Lattices with Few Distances*

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1. Introduction

An old problem in combinatorial geometry asks how to place a given number of distinct points in n-dimensional Euclidean space so as to minimize the total number of distances they determine [2], [3], [6], [8], [13], [14, Problem 17.1], [21], [23], [36]. Erdös in 1946 [13] considered configurations formed by taking all the points of a suitable lattice Λ that lie within a large region. The best lattices for this purpose are those that minimize what we shall call the *Erdös number* of the lattice, given by

$$E = F\, d^{1/n} \ , \tag{1}$$

where d is the determinant of the lattice and F, its *population fraction*, is given by

$$F = \lim_{x \to \infty} \frac{P(x)}{x} \ , \quad \text{if} \quad n \geq 3 \ , \tag{2}$$

where $P(x)$ is the population function of the corresponding quadratic form, i.e. the number of values not exceeding x taken by the form.(1) The Erdös number is the population fraction when the lattice is normalized to have determinant 1. It turns out that minimizing E is an interesting problem in pure number theory.

In this paper we prove all cases except $n=2$ (for which see Smith [37]) of the following proposition.(2)

The lattices with minimal Erdös number are (up to a scale factor) the even lattices of minimal determinant. For $n = 0, 1, 2, \ldots$ these determinants are (3)

$$1,\ 2,\ 3,\ 4,\ 4,\ 4,\ 3,\ 2,\ 1,\ 2,\ 3,\ 4,\ 4,\ 4,\ \ldots,$$

this sequence continuing with period 8.

For $n \le 10$ these lattices are unique:

$$A_0,\ A_1,\ A_2,\ A_3 \cong D_3,\ D_4,\ D_5,\ E_6,\ E_7,\ E_8,\ E_8 \oplus A_1,\ E_8 \oplus A_2\ ,$$

with Erdös numbers

$$1,\ 1,\ 2^{-3/2}\ 3^{1/4} \prod_{p \equiv 2(3)} \left[1 - \frac{1}{p^2}\right]^{-1/2} = 0.5533\ (p \text{ prime}),$$

$$\frac{11}{24}\ 4^{1/3} = 0.7276,\quad \frac{4^{1/4}}{2} = 0.7071,\quad \frac{4^{1/5}}{2} = 0.6598,\quad \frac{3^{1/6}}{2} = 0.6005,$$

$$\frac{2^{1/7}}{2} = 0.5520,\quad \frac{1}{2},\quad \frac{2^{1/9}}{2} = 0.5400,\quad \frac{3^{1/10}}{2} = 0.5581,$$

(rounded to 4 decimal places), while for each $n \ge 11$ there are two or more such lattices. (See Sect. 6, where we also give the runners-up in dimensions 3 and 4.)

Our methods do not apply when $n = 2$, since in this case the formula for the Erdös number is completely different. (The Erdös numbers of the simplest two-dimensional lattices can be evaluated from the information in [34], [35]. W. D. Smith informs us that he has recently settled this case [37].)

Remarks. (i) We note that the proposition is perhaps stated more simply in terms of the other prevailing notion of integrality: the answers are just the *integer-valued* quadratic forms that minimize the absolute value of the discriminant.

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(1) For $n \le 2$ these definitions must be modified. For $n = 0$ and 1 we set $E = 1$, while for $n = 2$ we replace (2) by $F = \lim_{x \to \infty} x^{-1} P(x) \sqrt{\log x}$.

(2) For $n = 2, 3, 4, 8, 16, 24, 32, \ldots$ this agrees with a conjecture of W. D. Smith (see [36],[37]).

(ii) When n is a multiple of 8 the proposition asserts that the lattices with minimal Erdös number are the even unimodular lattices.

(iii) The Erdös number of a form not proportional to a rational one is infinite, as will be shown in the Appendix. In the body of the paper we shall consider only rational forms.

(iv) The case $n = 3$ is the most difficult. The crucial number-theoretic result needed for our proof was first established by Peters [29] using the Generalized Riemann Hypothesis. The dependence on this hypothesis has very recently been removed by Duke and Schulze-Pillot [12] (see Section 2 below).

