On the Covering Multiplicity of Lattices*

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1. Introduction

A problem of recent interest (Sullivan [18]) is to find the smallest possible covering multiplicity (defined in the Abstract) of any n-dimensional lattice, and to calculate the covering multiplicities of various well-known lattices. Figure 1 shows an example. The covering multiplicity so defined is of use in estimating various integrals, since if I is the integral of some function over the whole space and I_B its integral over one of the covering balls then

$$I \le \sum_B I_B \le CM(\Lambda) I \ .$$

The figures appear at the end of the paper

In the present paper we determine the covering multiplicities of certain members of the families $I_n (n \ge 1)$, $A_n (n \ge 1)$, $D_n (n \ge 1)$, $E_n (n = 6, 7, 8)$ and their duals A_n^*, D_n^*, E_n^* for small values of n (the results are given in Table I) and of the Leech lattice Λ_{24} (for which the covering multiplicity is 25).

Table I
Covering multiplicities of root lattices and duals

n	I_n	A_n	A_n^*	D_n	D_n^*	E_n	E_n^*
1	1	1	1	1	1		
2	2	2	2	2	2		
3	4	4	3	4	3		
4	8	7	4	4	4		
5	16	15	5	11	6		
6	≥ 32	≥ 21	6	≥ 16	8	17	6
7	≥ 64	≥ 56	7	≥ 32	≥ 12	28	9
8	≥ 128	≥ 84	8	≥ 64	≥ 16	9	9
9	≥ 256	≥ 210	≥ 11	≥ 163	≥ 24		

In particular we will show that $CM(A_n^*) = n$ for $n \le 8$, and also that $CM(D_4) = 4$ and $CM(E_6^*) = 6$. Since the covering multiplicity of an n-dimensional lattice is at least n (see Theorem 2), this will establish our main result.

Theorem 1. The minimal covering multiplicity of an n-dimensional lattice is n if $n \le 8$.

We have also made experimental investigations of these lattices in higher dimensions, obtaining lower bounds on their covering multiplicities, some of which are shown in Table I. (We conjecture that the lower bounds shown in the table are in fact the true values.)

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The results in Table I (and more extensive experiments not reported here) lead us to conjecture that the minimal covering multiplicity exceeds n in all dimensions above 8.

Since there are usually many ways to perturb a lattice without changing its covering multiplicity, the lattices in Table I are of course not the only ones with these values of the covering multiplicity.

It is worth pointing out that there is a simple way to compare lattices that have the same covering multiplicity. This is by finding the largest sphere whose interior is wholly contained in the set of points covered $CM(\Lambda)$ times: we call this the *umbral sphere* of the lattice. It is illustrated in Figure 1 for the square lattice I_2. Let R_u be the radius of the umbral sphere when the lattice is scaled so as to have determinant 1. We can then ask: among all lattices with the minimal covering multiplicity, which has the smallest value of R_u? We will show how to compute R_u in Section 2 – see Eq. (4). For example, although both A_4^* and D_4 have covering multiplicity 4, the value of R_u for A_4^* is

$$\frac{\sqrt{\frac{2}{5}} - \sqrt{\frac{19}{50}}}{\left[\frac{1}{5}\right]^{1/8}} = 0.019582\ldots ,$$

while for D_4 it is

$$\frac{1 - \frac{\sqrt{3}}{2}}{4^{1/8}} = 0.112658\ldots .$$

Similarly the values of R_u for A_6^* and E_6^* (which both have covering multiplicity 6) are

$$\frac{\frac{2}{\sqrt{7}} - \sqrt{\frac{55}{98}}}{\left[\frac{1}{7}\right]^{1/12}} = 0.007973\ldots$$

and

$$\frac{\sqrt{\frac{2}{3}} - \sqrt{\frac{11}{18}}}{\left[\frac{1}{3}\right]^{1/12}} = 0.038093\ldots$$

respectively.

The n-dimensional cubic lattice I_n is of particular interest. Our investigations strongly suggest that the sequence of values of $CM(I_n)$ has a rather unusual behavior: $CM(I_n) = 2^{n-1}$ for all $n \le 33$, while $CM(I_n) > 2^{n-1}$ for all $n > 33$. Our best lower bounds on $CM(I_n)$ are shown in Table II. It follows from the work of Mazo and Odlyzko [12] that

$$CM(I_n) \sim 2.089097\ldots^{\,n + O(\sqrt{n})} \tag{1}$$

as $n \to \infty$. We discuss I_n in Section 3. The lattices $A_n, A_n^*, D_n, \ldots$ are dealt with in turn in Sections 4, . . . , ending with the Leech lattice Λ_{24} in Section 9.

We remark that, once the covering radius R of a lattice Λ has been found, it is trivial to obtain the *average* covering multiplicity. This is the average number of times a point is in the interior of the covering balls, which is equal to the *covering density* (or *thickness)* of Λ, given by

$$\frac{V_n \, R^n}{\sqrt{\det \Lambda}} , \tag{2}$$

Table II
Lower bounds to covering multiplicities
of cubic lattices I_n (r is defined in Section 3)

n	$CM(I_n) \geq$	r
1 to 33	2^{n-1}	$n-1$
34	$2^{33} + 10850970$	1
35	$2^{34} + 341772656$	2
36	$2^{35} + 1329309096$	3
37	$2^{36} + 7307575295$	0
38	$2^{37} + 17587471742$	1
39	$2^{38} + 41179889288$	2
40	$2^{39} + 94490796376$	3

where V_n is the volume of an n-dimensional sphere of radius 1 ([3, p. 31]). Finding $CM(\Lambda)$ appears to be a much more difficult problem.

For example, the average covering multiplicities of the lattices I_n, D_n and A_n^* are respectively

$$\left[\sqrt{\frac{\pi e}{2}} \right]^{n+o(n)} = 2.066365\ldots^{n+o(n)} ,$$

$$\left[\sqrt{\frac{\pi e}{2}} \right]^{n+o(n)} = 2.066365\ldots^{n+o(n)} ,$$

$$\left[\sqrt{\frac{\pi e}{6}} \right]^{n+o(n)} = 1.193016\ldots^{n+o(n)} . \tag{3}$$

We shall call a point of R^n which is in the interior of $CM(\Lambda)$ balls a *thickest point* for Λ.

Note that since only open balls are used in determining $CM(\Lambda)$, points of R^n that are maximally distant from Λ (the ‘‘deep holes’’ in Λ) are usually not thickest points. However, we will see several examples later where thickest points are deep holes in sublattices.

