since $K(n)$ grows rapidly, it is appropriate to describe its value by a tower of exponentials. For $b \ge 2$, any number $u \ge 1$ can be written in a unique way as a "tower"

$$u \;=\; b^{b^{\cdot^{\cdot^{\cdot^{b^{\omega}}}}}}\,, \tag{85}$$

with $0 < \omega = \omega(u) \le 1$. If this tower contains $h-1$ b's and one ω, we call h the base b *height* of u, denoted by $\mathrm{ht}(u)$. Then $\mathrm{ht}(u)$ is one more than the number of times one has to take logarithms to the base b of u until reaching a number $\omega \le 1$.

Examination of the data in Tables 1, 2, 5, 7, 11 (and in the more extended tables in [10]) suggests the following conjecture.

Conjecture 20. *It appears that:*
(i) If $b = 2$, $n \ge 2$, and $2^{i-1} < n \le 2^i$, then $\mathrm{ht}(n) = i + 3 = \lceil \log_2(n) \rceil + 3$;
(i) If $b = 3$, $n \ge 3$, and $5 \cdot 2^{i-1} < n \le 5 \cdot 2^i$, then $\mathrm{ht}(n) = i + 4 = \lceil \log_2(n/5) \rceil + 4$;
(iii) If $b \ge 4$ is even, $n \ge 2$, and $2^{i-1} < n \le 2^i$, then $\mathrm{ht}(n) = i + 2 = \lceil \log_2(n) \rceil + 2$;
(iii) If $b \ge 5$ is odd, $n \ge 2$, and $2^{i-1} < n \le 2^i$, then $\mathrm{ht}(n) = i + 1 = \lceil \log_2(n) \rceil + 1$.

For example, in base $b = 10$,

$$K(3) \;=\; 10^{13} + 1 \;=\; 10^{10^{10^{0.04686\ldots}}}, \tag{86}$$

which has height 4. The heights of $K(2)$ through $K(16)$ in base 10 (see (3) and Tables 1, 11) are $3, 4, 4, 5, 5, 5, 5, 6, 6, 6, 6, 6, 6, 6, 6$, respectively, in agreement with the conjecture.

There are two reasons for believing the conjecture. First, it is true in every case that we have checked. Second, from Section 6, $K(n)$ is very roughly equal to

$$b^{(K(\lceil n/2 \rceil) + K(\lfloor n/2 \rfloor))/(b-1)}, \tag{87}$$

which suggests that the height of the tower for $K(n)$ is one greater than the height of the tower for $K(\lceil n/2 \rceil)$, which would leads to the formulas in the conjecture. However, two difficulties arise when trying to make this argument rigorous. One is the fact that if u in (85) has height h, and $\omega(u)$ is very close to 1, b^u can have height $h+2$ instead of $h+1$. This seems not to happen with $K(n)$, but we cannot rule out the possibility, even for base 2. The second difficulty is that (87) ignores the choices that must be made (for $b \ge 4$) among the $K_i(n)$ when determining $K(n)$.

The following example shows the first of these difficulties in a simpler setting. Consider the sequence defined by the recurrence

$$a(1) = 0, \;\; a(n) = 2^{a(\lceil n/2 \rceil) + a(\lfloor n/2 \rfloor)} \text{ for } n \ge 2\,. \tag{88}$$

This is similar to the recurrence for the base 2 sequence $K(n)$ given in (28) and (29), except that the additive terms on the right-hand sides of those equations are missing. The initial values of $a(n)$ for $n = 1, 2, 3, \ldots$ are

$$0, 1, 2, 4, 8, 16, 64, 2^8, 2^{12}, 2^{16}, 2^{24}, 2^{32}, 2^{80}, 2^{128}, 2^{320}, 2^{512}, 2^{4352}, \ldots,$$

(A230863). The heights of $a(n)$ for $n = 2, \ldots, 10$ are $1, 1, 2, 3, 4, 4, 5, 5, 5, 5$. For $11 \le n \le 40$, if $9 \cdot 2^{i-1} < n \le 9 \cdot 2^i$ then $\mathrm{ht}(a(n)) = i + 5$, although here we do not know if this will hold for all n. (The difficulty lies in the fact that terms $a(n)$ for which the top entry in the tower, ω, is 1 or very close to 1, could disrupt the pattern of the heights.)

For small values of n, of course, there is no difficulty in computing the height of $K(n)$. From Theorems 7 and 8, for example, we have $\mathrm{ht}(K(2)) = 4$ if $b = 2$, or 3 if $b \geq 3$, and

$$\mathrm{ht}(K(3)) \ = \ \begin{cases} 5 & \text{if } b = 2, \\ 4, & \text{if } b = 3, \\ 4, & \text{if } b \geq 4 \text{ is even}, \\ 3, & \text{if } b \geq 5 \text{ is odd}. \end{cases} \tag{89}$$

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Appendix: Computations

The most difficult task in programming the algorithm in Section 6 is in handling the very large numbers that appear. As we saw in Section 9, to compute $K(100)$ in base 10, for example, we have to work with numbers that are of the order of a tower of 10's of height 9. This problem was solved using C++, and defining a special type of object (we called it the "sparse radix representation") which represents either an integer or has the algebraic form $\frac{1}{c}\left(\alpha_1 b^{d_1} + \alpha_2 b^{d_2} + \cdots + \alpha_k b^{d_k}\right)$, where c and the α_i are integers, with $c \geq 1$ and $1 \leq \alpha_i < b$, and d_i are objects of the same type, satisfying $d_1 > d_2 > \cdots > d_k$. Comparison of two numbers of this type with $c = 1$ is done recursively, starting by comparing the highest order terms, and if these are tied, comparing the next-to-highest terms, and so on. If the denominator c of either of the two numbers is not 1, the numbers are first multiplied by the least common multiple of their denominators and the coefficients are then reduced into the required range (e.g., βb^d with $\beta \geq b$ is replaced with $(\beta \bmod b) b^{d+\lfloor \beta/b \rfloor}$) and then the numbers are compared as before.

Many other computations were carried out using C or Maple. We also made extensive use of the following PARI/GP 2.8.0 program for computing $\mathrm{Gen}(u)$ and $F(u)$. This program, `Gen(u,b)`, uses the recurrence in (14) to compute $\mathrm{Gen}(u)$ in base b for $u \in \mathbb{N}$. For example, `Gen(10^13+1,10)` would return the three generators 9999999999892, 9999999999901, 10000000000000 of $10^{13}+1$ in base 10. The command `#Gen(u,b)` returns $F(u)$, the number of generators of $u \in \mathbb{N}$ in base b.

```
/* The PARI/GP 2.8.0 program Gen(u,b) */
```

```
{ Gen(u,b=10) = my(d,m,k);
  if(u<0 || u==1, return([]); );
  if(u==0, return([0]); );

  d = #digits(u,b)-1;
  m = u\b^d;

  while( sumdigits(m,b) > u - m*b^d,
    m--;
    if(m==0, m=b-1; d--; );
  );
  k = u - m*b^d - sumdigits(m,b);

  vecsort(concat(apply(x->x+m*b^d,Gen(k,b)),apply(x->m*b^d-1-x,Gen((b-1)*d-k-2,b))))
}
```