Although many books and papers have studied the numbers that are represented by quadratic forms (see for example [1], [4], [7], [11], [12], [15]-[20], [22], [24]-[35], [38], [40], [41]), we believe our results to be new.

2. Preliminaries to the proof

Our goal is to prove the proposition (3) for rational forms of dimension $n \geq 3$.

Let f be a positive-definite n-dimensional rational quadratic form with $n \geq 3$. By rescaling (which does not change the Erdös number E) we may assume that f is a *primitive* classically integral form, that is, f is classically integral but $\frac{1}{k} f$ is not, for $k = 2, 3, \ldots$.

To calculate E it suffices to consider the integers that are everywhere locally represented by f, that is to say, are represented over the p-adic integers $\mathbb{Z}_p$ for each p, or equivalently are represented by some form in the genus of f.

This is because for $n \geq 5$ a classical theorem of Tartakovsky [38], [7, p. 204] asserts that there are only finitely many numbers that are represented by f everywhere locally but not globally — the so-called *exceptions* for f. For $n = 4$ a theorem of Kloosterman, Tartakovsky and Ross & Pall tells us that there are only finitely many *primitive exceptions* (numbers not primitively represented by f although everywhere locally primitively represented by f) [7, p. 204], [20], [33], [38], [40, Th. 76].

For $n = 3$ there can be an infinite number of primitive exceptions. It was shown by Pall and Jones [28], [19, p. 188] that diag$\{3,4,9\}$ does not primitively represent any m^2 with $m \equiv 1$ (modulo 3). These numbers are represented by the form diag$\{1,3,36\}$, which is in the same genus but not in the same spinor genus. In fact the appropriate theorem for $n = 3$ is quite deep. It was shown by Peters [29] that, assuming the Generalized Riemann Hypothesis, the primitive exceptions belong to finitely many rational square classes,(3) and are finite in number in the case when the genus of the form contains just one spinor genus. Duke and Schulze-Pillot [12] have recently shown that the same result holds without the Generalized Riemann Hypothesis.(4)

So for all $n \geq 3$ the exceptions belong to finitely many rational square classes, which entails that the number of exceptions below x is small compared with the population function $P(x)$. (We thank the referee for pointing out that this weaker result was apparently already established by Watson [40], although no precise argument is given there. For the problem in 2 dimensions some relevant analytic results are given by Bernays [4] and Odoni [25].)

The above discussion permits us to calculate the Erdös number locally. For primes $p = 2,\ 3,\ 5,\ \ldots$ we define

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(3) A rational square class is a set of rational numbers of the form $\{k^2 a : k \in \mathbb{Q},\ k \neq 0\}$.

(4) We quote from a letter from Schulze-Pillot. ‘‘For ternary quadratic forms, the required result does not immediately follow from our corollary (on p. 56 of [12]), but can be proved in the same way. By Kneser’s result on spinor exceptions the (primitive) spinor exceptions do belong to finitely many square classes. Outside these classes the number of primitive representations by the spinor genus is the same as that by the genus and grows like $\sqrt{n}$. The difference between this and the number of primitive representations is obtained from the Fourier coefficients of the cusp form by Moebius inversion, and grows more slowly than $\sqrt{n}$. The result of our paper [12] can be summarized as *Peters’ results are true unconditionally*.’’

$$d_p \ = \ p\text{-part of } \ d \ = \ \det(f) \ ,$$

$$F_p \ = \ \text{proportion of } p\text{-adic integers represented by } f \ ,$$

$$E_p \ = \ d_p \ F_p^{1/n} \ ,$$

so that $d = \Pi d_p$, $F = \Pi F_p$, $E = \Pi E_p$.

For an odd prime p the form can be p-adically diagonalized, and written as

$$f_1 \oplus p \ f_p \oplus p^2 \ f_{p^2} \oplus \ \cdots \ , \tag{4}$$

where each f_q (for $q = p^k$) is of shape

$$f_q \ = \ \mathrm{diag}\{\alpha_1, \alpha_2, \ldots, \alpha_{n_q}\} \ , \tag{5}$$

the α_i are integers prime to p, and $n_q = \dim \ f_q$.