This work has involved considerable use of computers.

(1) We searched for thickest points in a lattice by choosing points at random from a fundamental parallelepiped, and computing the number of open balls of radius R that contain them. However it appears that for many lattices the probability that a random point is a thickest point is very small, so this approach is only successful in low dimensions.

(2) For some lattices we use MACSYMA [11] to determine how many open balls contain a point P, by computing the theta series with respect to P – see Sections 3 and 6.

(3) It is often necessary to determine if the open balls of radius R about lattice points $P_1, \ldots, P_M \in \Lambda$ have a common point. This can be answered by computing the radius r of the smallest sphere that contains $P_1, \ldots, P_M$. (There is a common point if and only if $r < R$.)

Although this *smallest containing sphere* problem has been studied by several authors [6]–[8], [13], [16, p. 248], [17], there does not seem to be a standard name for the containing sphere. We shall call it the *perisphere*[(1)], and similarly refer to the *pericenter* and *periradius* of $P_1, \ldots, P_M$. The square of the periradius is the *perinorm* of $P_1, \ldots, P_M$.

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(1) The prefix ‘‘peri’’ means ‘‘around’’.

Smallest containing sphere problems can be solved very efficiently using the AMPL [9] and MINOS [15] programs.(2)

(4) We also frequently use AMPL and MINOS to compute the distance from a given point P to a convex polytope Π (usually a fundamental simplex for the lattice in question – see for example the proof of Theorem 6).(3)

Since we make heavy use of the results from AMPL and MINOS in proving some of our theorems, it is appropriate to discuss the accuracy and rigor of these procedures. (a) AMPL [9] is simply a flexible front-end, which transforms the problem into the appropriate format for MINOS or other optimization programs. (b) MINOS [15] was developed to handle much larger problems than ours, which typically involve minimizing a function of 8 variables subject to 8 constraints. Running times on our problems were around 0.20 seconds, and the results were correct to at least 7 significant figures. (c) Our problems – see footnotes 3 and 4 – involve minimizing a quadratic function, usually of the form $x_1^2 + \cdots + x_n^2$, subject to linear constraints. MINOS solves such problems using a reduced-gradient algorithm, following Murtagh and Saunders [14]. Since our problems are so small, any particular solution found by MINOS could be verified by hand. The total number of calls to MINOS used in all our proofs was less than 10000. So although some of these proofs depend on computers, the same results could in principal be obtained without their help.

We assume the reader is familiar with the lattices mentioned (any other course would have doubled the length of the paper), although we do give definitions for the principal lattices. For further information see [1], [2], [5], [10] and especially [3].

The subscript on the name of a lattice gives its dimension, and a star indicates the dual lattice. We note that $I_n^* = I_n$, $E_8^* = E_8$ and $\Lambda_{24}^* = \Lambda_{24}$. Also $A_1 \cong A_1^* \cong D_1 \cong D_1^* \cong I_1$, $A_2^* \cong A_2$, $D_2 \cong I_2$, $D_3 \cong A_3$ and $D_4^* \cong D_4$, where $\cong$ indicates geometrically similar lattices.

The cell structures (i.e. the Voronoi and Delaunay polytopes) of the root lattices I_n, A_n, D_n, E_n and their duals are described in [3], [5], [19], [20]. The reader is particularly referred to [4], where all these cell structures are obtained in a systematic and uniform manner.

The *norm* or squared length $x \cdot x$ of a vector x will be denoted throughout by $N(x)$.

2. General results.

We first prove that the covering multiplicity of an n-dimensional lattice is at least n. This is a consequence of the following more general result.

Theorem 2. Let Σ be a set of points in $\mathbb{R}^n$ for which there are positive numbers a, R such that $\mathrm{dist}(s, t) \ge a$ for all $s, t \in \Sigma$, $s \ne t$, and such that the closed balls of radius R about the points of Σ cover $\mathbb{R}^n$. Then there is a point $w \in \mathbb{R}^n$ in the interior of at least n of these balls.

For the proof we need a lemma.

Lemma 3. Suppose $v_1, \ldots, v_M$ ($M \ge n$) are vectors in $\mathbb{R}^n$ such that any nonzero vector has positive inner product with at least one v_i. Then there is a vector $w \in \mathbb{R}^n$ with positive inner product with at least n of the v_i.

Proof of lemma. The proof is by induction on n. Choose one of the v_i's, v say, and let Π be the $(n-1)$-

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(2) An appropriate formulation of such a problem for AMPL and MINOS is the following. Let C be the pericenter and s the perinorm. Then set $t = s - C \cdot C$ and call AMPL and MINOS to minimize $t + C \cdot C$ subject to $t \ge P_i \cdot P_i - 2C \cdot P_i$, $i = 1, \ldots, M$. We thank Steve Fortune for this observation.

(3) Let Π be defined by the inequalities $\pi_i \cdot z \ge a_i$, $i = 1, \ldots, M$. Let z be the closest point of Π to P, and set $y = P - z$. Then we call AMPL and MINOS to minimize $y \cdot y$ subject to $\pi_i \cdot y \le \pi_i \cdot P - a_i$, $i = 1, \ldots, M$.

dimensional subspace orthogonal to v. Each vector in Π has positive inner product with at least one of the other v_i's, and so by the inductive hypothesis there is a vector $x \in \Pi$ with positive inner product with at least $n-1$ of them, say $x \cdot v_i = b_i > 0$, $1 \le i \le n-1$. Choose ε to satisfy $0 < \varepsilon < b_i / |v \cdot v_i| \neq 0$. Then $w = x + \varepsilon v$ is a vector with the desired properties. ■

Proof of Theorem 2. Let $R' \le R$ be the smallest number such that closed balls of radius R' around the points of Σ cover $\mathbb{R}^n$, and let P be a critical point for the covering, i.e. a point which would not be covered if the balls were shrunk further. We take P as the origin, and let $v_1, \ldots, v_M$ be the points in Σ at distance R' from P. The v_i satisfy the hypothesis of Lemma 3, and the point εw, for sufficiently small $\varepsilon > 0$, is in the interior of at least n of the balls. ■

We now show that the behavior illustrated in Figure 1 is typical of all lattices, in the sense that the regions inside sets of $CM(\Lambda)$ intersecting balls are always disjoint from each other (just as the open lunes in Figure 1 are disjoint).

