Four Icosahedra Can Meet at a Point *

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ABSTRACT

What is the maximal number of nonoverlapping copies of a regular polyhedron Π that can share a common vertex? The answer is shown to be 4 if Π is an icosahedron or dodecahedron, and is conjectured to be 7 for an octahedron and 20 for a tetrahedron. (For a cube the answer is trivially 8.)

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I. Upper Bounds

We wish to determine N_{pq}, the maximal number of copies of a regular polyhedron Π of type $\{p\,,\,q\}$ that can share a common vertex V but are otherwise disjoint. A small sphere around V intersects the polyhedra in N_{pq} nonoverlapping regular spherical q-gons of side $\pi - 2\,\pi/p$. Each such q-gon may be dissected into $2q$ right-angled spherical triangles, one of which is shown in Figure 1. This triangle has area

$$\mathrm{A} + \mathrm{B} + \mathrm{C} - \pi = B - \frac{\pi\,(q-2)}{2q} \ ,$$

where

$$\sin \mathrm{B} = \cos \frac{\pi}{q} \Big/ \cos \frac{\pi\,(p-2)}{2p}$$

cf. [3, pp. 489, 511], and so

$$N_{pq} \le 4\,\pi \big/ \{2q\,B - \pi\,(q-2)\} \ .$$

The Table gives the values of this bound and also the lower bounds established in the following section.

II. Lower Bounds

Eight cubes fit together around V with no wasted space. An arrangement of twenty tetrahedra is obtained by placing one inside each face of an icosahedron. Since the edge length of an icosahedron is slightly greater than its circum-radius, the tetrahedra only touch at the center.

An arrangement of seven octahedra can be obtained by placing a ring of five around the equator (each occupies a sector of arcos $\frac{1}{3} \approx 70^\circ\ 32'$, less than $2\,\pi/5$), plus one each at the North and South poles. There are two natural candidates for configurations of eight octahedra — at the eight vertices of a twisted cube or square antiprism, or in two C-shaped strips of four that interlock as in a tennis ball — but in both cases it seems the octahedra must overlap. We conjecture that $N_{33} = 20$ and (with less conviction) that $N_{34} = 7$.

An arrangement of $N_{53} = 4$ nonoverlapping dodecahedra can be obtained by placing them at the vertices of a regular tetrahedron. To see this we must verify that the corresponding equilateral spherical triangles (described in §I) do not overlap. The inradius of these triangles is denoted by b in Figure 1, and satisfies

$$\begin{aligned} \sin b &= \tan \frac{3\pi}{10} \cot \frac{\pi}{3} \\ &= \frac{\tau^{3/2}}{\sqrt{3}\ 5^{1/4}} = 0.7946\ldots , \end{aligned}$$

where $\tau = (1 + \sqrt{5})/2$ (the value of $\tan 3\pi/10$ can be obtained from Table 3 of [2]). On the other hand the spherical Voronoi cells of the vertices of a regular tetrahedron are also equilateral spherical triangles, with slightly larger inradius β given by $\sin \beta = \sqrt{2}/\sqrt{3} = 0.8165\ldots$. So the dodecahedra do not overlap.

The final arrangement, of $N_{35} = 4$ nonoverlapping icosahedra, is the least obvious. We work instead with four pentagonal pyramids (one of which is shown in Figure 2), obtained by ‘‘cutting the tops off’’ the icosahedra by planes through five vertices. The vertices of the i-th pyramid are labeled $A_i, \ldots, E_i, V_i$ $(i = 1, \ldots, 4)$. We first make pyramids 1 and 2 share a triangular face by setting $V_1 = V_2$, $A_1 = B_2$, $B_1 = A_2$, and similarly for pyramids 3 and 4: $V_3 = V_4$,

$A_3 = B_4$, $B_3 = A_4$. The two pairs are now brought together by requiring that $V_1 = V_3$, planes $V_1 A_1 B_1$ and $V_3 D_3 D_4$ coincide, the line $V_1 C_1$ lies in the face $V_3 C_3 D_3$, and the line $V_2 E_2$ lies in the face $V_3 D_3 E_3$. (Then C_3 is close to C_1, D_3 to $A_2 = B_1$, E_3 to E_2, and D_4 to $A_1 = B_2$.)

Coordinates for the final arrangement are as follows. (We use standard coordinates for the second icosahedron [1, p. 52]; the others are obtained by applying appropriate reflections and rotations. Only three decimal places are given, although the calculations were performed with ten-place accuracy.) $V_1 = V_2 = \cdots = (0,0,0)$, $A_2 = B_1 = (-\tau, \tau-1, -1)$, $B_2 = A_1 = (-\tau, \tau-1, 1)$, $C_2 = (1-\tau, -1, \tau)$, $D_2 = (0, -2, 0)$, $E_2 = (1-\tau, -1, -\tau)$, $C_1 = \frac{1}{3}(\tau-1, 4\tau-3, -3\tau)$, $D_1 = \frac{1}{3}(4, 4\tau-2, 0)$, $E_1 = \frac{1}{3}(\tau-1, 4\tau-3, 3\tau)$, $A_3 = B_4 = (1.260, -1.552, -0.0763)$, $B_3 = A_4 = (1.973, 0.317, -0.0763)$, $C_3 = (0.736, 1.451, -1.163)$, $D_3 = (-0.743, 0.284, -1.835)$, $E_3 = (-0.419, -1.572, -1.163)$, $C_4 = (-0.324, -1.608, 1.144)$, $D_4 = (-0.589, 0.225, 1.898)$, $E_4 = (0.831, 1.415, 1.144)$. The centers of icosahedra 1,2,3,4 are respectively $\frac{1}{3}(-\tau, 4\tau-1, 0)$, $(-\tau, -1, 0)$, $(1.015, -0.388, -1.561)$, $(1.140, -0.435, 1.459)$.

It can now be verified that this is a nonoverlapping arrangement of four icosahedra around a common point. Figure 3 shows the corresponding arrangement of four spherical pentagons on a sphere.

List of Figure Captions

Fig. 1.

Fig. 2.

Fig. 3. Four spherical pentagons, corresponding to an arrangement of four nonoverlapping icosahedra around a common vertex.

Table

Name	Symbol	Angle B	$N_{pq} \leq$	$N_{pq} \geq$
Tetrahedron	{3,3}	0.6155	22.79	20
Octahedron	{3,4}	0.9553	9.24	7
Cube	{4,3}	0.7854	8	8
Icosahedron	{3,5}	1.2059	4.77	4
Dodecahedron	{5,3}	1.0172	4.24	4

References

1. H. S. M. Coxeter, *Regular Polytopes*, Dover, N. Y., 3rd. ed., 1973.

2. H. S. M. Coxeter, M. S. Longuet-Higgins and J. C. P. Miller, *Uniform polyhedra*, Phil. Trans. Royal. Soc. London **A 246** (1954), 401-450.

3. T. M. MacRobert and W. Arthur, *Trigonometry, Part IV: Spherical Trigonometry*, Methuen, London, 1938.