Since we can multiply by p-adic squares, all that is important about each α_i is whether it is a quadratic residue modulo p, denoted u_+, or a nonresidue, denoted u_-. If the form p-adically represents $p^k u_+$ (or $p^k u_-$) then it automatically represents $p^{k+2m} u_+$ (or $p^{k+2m} u_-$) for all $m = 0, 1, \ldots$.

For a p-adic integer b we define

$$S(b) \ = \ \{ bc^2 \ : \ c \ \text{any nonzero } p\text{-adic integer} \} \ .$$

We say that a form p-adically represents (at least)

$$\begin{array}{l} a \ \text{ classes beginning at } 1 \ , \\ b \ \text{ classes beginning at } p \ , \\ c \ \text{ classes beginning at } p^2 \ , \\ \quad \cdots \ , \end{array}$$

if its value set includes $a + b + c + \ \cdots$ *disjoint* sets of the form

$$S(u_1), \ldots, S(u_a) \; ,$$
$$S(pv_1), \ldots, S(pv_b) \; ,$$
$$S(p^2 w_1), \ldots, S(p^2 w_c) \; ,$$
$$\cdots$$

where the u_i, v_j, w_k, ... are p-adic units.

We postpone further consideration of odd primes p to Sect. 4, where we shall see that $E_p \geq 1$, and $E_p > 1$ if $p \mid \det(f)$.

3. The 2-adic analysis

For $p = 2$ the decomposition of the form is somewhat different. Any form is 2-adically a direct sum of 2-dimensional forms $2^k \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $2^k \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$ and 1-dimensional forms of shape $(2^k u)$, where u is a 2-adic unit whose value is only important up to squares of 2-adic integers (we denote the 2-adic units u by u_1, u_3, u_5, u_7 according as $u \equiv 1, 3, 5, 7$ modulo 8). By collecting terms with the same 2^k we obtain a 2-adic decomposition

$$f_1 \oplus 2\, f_2 \oplus 4\, f_4 \oplus 8\, f_8 \oplus \cdots \; , \tag{6}$$

and again set $n_q = \dim\, f_q$. The form f_q is said to be Type II (or even) if it only represents even numbers, otherwise Type I (or odd).

This decomposition is not unique (see for example [9, Chap. 15]). We shall make use of the following facts.

(i) If f_q is Type I it may be taken to be a diagonal form.

(ii) If f_q is Type II it may be taken to be a direct sum of 2-dimensional forms $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, the first of which can be taken to be $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ if $\dim\, f_q > 2$ or either of $f_{q/2}, f_{2q}$ is Type I.

The form $2xy$, with matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$, visibly represents all even numbers, i.e. the sets $S(2u_1)$,

$S(2u_3)$, $S(2u_5)$, $S(2u_7)$, $S(4u_1)$, $S(4u_3)$, $S(4u_5)$, $S(4u_7)$, using the notation introduced in the previous section, that is to say,

4 classes beginning at 2, 4 classes beginning at 4,

or, as we shall abbreviate it,

$$[\ 4 \ @ \ 2, \quad 4 \ @ \ 4\ ]\ .$$

On the other hand the form $2(x^2+xy+y^2)$, with matrix $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, 2-adically represents $2,\ 6,\ 14,\ 26$, which belong to the sets $S(2u_1)$, $S(2u_3)$, $S(2u_5)$, $S(2u_7)$, so we write

$$[\ 4 \ @ \ 2\ ]\ ,$$

but this form does not represent any class $S(2^{2k}u_i)$.