Theorem 4. For each set 6 of $CM(\Lambda)$ lattice points for which the corresponding balls have a common intersection, let $U(6)$ denote their open intersection. Then the regions $U(6)$ are disjoint.

Proof. A point P of the boundary of $U(6)$ is not in the interior of any ball with center $x \notin 6$, or else some point near P would be contained in more than $CM(\Lambda)$ balls. On the other hand P is not in the interior of at least one of the original balls. Therefore points on the boundary of $U(6)$ are not covered $CM(\Lambda)$ times. ■

Theorem 4 implies that we can compute the umbral spheres for Λ (defined in Section 1) as follows. Let ρ be the smallest periradius of any such set 6. Then the umbral spheres have radius $R - \rho$, where R is the covering radius of the lattice, and so

$$R_u = \frac{R - \rho}{(\det \Lambda)^{1/(2n)}} . \tag{4}$$

For many of the lattices we consider (although not for E_7 or Λ_{24}), all sets 6 have the same periradius.

3. The n-dimensional cubic lattice $\mathbf{I_n}$.

I_n consists of the vectors $x_1 \, x_2 \ldots x_n$ with all $x_i \in \mathbb{Z}$, and has covering radius $R = \sqrt{n}/2$ (see [3, Chapter 4, §5)]. Clearly $CM(I_1) = 1$, so we may assume $n \ge 2$. The point $0 \; ½ \; ½ \cdots ½$ is in the interior of the balls of radius R centered at all lattice points $0{*}{*}\cdots{*}$, where $*$ indicates a coordinate that is 0 or 1, and so

$$CM(I_n) \ge 2^{n-1} . \tag{5}$$

The numerical evidence in Table II suggests that equality may hold in (5) if and only if $n \le 33$. However we can establish the equality only for smaller values of n.

Theorem 5.

$$CM(I_n) = 2^{n-1} \quad \text{for} \quad n \le 5 . \tag{6}$$

Proof. We give the proof when $n = 5$, the other cases being similar and easier. We may assume that a thickest point has the form $P = v \; w \; x \; y \; z$, where $0 \le v \le w \le x \le y \le z \le ½$. It is easy to check by hand that only 37 lattice points are within norm $R^2 = 5/4$ of such a point, namely the 32 points $* * * * *$ (where $* = 0$ or 1) and the 5 points $(-1 \; 0 \; 0 \; 0 \; 0)$, where the parentheses indicate that all cyclic shifts of the parenthesized coordinates are to be included. We must show that no subset 6 of 17 of these 37 points has perinorm $< 5/4$.

A basic principle which we will use repeatedly is that a set of balls of radius R with a common interior point cannot include two balls with centers A and B satisfying

$$N(A - B) \ge 4R^2 . \tag{7}$$

(For the interiors of two such balls are disjoint.)

Case 1, 6 contains $-1\ 0\ 0\ 0\ 0$ and $0\ -1\ 0\ 0\ 0$. By the above principle the other points in 6 can only include the 10 points $1\ 0\ 0\ 0\ 0$, $0\ 1\ 0\ 0\ 0$ and $0\ 0\ *\ *\ *$, and so $|6| \le 12$.

Case 2, 6 contains no vector with negative components. In this case there are 32 candidates for 6, the points $*\ *\ *\ *\ *$. We form what we shall call the *exclusion graph* for these points: there is a node for each point, and two nodes are joined by an edge if and only if the corresponding points *A* and *B* satisfy (7). Then $|6|$ clearly cannot exceed the *independence number* of this graph (the maximal size of an independent set of nodes).

In the present example the graph consists of 16 disjoint edges, and so $|6| \le 16$.

Case 3, 6 contains a single vector with a negative coordinate, say $-1\ 0\ 0\ 0\ 0$. The only other vectors that can occur in 6 are the 17 vectors $1\ 0\ 0\ 0\ 0$ and $0\ *\ *\ *\ *$, and we cannot have both $1\ 0\ 0\ 0\ 0$ and $0\ *\ *\ *\ *$. If $1\ 0\ 0\ 0\ 0$ does not occur we are back to Case 2. Otherwise 6 consists of the 17 vectors $-1\ 0\ 0\ 0$ and $0\ *\ *\ *\ *$, whose perinorm (found by AMPL and MINOS, see footnote 3) is $13/9 > 5/4$. Thus no 17-subset 6 exists, and so $CM(I_5) = 16$. ■

To obtain better lower bounds than (6) for larger values of *n* we make use of theta series. Points of the form $P = 0^{n-r} \frac{1}{2}^{r}$ for $0 \le r \le n-1$ appear to be good candidates for thickest points in I_n. The theta series of I_n with respect to *P* is ([3, p. 106])

$$\begin{aligned} & \sum_{x \in I_n} q^{N(x-P)} \\ &= \theta_2(q)^r \theta_3(q)^{n-r} \\ &= \sum A_t q^t \quad (\text{say})\ , \end{aligned} \tag{8}$$

where

$$\theta_2(q) = \sum_{m=-\infty}^{\infty} q^{(m+1/2)^2}\ ,\ \theta_3(q) = \sum_{m=-\infty}^{\infty} q^{m^2}\ ,$$

and A_t denotes the number of lattice points at squared distance *t* from *P*.

We use MACSYMA [11] to calculate $\sum_{t<R} A_t$ (the number of balls containing *P*) and to maximize this over *r*. The results for $n \le 40$, together with a best choice for *r*, are shown in Table II.

It follows from the work of Mazo and Odlyzko [12] that, as $n \to \infty$,

$$CM(I_n) \sim 2.089097\ldots^{n+O(\sqrt{n})} \tag{9}$$

This is the case $\alpha = 1/4$ of [12]; the particular constant needed for (9) is not given in Table 1 of [12] but was kindly supplied by Andrew Odlyzko. In [12] it is also shown that any point of the form $0^{n-r}\frac{1}{2}^{r}$ for fixed *r* is in $c^{n+o(n)}$ balls as $n \to \infty$, where *c* is strictly less than the constant in (9).

4. The lattice $\mathbf{A_n}$.

A_n consists of those points $x_0\, x_1 \ldots x_n \in I_{n+1}$ satisfying $\Sigma x_i = 0$, and has squared covering radius $R^2 = a(n+1-a)/(n+1)$, where $a = [(n+1)/2]$ (see [3, Chapter 4, §6.1]). It is trivial to see that $CM(A_1) = 1$, $CM(A_2) = 2$; and the face-centered cubic lattice $A_3 \cong D_3$ will be discussed in Section 6.