Each of u_1, u_3, u_5, u_7 symbolizes one-eighth of the 2-adic integers, and so a set $S(u_i)$ accounts for

$$\frac{1}{8}\left[1 + \frac{1}{4} + \frac{1}{16} + \cdots\right] = \frac{1}{6}$$

of the 2-adic integers. Similarly a set $S(2^k u_i)$ accounts for

$$\frac{1}{2^k 6}$$

of the 2-adic integers. Therefore a form that represents $[\ a \ @ \ 1,\ b \ @ \ 2,\ c \ @ \ 4, \ldots\ ]$ accounts for (at least) a fraction

$$X = \frac{a}{6} + \frac{b}{12} + \frac{c}{24} + \cdots$$

of the 2-adic integers. If the 2-part of the determinant is at least Y, we write

$$[a \ @ \ 1,\ b \ @ \ 2,\ c \ @ \ 4, \ldots\ ] \rightarrow (X,\ Y)\ ,$$

and then $F_2 \geq X$, $E_2 \geq X\ Y^{\frac{1}{n}}$. Since $E_p \geq 1$ for p odd (see Sect. 4), we have

$$E \geq E_2 \geq X\, Y^{\frac{1}{n}} \ . \tag{7}$$

The contribution to E from the 2-adic part of the form is now analyzed in Tables I and II. In every case except those labeled (∗) the bound (7) is greater than or equal to

$$\frac{11}{24}\, 4^{\frac{1}{3}} \quad \text{if} \quad n = 3, \qquad \frac{1}{2}\, 4^{\frac{1}{n}} \quad \text{if} \quad n \geq 4 \ , \tag{8}$$

and establishes the desired result (3). The starred cases are dealt with in Sect. 5.

Tables I and II appear at the end of the paper.

Notes on Tables I and II

(i) We must have $n_1 > 0$, or else f is not primitive.

(ii) The calculation of the fraction of 2-adic integers represented by these forms [the information given in square brackets] is straightforward. Two examples will illustrate the method.

If f is as described by the first line of Table I, then f_1 contains a 3-dimensional form $g = \text{diag}\,\{u_a, u_b, u_c\}$, where the u_i are 2-adic units. By multiplying by a 2-adic unit we can assume $u_a u_b u_c = 1$, so u_a, u_b, u_c are one of

$$1,1,1 \qquad 1,3,3 \qquad 1,5,5 \qquad 1,7,7 \qquad 3,5,7$$

in some order. It is easy to check that each of these represents at least [$3\ @\ 1,\ 4\ @\ 2$]. For example diag $\{1,7,7\}$ represents $1 = u_1$, $11 = u_3$, $29 = u_5$, $7 = u_7$, $18 = 2u_1$, $70 = 2u_3$, $74 = 2u_5$, $14 = 2u_7$. On the other hand diag $\{1,1,1\}$ shows (by the 3-squares theorem [1], [22]) that we do not always get [$4\ @\ 1$]. By using the techniques of (for example) [9, Chap. 15], the number of cases could be further reduced, but this is immaterial.

If f is as described by the third line of Table I then (2-adically) it contains a 4-dimensional form $g = \mathrm{diag}\ \{u_a, u_b\} \oplus 2h$, where h is $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ or $\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$, so that $2h$ represents all multiples of 4. By rescaling we can also assume that u_a, u_b are one of $1,1$ $1,3$, $1,5$ or $1,7$. If $1,1$ then we see that g represents $1,5,2,6,10,14$, so $[\ 2 @ 1,\ 4 @ 2\ ]$; if $1,3$ we get $1,3,5,7$, so $[\ 4 @ 1\ ]$; and similarly in the other cases.

(iii) The forms with smallest Erdös numbers all occur in Table II: in line 4 if $n = 3$ (D_3), in line 3 if $n = 4$ (D_4), and in line 1 if $n \geq 5$ ($D_5, E_6, E_7, \ldots$).

4. The p-adic analysis when p is odd

If p (an odd prime) does not divide $\det(f)$ then $E_p = 1$. (For then the form contains a direct summand diag $\{u, u', u''\}$, which for p odd is well-known to represent all p-adic integers.)

We now suppose p is an odd prime dividing $\det(f)$. Since either symbol u_6 denotes a proportion $\frac{p-1}{2p}$ of the p-adic integers, a set $S(u_6)$ accounts for a proportion

$$\frac{p-1}{2p} \left[1 + \frac{1}{p} + \frac{1}{p^2} + \cdots \right] = \frac{p}{2p+2}$$

of these integers, so that a form which represents

$$[\ a @ 1\ ,\ \ b @ p\ ,\ \ c @ p^2\ , \ldots\ ]$$

has

$$F_p \geq \frac{p}{2p+2} \left[a + \frac{b}{p} + \frac{c}{p^2} + \cdots \right] .$$

We split the analysis into four cases as shown in Table III.