For A_{2t} the point

$$\left[\frac{t+2}{2t+1}\right]^{t-1} \left[\frac{-t+1}{2t+1}\right]^{t+2}$$

is certainly contained in the balls centered at points of the form

$$x_1\ \ldots\ x_{t-1}\ ;\ -x_1\ \ldots\ -x_{t-1}\ 0\ldots0\ ,$$

$x_i = 0$ or 1 (and in other balls when *n* is large), so that

$$CM(A_{2t}) \geq \sum_{i=0}^{t-1} \binom{t-1}{i} \binom{t+2}{i} = \binom{2t+1}{t-1} . \tag{10}$$

Similarly the point

$$\left[\frac{t+2}{2t+2} \right]^{t} \left[\frac{-t}{2t+2} \right]^{t+2}$$

shows that

$$CM(A_{2t+1}) \geq \sum_{i=0}^{t} \binom{t}{i} \binom{t+2}{i} = \binom{2t+2}{t} . \tag{11}$$

Equations (10) and (11) justify the lower bounds in Table I for $n \geq 5$. When $n = 4$, (10) can be strengthened, since the point $\frac{3}{7}^{\,2} \; -\frac{2}{7}^{\,3}$ is contained in seven spheres.

Theorem 6. The covering multiplicities of $A_1 , \ldots , A_5$ are respectively $1, 2, 4, 7, 15$.

Proof. We begin with the proof for A_4; this is the prototype of many later proofs.

A fundamental simplex for the (infinite) affine Weyl group of type A_n is defined by the inequalities

$$x_0 \leq x_1 \ , \ x_1 \leq x_2 , \ldots , x_{n-1} \leq x_n \ , \ x_n \leq x_0 + 1 \tag{12}$$

(see [1], [5], and especially [3, Chapter 21, Figure 21.1]). We consider a thickest point P for A_4 contained in the fundamental simplex. In particular, P is in the ball centered at the origin. The center of any other ball containing P is at squared distance $< 4R^2 = 24/5 = 4.8$ from the origin. Since the theta series of A_4 (with respect to the origin) is

$$1 + 20q^2 + 30q^4 + \cdots ,$$

there are 51 possible balls to consider.

We now use AMPL and MINOS (see footnote 4) to compute the squared distance from each of these points to the fundamental simplex. For only 12 of the 51 points is this less than $R^2 = 6/5$. The exclusion graph of these 12 points (defined in the previous section) is shown in Figure 2. It is easily seen (by eye) that the independence number of this graph is 8, and that there are exactly two independent sets of size 8. However (from AMPL and MINOS), the perinorm of each such set is exactly $6/5$. Thus only seven open balls can intersect, and $CM(A_4) = 7$.

The argument for A_5 is similar. Now $R^2 = 3/2$, $4R^2 = 6$, and the theta series is $1 + 30q^2 + 90q^4 + 140q^6 + \cdots$, so there are 261 vectors to be considered. Only 27 of them are at distance $< R$ from the fundamental simplex, and we form the 27-node exclusion graph, in which we must show that there is no maximal independent set of size ≥ 16. There are five nodes with too high a degree to be part of such a set, and (after their removal) six isolated nodes which *must* be part of such a set. But the remaining 16-node graph contains 8 disjoint edges. ■

5. The lattice $\mathbf{A}_\mathbf{n}^*$

The dual lattice A_n^* is the union of $n+1$ cosets $[i] + A_n$ of A_n, where

$$[i] = \left[\frac{-i}{n+1} \right]^{j} \left[\frac{j}{n+1} \right]^{i} , \tag{13}$$

and $i + j = n+1$, $0 \leq i \leq n$. The Delaunay cells are copies of the fundamental simplex for A_n, for instance that with vertices $[i]$, $0 \leq i \leq n$. (This simplex is the same as that defined in (12).) The covering radius R of A_n^* is given by

$$R^2 = \frac{n(n+2)}{12(n+1)} .$$

(For further information about this lattice see [3, Chapter 4, §6.6] and [4].)

Theorem 7.

$$CM(A_n^*) = n \quad \text{for} \quad n \le 8 \ .$$

Proof. By Theorem 2, $CM(A_n^*) \ge n$. We may assume that a thickest point P is in the fundamental simplex with vertices $[i]$, $0 \le i \le n$. Let 6 be the set of centers of a collection of $CM(A_n^*)$ open balls containing P. For $n \le 5$ we have $4R^2 < 2$, and so 6 cannot contain two points from the same coset $[i] + A_n$, implying $|6| \le n + 1$. (This is because if $u, v \in [i] + A_n$, $u \ne v$, then $u - v \in A_n$ and so $N(u - v) \ge 2$.) Furthermore (still for $n \le 5$) we cannot have $|6| = n + 1$, for the $n + 1$ closed balls centered at the vertices $[i]$, $0 \le i \le n$, meet at the center of the simplex but do not have a common interior point. Thus

$$CM(A_n^*) \le n \quad \text{for} \quad n \le 5 \ .$$

We now discuss the case $n = 8$, omitting $n = 6$ and 7 which are similar and easier.

Since $R^2 = 20/27$, $4R^2 = 80/27$, and the theta series of A_8^* begins

$$1 + 18q^{8/9} + 72q^{14/9} + 240q^2 + 252q^{20/9} + 504q^{26/9} + \cdots ,$$

we must consider a population Ω of $1 + 18 + \cdots + 504 = 1087$ lattice vectors. These are distributed among the nine cosets as shown in Table III.

Table III. Vectors of norm $< \dfrac{80}{27}$ in A_8^*

Coset	Norm	Number	Norm	Number
0	0	1	2	72
1	8/9	9	26/9	252
2	14/9	36	–	–
3	2	84	–	–
4	20/9	126	–	–
5	20/9	126	–	–
6	2	84	–	–
7	14/9	36	–	–
8	8/9	9	26/9	252

We must show that no 9-subset 6 of Ω corresponds to a set of intersecting spheres. There cannot be just one vector from each coset, by the argument at the end of the proof for $n = 5$, so there must be ≥ 2 vectors from some coset. We may also assume if we wish that $0 \in 6$. We proceed to establish several properties of 6.