Table III

Analysis of the p-adic form $f_1 \oplus p\, f_p \oplus p^2 f_{p^2} \oplus \cdots$, where $n_q = \dim\, f_q$.

$n_1 \geq 3$ or $n_1 = 2,\, n_p \geq 2 \,:\, [\; 2 \,@\, 1,\; 2 \,@\, p \;] \to (1, p)$

$n_1 = 2,\, n_p \leq 1 \,:\, [\; 2 \,@\, 1 \;] \to \left[\dfrac{p}{p+1},\; p^{2n-5} \right]$

$n_1 = 1,\, n_p \geq 2 \,:\, [\; 1 \,@\, 1,\; 2 \,@\, p \;] \to \left[\dfrac{p+2}{2p+2},\; p^{n-1} \right]$

$n_1 = 1,\, n_p \leq 1 \,:\, [\; 1 \,@\, 1 \;] \to \left[\dfrac{p}{2p+2},\; p^{2n-3} \right]$

Table III implies that E_p is at least

$$p^{\frac{1}{n}}\,,\;\; \frac{p}{p+1}\, p^{2-\frac{5}{n}}\,,\;\; \frac{p+2}{2p+2}\, p^{1-\frac{1}{n}}\,,\;\; \frac{p}{2p+2}\, p^{2-\frac{3}{n}} \tag{9}$$

in the four cases, the first arising only when $F_p = 1$. As claimed, all four quantities exceed 1 (when $n \geq 3$, $p \geq 3$). Furthermore, if d_p exceeds the lower bound used in the argument, E_p increases by a factor of $p^{1/n}$.

5. Completion of the proof

It only remains to deal with the starred cases in Table II. We first state a lemma.

Lemma. *The smallest determinant of an even n-dimensional lattice is*

$$1,\, 2,\, 3,\, 4,\, 4,\, 4,\, 3,\, 2,\, 1$$

according as n $\equiv 0, 1, 2, \ldots, 8$ (*modulo* 8).

Proof. We remark that the assertion for $n \leq 8$ follows from the well-known fact [5], [39] that the absolutely extreme lattices (suitably scaled) are even lattices with these determinants. Alternatively, and for all n, the assertion can be checked immediately from the list of possible genera of small determinant given in Table 15.4 of [9]. For example, if $n \equiv 3, 4$ or 5 (mod 8), then the determinant cannot be less than 4 (for if so then by Table 15.4 the signature is $0, \pm 1$ or

6 2 mod 8, which is a contradiction), while the lattices $mE_8 \oplus D_3$, $mE_8 \oplus D_4$, $mE_8 \oplus D_5$ show that determinant 4 is possible.

We now return to the proof of the proposition in the starred case, noting that $F_2 = 1/2$. We argue as follows, in each case obtaining a lower bound on E which is greater than or equal to (8).

If $d_2 \geq 4$ then $E_2 \geq \frac{1}{2}\, 4^{1/n}$.

Therefore d_2 is 1 or 2. If all F_p (for p odd) are equal to 1, then $E = \frac{1}{2}\, d^{1/n}$, and the result follows from the lemma.

Otherwise we have $F_p \neq 1$ for some odd prime p, and E contains a factor E_p which (since $F_p \neq 1$) is at least one of the last three quantities in (9). All three quantities are increasing functions of n (for $n \geq 3$) and p (for $p \geq 3$).

If $n \geq 5$ these quantities are at least

$$\frac{9}{4} = 2.25, \quad \frac{5}{8}\, 3^{4/5} = 1.5051, \quad \text{or} \quad \frac{9}{8}\, 3^{2/5} = 1.7458,$$

and so $E \geq E_2 E_p \geq \frac{1}{2}\, 1.5051 \geq \frac{1}{2}\, 4^{1/n}$.

If $p \geq 5$ then E_p is at least (if $n = 4$)

$$\frac{5}{6}\, 5^{3/4} = 2.7864, \quad \frac{7}{12}\, 5^{3/4} = 1.9505, \quad \frac{5}{12}\, 5^{5/4} = 3.1153,$$

or (if $n = 3$)

$$\frac{5}{6}\, 5^{1/3} = 1.4250, \quad \frac{7}{12}\, 5^{2/3} = 1.7057, \quad \frac{25}{12} = 2.0833,$$

and we have (since d_2 must be 2 when $n = 3$)

$$E \geq E_2 E_p \geq \frac{1}{2} \cdot \frac{5}{6}\, 5^{3/4} = 1.3932, \quad (n=4),$$

$$E \geq E_2 E_p \geq \frac{1}{2}\, 2^{1/3} \cdot \frac{5}{6}\, 5^{1/3} = 0.8979, \quad (n=3)\ .$$

Therefore the only odd prime present is 3, and n is 3 or 4. If d_3 is greater than the value used in (9) then E_3 increases by $3^{1/n}$ and we are done. Therefore we are in the situation described by one of the last three lines of Table III (not the first line, since $F_3 > 1$), and d_3 has the least possible value in each case.

If $n=4$ then $d_2=1$ and 3-adically the form is one of

$$\mathrm{diag}\{6\,1,\ 6\,1,\ 6\,3,\ 6\,9\},\quad \mathrm{diag}\{6\,1,\ 6\,3,\ 6\,3,\ 6\,3\},\quad \mathrm{diag}\{6\,1,\ 6\,3,\ 6\,9,\ 6\,9\}\ .$$

However such forms cannot exist globally, since they do not satisfy the product formula. (In the notation of [9], Chap. 15, the 3-excess is an odd multiple of 2.)

If $n=3$ then $d_2=2$ and 3-adically the form is one of

$$(\mathrm{a})\ \ \mathrm{diag}\{6\,1,\ 6\,1,\ 6\,3\},\quad (\mathrm{b})\ \ \mathrm{diag}\{6\,1,\ 6\,3,\ 6\,3\},\quad (\mathrm{c})\ \ \mathrm{diag}\{6\,1,\ 6\,3,\ 6\,9\}\ ,$$

which must be examined individually.

(a) This form represents at least [2 @ 1, 1 @ 3], so $F_3 \geq 7/8$ and

$$E \geq \frac{1}{2}\, 2^{\frac{1}{3}} \cdot \frac{7}{8}\, 3^{\frac{1}{3}} = 0.7950\ .$$

Equality holds for the form $A_2 \oplus A_1$, which has the second-smallest Erdös number in 3 dimensions.

(b) This is eliminated directly by Eq. (9):

$$E \geq \frac{1}{2}\, 2^{\frac{1}{3}} \cdot \frac{5}{8}\, 3^{\frac{2}{3}} = 0.8190\ .$$

Equality holds for the form $A_2 \oplus 6_1$, which has the third-smallest Erdös number in 3 dimensions.

(c) The form represents at least [$1 @ 1, 1 @ 3, 1 @ 9$], so $F_3 \geq 13/24$, and

$$E \geq \frac{1}{2} 2^{\frac{1}{3}} \cdot \frac{13}{24} 27^{\frac{1}{3}} = 1.0237 .$$

This completes the proof.

6. Lattices with the smallest Erdös numbers

Using the methods of [10] it is easy to show that there is a unique lattice L_n with minimal Erdös number in each dimension $n \leq 10$, namely

$$A_0, A_1, A_2, A_3 \cong D_3, D_4, D_5, E_6, E_7, E_8, E_8 \oplus A_1, E_8 \oplus A_2 .$$

Then $L_{m+8} = E_8 \oplus L_m$ is an example in dimension $m+8$. A second sequence of examples is given (using the notation of [10]) by

$$L'_{16-m} = L_m^{\perp} \text{ in } D_{16}^{+} \quad (m \leq 5),$$
$$L'_{16+m} = L_m \oplus D_{16}^{+} .$$

In view of the isomorphism $A_3 \cong D_3$, this actually gives two distinct lattices in dimension 13. In fact these two sequences give all the lattices with minimal Erdös numbers in dimensions up to 17. The number of such lattices is

$$1 \text{ for } n \leq 10,$$
$$2 \text{ for } n = 11, 12, 14, 15, 16, 17,$$
$$3 \text{ for } n = 13,$$

and at least 4 for $n \geq 18$.