(i) 6 does not contain four vectors u, v, u, x from the same coset.
Proof. We may take $u = 0$. Then v, w, x and their differences must have norm 2, say $v = 1\ {-1}\ 0^7$, $w = 1\ 0\ {-1}\ 0^6$, $x = 1\ 0^2\ {-1}\ 0^5$. But the perinorm of this set is $3/4 > 20/27$.

(ii) If $0 \in 6$, then 6 does not contain a vector of norm 26/9.

Proof. We take the particular norm 26/9 vector

$$u = -\frac{10}{9} \left[\frac{8}{9} \right]^2 \left[-\frac{1}{9} \right]^6$$

and examine every triple $0, u, v$, where $v \in \Omega$, $v \ne 0$, $v \ne u$. There are only six choices for v for which the perinorm of the triple is $< 20/27$. So if $u \in 6$, $|6| \le 8$.

(iii) If 6 contains two vectors from coset $[i] + A_8$, there is at most one vector from $[i + 1] + A_8$ and at most one from $[i - 1] + A_8$.

Proof. Suppose $u = 0^9$, $v = 1\ 0^7\ {-1} \in 6$. Then the only compatible vector from $[1] + A_8$ is $\frac{8}{9} \left[-\frac{1}{9} \right]^8$.

(iv) 6 does not contain vectors u, v, w with $u - v \in A_8$, $v - w \in A_8$ and $N(u - v) = N(v - w) = N(w - u) = 2$.
Proof. We may take $u = 0^9$, $v = 1\ -1\ 0^7$, $w = 1\ 0\ -1\ 0^6$. There are 10 possible vectors $x \in \Omega$ such that the perinorm of u, v, w, x is $< 20/27$. By use of (i) and (ii) these 10 are reduced to 5, and so $|6| \le 8$.

(v) 6 does not contain three vectors from the same coset.

This follows from (ii) and (iv).

(vi) If 6 contains two vectors from coset $[i] + A_8$ then it contains no vectors from $[j] + A_8$ if $|i - j| \equiv 4 \pmod 9$.
Proof. We may assume 6 contains $u = 0^9$, $v = 1\ 0^7\ -1$, $w = \left[\frac{5}{9}\right]^4 \left[-\frac{4}{9}\right]^5 \in [4] + A_8$. There are 12 vectors x that can be adjoined, but none belong to cosets [4], [5] or [6]. Therefore 6 must contain two vectors x, x' from one of cosets [2], [3] or [7]. However none of the resulting sets u, v, w, x, x' can be extended by a sixth vector x''.

It is now easy to see that there is no way to distribute 9 points among the 9 cosets while satisfying (iii), (v) and (vi). ■

The perinorm of the points $[1], \ldots, [n]$ is

$$\rho^2 = \frac{(n-1)(n^2+4n+6)}{12(n+1)^2} .$$

This determines the radius of the umbral sphere for A_n^*, $n \le 8$ (see Section 1 and Eq. (4)).

For larger values of n we have $CM(A_9^*) \ge 11$ (by considering the point $\frac{1}{10}^{5} \ -\frac{1}{10}^{5}$), $CM(A_{10}^*) \ge 12$ (from $0^5\ \frac{5}{6}\ -\frac{1}{6}^{5}$), $CM(A_n^*) \ge 2n + 3$ for $n \ge 11$ (from 0^{n+1}), and $CM(A_{14}^*) \ge 34$ (from $\frac{9}{14}^{5}\ 0\ -\frac{5}{14}^{9}$). We do not know the asymptotic behavior of $CM(A_n^*)$, although (3) gives a lower bound.

6. The lattice D_n

D_n consists of those points $x_1 \ldots x_n \in I_n$ for which $x_1 + \cdots + x_n$ is even. Since $D_1 \cong I_1$, $D_2 \cong I_2$ we shall assume $n \ge 3$. The covering radius R is 1 (if $n \le 4$) or $\sqrt{n}/2$ (if $n \ge 4$). (For further information about D_n see [3, Chapter 4, §7.1].)

Points of the form $P = 0^{n-r} a^r$, where $a = ½ - \varepsilon$, $\varepsilon > 0$ is small, and $0 \le r \le n$ appear to be the best candidates for thickest points. When $r = 1$ this yields

$$CM(D_n) \ge 2^{n-2} \quad (n \ge 3) , \tag{14}$$

although if $n \equiv 1 \pmod 4$ we get a slightly better result by taking $r = 0$, namely

$$CM(D_{4t+1}) \ge 1 + \begin{bmatrix} 4t+1 \\ 2 \end{bmatrix} + \begin{bmatrix} 4t+1 \\ 4 \end{bmatrix} + \cdots + \begin{bmatrix} 4t+1 \\ 2t \end{bmatrix} . \tag{15}$$

When n is large these lower bounds fail to count all the balls containing P, and (just as for I_n) we get better bounds by using the theta series of the lattice with respect to P (compare Eq. (8)). For D_n this can be shown to be equal to

$$\frac{1}{2} \{ \psi_3(q)^r \theta_3(q)^{n-4} + \psi_4(q)^r \theta_4(q)^{n-r} \} , \tag{16}$$

where

$$\psi_3(q) = \sum_{m=-\infty}^{\infty} q^{(m-a)^2} \ , \quad \psi_4(1) = \sum_{m=-\infty}^{\infty} (-1)^m q^{(m-a)^2} \ ,$$

$$\theta_4(q) = \sum_{m=-\infty}^{\infty} (-1) q^{m^2} \ .$$

Again we use MACSYMA to determine the number of balls containing P and to optimize over r. We have computed this bound for $n \le 40$ and show the values for $n \le 20$ in Table IV. (The pattern in the third column breaks down for $n \ge 25$.)

Table IV
Lower bounds to covering
multiplicities of the lattices D_n

n	$CM(D_n) \ge$	r
3	$2+2$	3
4	2^2	3
5	2^3+3	5
6	2^4	5
7	2^5	6
8	2^6	7
9	2^7+35	9
10	2^8	9
11	2^9	10
12	2^{10}	11
13	$2^{11}+462$	13
14	2^{12}	13
15	2^{13}	14
16	2^{14}	15
17	$2^{15}+6435$	17
18	2^{16}	17
19	2^{17}	18
20	2^{18}	19

We suspect that many of the lower bounds in Table IV are exact, although we can only prove this for $n \le 5$.

Theorem 8. The covering multiplicities of D_3, D_4 and D_5 are respectively 4, 4 and 11.

Proof. The face-centered cubic lattice D_3 has two kinds of Delaunay polyhedra, tetrahedra and octahedra, and $R = 1$. We can now check that the ball centered at any vertex of a tetrahedral or octahedral Delaunay polyhedron intersects only those faces of the polyhedron that involve that vertex – we omit the details. It follows that points in the Delaunay octahedra are covered at most 3 times and points in the Delaunay tetrahedra at most 4 times, so that $CM(D_3) = 4$.

For D_4 and D_5 we assume that a thickest point belongs to the fundamental simplex defined by

$$0 \le x_1 + x_2 \ , \ x_1 \le x_2 \ , \ x_2 \le x_3 \ ,$$
$$\ldots \ , \ x_{n-1} \le x_n \ , \ x_{n-1} + x_n \le 1 \ . \tag{17}$$

([3, Chapter 21, Fig. 21.2]). The proofs now follow those for A_4 and A_5.

In the case of D_4 the exclusion graph consists of a triangle and three isolated vertices, and so $CM(D_4) \le 4$. Each Delaunay polytope for D_4 is an 8-vertex orthoplex (also called a cross-polytope or generalized octahedron, cf. [4], [5]). There is a unique way for four balls to intersect: they must be centered at the vertices of a tetrahedral face of an orthoplex cell (for example the four vertices $(1, 0, 0, 0)$, with pericenter $\frac{1}{4} \ \frac{1}{4} \ \frac{1}{4} \ \frac{1}{4}$ and perinorm $3/4$).

In the case of D_5 the exclusion graph has 20 nodes and contains precisely two maximal independent sets of size 12, both of which have perinorm $5/4$. ■

Asymptotically it seems very plausible (although we have not tried to carry this out) that the arguments of [12] can be modified to show that (9) also holds for D_n.

7. The lattice $\mathbf{D_n^*}$

For the dual lattice D_n^* see [3, Chapter 4, §7.4] and [4]. Since $D_3^* \cong A_3^*$ and $D_4^* \cong D_4$ we shall assume $n \geq 5$. Consideration of the points

$$\left[\frac{1}{2} - \varepsilon\right]^{t-1} \ \frac{1}{4}^{\,2} \ 0^{t-1} \ ,$$

$$\left[\frac{1}{2} - \varepsilon\right]^{t} \ \left[\frac{1}{4} + \varepsilon\right]^{1} \ 0^{t} \ ,$$

$$\left[\frac{1}{2}\right]^{2n} \ \left[\frac{1}{4} - \varepsilon\right]^{1} \ \varepsilon \ 0^{2u-1} \ ,$$

shows that

$$CM(D_{2t}^*) \ \geq \ 2t \ , \tag{18}$$

$$CM(C_{2t+1}^*) \ \geq \sum_{i=0}^{[(t-1)/2]} \begin{bmatrix} t \\ i \end{bmatrix} + 2^t \ , \ \ (t \text{ odd}) \tag{19}$$

$$CM(D_{4u+1}^*) \ \geq \ 3 \cdot 2^{2u-1} \ , \tag{20}$$

respectively, and yields the lower bounds in Table I.

Theorem 9.

$$CM(D_5^*) \ = \ 6 \ , \ CM(C_6^*) \ = \ 8 \ .$$

Proof. The method used to prove $CM(A_4) \ = \ 7$ works very easily here. The exclusion graphs for D_5^* and D_6^* contain respectively 8 and 12 nodes. In both cases these nodes are subsets of the vertices of the Delaunay polytope (cf. [4]). In the case of D_6^* for example the Delaunay polytope is the 16-vertex cross-join of two 3-dimensional cubes in orthogonal spaces. There is a unique arrangement of eight intersecting balls, whose centers are obtained by taking 4 non-opposite vertices out of the 8 vertices of each of the two cubes. ■

8. The lattices $\mathbf{E_n}$ and $\mathbf{E_n^*}$ $(\mathbf{n \ = \ 6, 7, 8})$

E_8 is the union of D_8 and the translate ½½½½½½½½ $+ \ D_8$. The covering radius R is 1, and there are two kinds of Delaunay polytopes, orthoplexes centered at deep holes such as $1 \ 0^7$, and simplexes centered at shallow holes such as $\frac{1}{3}^{\,7} \ \ -\frac{1}{3}$ (see [3, Chapter 4, § 8.1], [4], [5]).

Theorem 10.

$$CM(E_8) \ = \ 9 \ .$$

Proof. The point $\frac{1}{3}^{\,7} \ \ -\frac{1}{3}$ is contained in the balls centered at the 9 vertices 0^8, $1/2^8$, $(1/2^6 \ -1/2) \ -1/2$ of the Delaunay simplex, so $CM(E_7) \geq 9$. To obtain an upper bound we observe, as in the corresponding proof for D_3 (Theorem 8), that the covering ball centered at any lattice point is contained in the union of the Delaunay polytopes of which that point is a vertex. Points in a Delaunay orthoplex are therefore covered at most 8 times, and points in a Delaunay simplex at most 9 times. ■

E_7 may be defined to consist of those vectors $x_0 \, x_1 \ldots x_7 \in E_8$ for which $\Sigma x_i = 0$, and has $R^2 = 3/2$. The dual lattice E_7^* is generated by E_7 and the vector

$$\left[\frac{3}{4} \right]^2 \left[-\frac{1}{4} \right]^6 ,$$

and has $R^2 = 7/8$ (see [3, Chapter 4, § 8.2], [4], [5], [20]). The theta series of E_7 and E_7^* are

$$1 + 126q^2 + 756q^4 + 2072q^6 + \ldots , \tag{21}$$

$$1 + 56q^{3/2} + 126q^2 + 576q^{7/2} + \ldots , \tag{22}$$

respectively.

Theorem 11.

$$CM(E_7) = 28 , \; CM(E_7^*) = 9 .$$

Proof. (i) The Delaunay polytopes for E_7 are simplexes and Hesse polytopes 3_{21} (see [3], [4], [5]). To understand the Hesse polytope it is helpful to think of an icosahedron, whose 12 vertices lie in four layers of sizes $1 + 5 + 5 + 1$. The 56 vertices of the Hesse polytope similarly fall into four layers of sizes $1 + 27 + 27 + 1$, as shown in Table V (cf. [4, Table VIII]). The Schläfli polytope 2_{21} is described below when we discuss E_6.

Table V. Sections of the Hesse polytope 3_{21}.