Our methods can be modified to find all lattices in a given dimension with Erdös number less than a given bound. The p-adic localizations can be found by analyzing larger trees of possible cases, and all forms with given localizations can then be enumerated using the methods of [10]. In this way we found all 3-dimensional lattices with $E \le 1.02$ and all 4-dimensional lattices with $E \le 1.05$. The results are shown in Tables IV and V. Below each matrix we give its Erdös number, both in the form $F\, d^{1/n}$ and numerically, rounded to four decimals.

Table IV

The 3-dimensional lattices with smallest Erdös numbers

$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 2 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 6 \end{matrix}$	$\begin{matrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{matrix}$
$\frac{11}{24}\, 4^{1/3}$	$\frac{7}{16}\, 6^{1/3}$	$\frac{5}{16}\, 18^{1/3}$	$\frac{5}{6}\, 1^{1/4}$
0.7276	0.7950	0.8190	0.8333
$\begin{matrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 8 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 24 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 4 \end{matrix}$
$\frac{1}{3}\, 16^{1/3}$	$\frac{21}{64}\, 24^{1/3}$	$\frac{15}{64}\, 72^{1/3}$	$\frac{11}{24}\, 10^{1/3}$
0.8399	0.9465	0.9750	0.9874
$\begin{matrix} 4 & 2 & 0 \\ 2 & 4 & 0 \\ 0 & 0 & 1 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 4 & 1 \\ 0 & 1 & 2 \end{matrix}$	$\begin{matrix} 2 & 1 & 0 \\ 1 & 2 & 0 \\ 0 & 0 & 12 \end{matrix}$	
$\frac{7}{16}\, 12^{1/3}$	$\frac{7}{16}\, 12^{1/3}$	$\frac{11}{36}\, 36^{1/3}$	
1.0016	1.0016	1.0089	

Table V

The 4-dimensional lattices with smallest Erdös numbers

$\begin{matrix}2&0&0&1\\0&2&0&1\\0&0&2&1\\1&1&1&2\end{matrix}$	$\begin{matrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&2\end{matrix}$	$\begin{matrix}2&1&0&0\\1&2&1&0\\0&1&2&0\\0&0&0&2\end{matrix}$	$\begin{matrix}2&1&0&0\\1&2&0&0\\0&0&2&1\\0&0&1&2\end{matrix}$
$\frac{1}{2}\,4^{1/4}$	$\frac{1}{2}\,5^{1/4}$	$\frac{1}{2}\,8^{1/4}$	$\frac{1}{2}\,9^{1/4}$
0.7071	0.7477	0.8409	0.8660
$\begin{matrix}2&1&0&0\\1&2&0&0\\0&0&2&0\\0&0&0&2\end{matrix}$	$\begin{matrix}2&1&1&0\\1&2&1&1\\1&1&2&1\\0&1&1&4\end{matrix}$	$\begin{matrix}2&1&0&0\\1&2&1&0\\0&1&2&1\\0&0&1&4\end{matrix}$	$\begin{matrix}1&0&0&0\\0&1&0&0\\0&0&1&0\\0&0&0&1\end{matrix}$
$\frac{1}{2}\,12^{1/4}$	$\frac{1}{2}\,12^{1/4}$	$\frac{1}{2}\,13^{1/4}$	$1^{1/4}$
0.9306	0.9306	0.9494	1.0000
$\begin{matrix}2&1&0&0\\1&2&1&0\\0&1&2&0\\0&0&0&4\end{matrix}$	$\begin{matrix}2&1&0&0\\1&2&0&0\\0&0&6&3\\0&0&3&6\end{matrix}$	$\begin{matrix}4&1&2&2\\1&4&2&2\\2&2&4&1\\2&2&1&4\end{matrix}$	$\begin{matrix}4&-1&-1&-1\\-1&4&-1&-1\\-1&-1&4&-1\\-1&-1&-1&4\end{matrix}$
$\frac{1}{2}\,16^{1/4}$	$\frac{1}{3}\,81^{1/4}$	$\frac{1}{3}\,81^{1/4}$	$\frac{3}{10}\,125^{1/4}$
1.0000	1.0000	1.0000	1.0031