Coordinates	Number	Shape
$\frac{3}{4} \left(-\frac{1}{4}\right)^6 \frac{3}{4}$	1	zenith
$\frac{3}{4} \left[\frac{3}{4} \left(-\frac{1}{4}\right)^5 \right] -\frac{1}{4}$	6	27 vertices,
$\frac{1}{4} \left[\left(-\frac{3}{4}\right)^2 \left(\frac{1}{4}\right)^4 \right] \frac{1}{4}$	15	forming a
$-\frac{1}{4} \left[\frac{3}{4} \left(-\frac{1}{4}\right)^5 \right] \frac{3}{4}$	6	Schläfli polytope
$\frac{1}{4} \left[-\frac{3}{4} \left(\frac{1}{4}\right)^5 \right] -\frac{3}{4}$	6	27 vertices,
$-\frac{1}{4} \left[\left(\frac{3}{4}\right)^2 \left(-\frac{1}{4}\right)^4 \right] -\frac{1}{4}$	15	forming a
$-\frac{3}{4} \left[-\frac{3}{4} \left(\frac{1}{4}\right)^5 \right] \frac{1}{4}$	6	Schläfli polytope
$-\frac{3}{4} \left(\frac{1}{4}\right)^6 -\frac{3}{4}$	1	nadir

The vectors in Table V are in fact the minimal vectors of E_7^*. The pericenter of the first 28 vectors in the table is

$$\frac{1}{4} \left(-\frac{1}{12}\right)^6 \frac{1}{4}$$

and their perinorm is $4/3$, showing that $CM(E_7) \geq 28$. However, there are several inequivalent ways to choose a set of 28 overlapping balls. We conjecture that the above set has the smallest perinorm.

To prove $CM(E_7) \le 28$ we follow our usual method. From the theta series (21) there are 2955 vectors of norm $< 4R^2 = 6$ to be considered. The fundamental simplex is defined by

$$x_0 \le x_1 \ , \ x_1 \le x_2 , \ldots , x_6 \le x_7 \ ,$$
$$x_4 + \cdots + x_7 - x_0 - \cdots - x_3 \le 1 \ . \tag{23}$$

The exclusion graph has 42 nodes, and we must consider all maximal independent sets 6 of size ≥ 29. There is a set (6_1 say) of 12 isolated nodes. It is easy to find by hand a set of 13 disjoint edges, and in four distinct ways. Hence there are at most $42 - 13 = 29$ nodes in 6. Furthermore an independent set of size 29 must contain every node not involved in these 13 edges. In this way we find a set (6_2) of six further nodes that must belong to 6. Furthermore the six nodes (6_3) joined to 6_2 are not in 6. This reduces the graph to $42 - 12 - 6 - 6 = 18$ nodes in which we must find all independent sets of size $29 - 12 - 6 = 11$. In the reduced graph there is a node x of degree 8 which therefore cannot belong to 6, and a node y of degree 6. If $y \in 6$ there is a unique choice for 6, which has perinorm $14/9 > 3/2$; and if $y \notin 6$ then $|6| \le 28$.

(ii) The proof for E_7^* is much easier. From (22) there are 759 vectors to be considered. The exclusion graph has 15 nodes and is shown in Figure 3.

There is a unique maximal independent set, of size 9, consisting of the points

$$(1) \qquad -\frac{7}{4} \ \frac{1}{4}^{7}$$
$$(2) \qquad -\frac{5}{4} \ -\frac{1}{4}^{4} \ \frac{3}{4}^{3}$$
$$(3) \qquad -1 \ 0 \ (1 \ 0^2)$$
$$(4) \qquad -\frac{3}{4} \left[-\frac{3}{4} \ \frac{1}{4}^{3} \right] \ \frac{1}{4}^{3}$$

with pericenter

$$-\ 1 \ 0^4 \ \frac{1}{3}^{3}$$

and periradius $5/6 < 7/8$. Thus $CM(E_7^*) = 9$. ■

The proof also shows that the configuration of 9 overlapping balls in E_7^* is unique.

The points of E_7^* fall into two orbits under the corresponding affine Weyl group, as indicated by the stars in Figure 4(a), using the notation of [4]. The above 9 points form two regular simplexes of dimensions 2 and 5, as shown in Figure 4(b), so their convex hull is the *cross-join* of these two simplexes.

E_6 may be defined to consist of those vectors $x_0 \, x_1 \ldots x_7 \in E_8$ for which $x_0 + x_7 = x_1 + \cdots + x_6 = 0$, and has $R^2 = 4/3$. The dual lattice E_6^* is generated by E_6 and the vector

$$0 \ -\frac{2}{3}^{2} \ \frac{1}{3}^{4} \ 0 \ ,$$

and has $R^2 = 2/3$ (see [3, Chapter 4, §8.3], [4], [5], [19]). The theta series of E_6 and E_6^* are

$$1 + 72q^2 + 270q^4 + \cdots , \tag{24}$$
$$1 + 54q^{4/3} + 72q^2 + \cdots , \tag{25}$$

respectively.

Theorem 12.

$$CM(E_6) = 17 \ , \quad CM(E_6^*) = 6 \ .$$

Proof. (i) The Delaunay polytopes for E_6 are Schläfli polytopes in two orientations [4]. The Schläfli polytope has 27 vertices, which may be taken to be the following:

$$(6) \qquad \frac{1}{2}\left[\frac{5}{6} \; -\frac{1}{6}^{5}\right] -\frac{1}{2} \; ,$$

$$(15) \qquad 0 \; \left[-\frac{2}{3}^{2} \; \frac{1}{3}^{4}\right] \; 0 \; ,$$

$$(6) \qquad -\frac{1}{2}\left[-\frac{1}{6}^{5} \; \frac{5}{6}\right] - \frac{1}{2} \; .$$

Each vertex has 16 neighbors, and the perinorm of such a set of 17 vectors is $5/4 < 4/3$, showing that $CM(E_6) \geq 17$. We will see that this is the only way a point can be covered 17 times.

To obtain an upper bound we consider the 343 vectors (from (24)) of norm $< 4R^2 = 16/3$. The fundamental simplex for E_6 may be defined by the inequalities

$$x_1 \leq x_2 \; , \; x_2 \leq x_3 , \ldots , x_5 \leq x_6 \; ,$$

$$x_0 - x_7 \leq 1 \; , \; x_0 + \cdots + x_3 \geq x_4 + \cdots + x_7 \; . \tag{26}$$

However, this simplex is cut into two equal parts by the wall of one of the Schläfli polytopes, and so we may assume that a thickest point *P* belongs to the semi-simplex defined by (26) and the extra inequality

$$2(x_1 + x_6) \geq x_2 + x_3 + x_4 + x_5 \; . \tag{27}$$

We now use AMPL and MINOS (see footnote 4) to show that just 20 of the 343 vectors are within *R* of the semi-simplex. A second application of AMPL and MINOS (see footnote 3) shows that the perinorms of all 18-subsets of this set of 20 are $\geq 4/3$. Hence $CM(E_7) \leq 17$. Another calculation of the same type shows that there is a unique set of 17 out of the 20 with perinorm $< 4/3$.

(ii) There is only one type of Delaunay polytope for E_6^*, which is the convex hull of three equilateral triangles in three orthogonal planes – we call this the *cross-join* of three triangles. It has 9 vertices, for example

$$\begin{array}{ll}
(3) & 0 \; \left[-\frac{2}{3} \;\; \frac{1}{3} \;\; \frac{1}{3}\right] \; 0 \;\; 0 \;\; 0 \;\; 0 \\[2ex]
(3) & 0 \;\; 0 \;\; 0 \;\; 0 \; \left[\frac{2}{3} \; -\frac{1}{3} \; -\frac{1}{3}\right] \; 0 \\[2ex]
(3) & \left\{ \begin{array}{cccccccc}
\frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{2} \\[1ex]
-\frac{1}{2} & \frac{1}{6} & \frac{1}{6} & \frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & -\frac{1}{6} & \frac{1}{2} \\[1ex]
0 & -\frac{1}{3} & -\frac{1}{3} & -\frac{1}{3} & \frac{1}{3} & \frac{1}{3} & \frac{1}{3} & 0
\end{array} \right.
\end{array} \tag{28}$$

whose center is the hole

$$0 \; \frac{1}{3} \; \frac{1}{3} \; \frac{1}{3} \; - \frac{1}{3} \; - \frac{1}{3} \; - \frac{1}{3} \; 0 \; .$$

The best we can do with these centers is to pick two out of each triple, so $CM(E_6^*) \geq 6$. For example, taking the first two of each triple in (28), we find that the pericenter is

$$0 \;\; 0 \;\; 0 \; \frac{1}{6} \; 0 \;\; 0 \; -\frac{1}{6} \; 0$$

and the perinorm is $11/18$. These six centers form what we call the *separated join* of three intervals, and

Figures 4(c) and (d) are analogous to Figures 4(a) and (b).

To obtain an upper bound we consider the 127 vectors (from (25)) of norm $< 4R^2 = 8/3$. Of these just nine are within R of the fundamental simplex (26). After a suitable translation of coordinates those nine coincide with (28). ■

9. The Leech lattice Λ_{24}

For the definition and properties of this lattice see [3]. The covering radius $R = \sqrt{2}$ and there are 307 distinct types of hole, namely 23 types of deep hole and 284 types of shallow hole ([2], [3, Chapters 23,25]).

Theorem 13.

$$CM(\Lambda_{24}) = 25 \ .$$

We begin with a lemma.

Lemma 14. Let h be a hole in Λ_{24}, Π the Delaunay polytope containing h, and v_1 a vertex of Π. If the closed ball of radius R around h intersects the boundary of Π at some point of a face π, then v_1 is a vertex of π.

Proof of Lemma 14. Let the vertices of Π be $v_1, \ldots, v_M$. As in [3, Chapter 23, p. 481] we may choose a point c (outside the space if h is a shallow hole) so that $N(v_i - c) = 2$, $(v_i - c) \cdot (v_j - c) = 0, -1$ or -2, for $i, j = 1, \ldots, M$, $i \neq j$. Any point P on a face not containing v_1 can be written as

$$P - c = \sum_{i=2}^{M} \lambda_i (v_i - c) \ , \quad \lambda_i \geq 0 \ , \quad \sum_{i=2}^{M} \lambda_i = 1 \ .$$

Then $N(P - v_1) = N(P - c) - 2(P - c) \cdot (v_1 - c) + N(v_1 - c) \geq N(P - c) + 2 > 2$. ■

Lemma 14 implies that the ball of radius R around v_1 is strictly inside the union of all Delaunay polytopes that have v_1 as a vertex.

Proof of Theorem 13. (i) Suppose P is in a Delaunay polytope containing a shallow hole. All such polytopes are simplexes. By the Lemma, P is contained in at most the 25 balls centered at the vertices of the simplex. The shallow hole itself is contained in these 25 balls. The perinorm of these 25 vertices ranges from $2 - \frac{2}{25}$ to $2 - \frac{1}{4900}$ [3, p. 521].

(ii) Suppose P is in a Delaunay polytope containing a deep hole h. In this case the point c used in proving the Lemma coincides with h. Let $v_1, \ldots, v_\mu \in \Lambda_{24}$ be the vertices of a component of the *hole diagram* [3, p. 481]. Then P is not in the interior of all μ balls of radius R centered at $v_1, \ldots, v_\mu$. For if it were then we have $N(P - v_i) < 2$, $i = 1, \ldots, \mu$, hence

$$N(P - h) < 2(P - h) \cdot (v_i - h) \ . \qquad (29)$$

But [3, p. 483] there are positive integers $m_1, \ldots, m_\mu$ such that $\sum_{i=1}^{\mu} m_i (v_i - h) = 0$. Therefore, multiplying (29) by m_i and summing, we have $\sum_{i=1}^{\mu} m_i N(P - h) < 0$, which is impossible. It follows that the number of open balls containing P is at most the number of vertices of the Delaunay polytope, minus the number of components of the hole diagram. This quantity is equal to 24, independent of the type of deep hole [3, Table 23.1]. ■

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On the Covering Multiplicity of Lattices*

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ABSTRACT

Let the lattice Λ have covering radius R, so that closed balls of radius R around the lattice points just cover the space. The covering multiplicity $CM(\Lambda)$ is the maximal number of times the interiors of these balls overlap. We show that the least possible covering multiplicity for an n-dimensional lattice is n if $n \le 8$, and conjecture that it exceeds n in all other cases. We determine the covering multiplicity of the Leech lattice and of the lattices I_n, A_n, D_n, E_n and their duals for small values of n. Although it appears that $CM(I_n) = 2^{n-1}$ if $n \le 33$, as $n \to \infty$ we have $CM(I_n) \sim 2.089\ldots^n$. The results have application to numerical integration.

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