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Appendix

In this appendix we show that a positive definite quadratic form $f(x) = \Sigma\, x_i a_{ij} x_j$ of dimension $n \geq 3$ and finite Erdös number E must be proportional to a form in which all the a_{ij} are rational.

Let $\alpha_1, \ldots, \alpha_k$ be a maximal subset of the a_{ij} that is linearly independent over the rationals. For any given $\varepsilon > 0$ we can replace $\alpha_2, \ldots, \alpha_k$ by nearby rational multiples of α_1 so as to obtain a positive definitive quadratic form f_ε with determinant

$$d_\varepsilon \leq d(1+\varepsilon), \quad \text{where} \quad d = \det f \;,$$

and such that

$$\frac{f(x)}{1+\varepsilon} \leq f_\varepsilon(x) \leq f(x)(1+\varepsilon)$$

for all vectors *x*.

The Erdös number E_ε of f_ε is the limit as $R \to \infty$ of

$$d_\varepsilon^{1/n} \; \frac{N_{\varepsilon,R}}{R} \;,$$

where $N_{\varepsilon,R}$ is the cardinality of

$$\{\; f_\varepsilon(x) \; : \; f_\varepsilon(x) \leq R \;\} \;.$$

But since $f(x) = f(y)$ implies $f_\varepsilon(x) = f_\varepsilon(y)$, this cardinality is at most that of the set

$$\{\; f(x) \; : \; f_\varepsilon(x) \leq R \;\} \;,$$

which is contained in

$$\{\; f(x) \; : \; f(x) \leq R(1+\varepsilon) \;\} \;,$$

whose size for large R is approximately

$$E\ d^{-1/n}\ R(1+\varepsilon)\ .$$

It follows that

$$E_\varepsilon \ \le \ \left[\frac{d_\varepsilon}{d}\right]^{1/n} E(1+\varepsilon) \ \le \ E(1+\varepsilon)^2\ .$$

In particular, if $\varepsilon \le 1$, f_ε must be projectively equivalent to (i.e. a scalar multiple of) one of the finitely many rational forms with Erdös number at most $4E$. In other words, these f_ε lie in a known finite subset of the (compact) space of positive definite quadratic forms considered projectively.

Now letting $\varepsilon \to 0$, so that $f_\varepsilon \to f$, we see that f must also be in this finite set and so is projectively rational. This completes the proof.

For $n = 2$ we can obtain a stronger result. We show that if f is not projectively rational then the values of f coincide in sets of most 4 (say $f(6\ x) = f(6\ y) = \alpha$). Since the number of x with $f(x) \le N$ is approximately $d^{-1/2}\,\pi N$, the number of distinct values of $f(x) \le N$ is at least

$$\frac{1}{4}\ d^{-1/2}\ \pi N\ .$$

(Hence the expression defining the Erdös number – see footnote 1 – tends to infinity quite rapidly.)

If there are more than four vectors x with $f(x) = \alpha$ then we can select three such vectors of the form $x, y, z = rx + sy$, where r, s are rational numbers with $rs \ne 0$. Then x and y are a rational basis with respect to which f has matrix

$$\begin{bmatrix} b_{11} & b_{12} \\ b_{12} & b_{22} \end{bmatrix} \quad (\text{say})\ .$$

But now the equations

$$\alpha = b_{11} = b_{22} = r^2 b_{11} + 2rsb_{12} + s^2 b_{22}$$

entail that the b_{ij} are all rational multiples of α, and f is projectively rational.

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Lattices with Few Distances*

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ABSTRACT

The n-dimensional lattices that contain fewest distances are characterized for all $n \neq 2$.

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